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# A GENERALIZATION OF FIRST DIRECT METHOD OF PURSUIT FOR DIFFERENTIAL INCLUSIONS

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To solve the problem of pursuit in linear differential games, L.S. Pontryagin suggested two direct methods. Direct methods are of great importance in the development of the theory of differential games and in control theory under the conditions of uncertainty. It turned out to be useful also in solving the problem of control synthesis. Pontryagin direct methods have proved themselves as an effective means for solving problems of pursuit- evasion and control. These use integrals, having a number of significant differences from the classical integral. One of the differences consists in the use of multivalued mapping. Pontryagin's second direct method, based on concept of the alternating integral, which has no analogs in integration of real function. In definition of alternating integral participate of integration of setvalued mappings and geometric difference (Minkovski difference) of sets. These operations make difficulties for computation of alternating integral. From this point of view, the integral used by the first direct method has a simpler construction. Therefore, the question naturally arises of generalization the first direct method of pursuit. In this paper it will be studied a generalization of the first direct method for pursuit games, being described by differential inclusions $\dot{z} \in -F(t, v)$, where $F$ is a continuous multivalued mapping. This method will be called the modified first direct method of pursuit for differential inclusions. In particular, the class of stroboscopic strategies, the trajectory of the system are determined. For these classes games, it is proved that if the starting point belongs to the modified first integral (the integral from the multivalued mapping, which is present in the definition of the modified fist direct metod), then this is necessary and sufficient condition for completing the game in a fixed time instant in the class of stroboscobic strategies. The problem of computation this integral is important. In the present article it has also been proved that the union operations in the definition of the modified first integral can be narrowed down to the class of compact-valued mappings

**Keywords:** differential inclusion, differential games, crossection, stroboscopic strategy, admissible control, evader, pursuer, pursuit partition, nearly stroboscopic strategy.

In the present paper it will be studied a generalization of the first direct method for pursuit games, being described by differential inclusions $\dot{z} \in -F(t, v)$, where $F$ is a continuous multivalued mapping. This method will be called the modified first direct method of pursuit for differential inclusions. On the basis of the integral of the modified first direct method of pursueit, a necessary and sufficient condition for the completion of the game in the class of stroboscopic strategies of the pursuer in a fixed time instant is obtained.

Futher we shall use the following notations: $I = [0, \tau]$ is the fixed closed interval of time; $\Delta$ is a subsegment of $I$; $|\Delta|$ is the lenth of $\Delta$; $K^d$ ($C^d$, respectively) is the collection of all nonempty compact (closed) subsets of $R^d$. If the set $A$ is convex , we will write $A \in coK^d$ ($A \in coC^d$, respectively); $H = \{z \in R^d \,\big|\, |z| \le 1\}$ is the unit closed ball in $R^d$. $\omega = \{0 = \tau_0 < \tau_1 < ... < \tau_{n-1} < \tau_n = \tau\}$ is the partition of the segment $I$; $\Omega$ is the collection of all partition of the segment $I$; $\Delta_i = [\tau_{i-1}, \tau_i]$; $|\Delta_i| = \delta_i = \tau_i - \tau_{i-1}$; $|\omega| = \max\limits_{1 \le i \le n} \delta_i$ is the diameter of the partition $\omega$; $\int_i$ is an integral over the interval $\Delta_i$; If $A$ is a subset of the Euclidean space, then $A[\Delta]$ is the aggregate of all measurable functions $a(\cdot): \Delta \to A$. Denoted by $AC(J)$ the aggregate of all absolutely continuous functions $z(\cdot): J \to R^d$.

We considered the controlled differential inclusion

$$\dot{z} \in -F(t, v), \tag{1}$$

where $z \in R^d, t \in I = [0, \tau], v \in Q, Q \in K^q, F: I \times Q \to coK^d$ is a continuous mapping. There is also given subset $M, M \in coC^d$ which is called terminal set of system (1).

The pursuit problem in L.S. Pontryagin's approach is posed as follows:

- Let be chosen the class of strategies of the pursuer $U$ and the initial point $z_0$, a positive number $\tau$ be given. Is it possible from a point $z_0$ to complete the pursuit at the time $\tau$ in the game (1) for the class of strategies $U$ ?
- To solve this problem L.S. Pontryagin has suggested two direct methods of pursuit in a linear differential game [1].

Let $\alpha(\cdot): I \to R^+$ be a nonnegative measurable function that satisfies the condition $\int_0^\tau \alpha(t)dt = 1$ and $M \in coC^d$. Then the following equality

$$\int_0^\tau \alpha(t) M dt = M. \tag{2}$$

holds [2].

Modifications of Pontryagin's first direct method of pursuit based on the formula (2) were given in [2, 3].

If for any $B \in C^d$ a set $\{A(t) \cap B \ne \varnothing,\ t \in \Delta\}$ is closed then multivalued mapping $A(\cdot): I \to C^d$ is called measurable.

A function $a(\cdot): \Delta \to R^d$, $a(t) \in A(t)$ almost everywhere (a.e.) on $\Delta$ is called a single-valued cross-section of mapping $A(t)$ and $L[A(\cdot), \Delta]$ is denoted the family of all integrable single-valued cross-sections of mapping $A(t)$ on $\Delta$.

The set

$$\int_\Delta A(t)dt = \left\{ \int_\Delta a(t)dt,\ a(t) \in L[A(\cdot), \Delta] \right\}$$

is called the integral of the measurable multivalued mapping $A(\cdot): I \to C^d$.

Let $cl\Phi$ ( $\text{comp}\,\Phi$ , respectively) be an aggregate of all measurable closed-valued (respectively compact-valued) mappings $A(\cdot): I \to C^d$ ( $A(\cdot): I \to K^d$ , respectively) that satisfy the condition

$$\int_0^\tau A(t)dt \subset M. \quad (3)$$

The first direct pursuit method based on formula (2) was developed in [4, 5].

We construct an integral

$$W[A(\cdot), \tau] = \int \bigcap [A(t) + F(t, v)]\, dt$$

for every $A(\cdot) \in cl\Phi$ .

We will call every function $v(\cdot) \in Q(I)$ as evader admissible control.

Let $P_B = \{U\}$ be the aggregate of all Borel measurable cross-sections $U: I \times Q \to R^d$ of the mapping $F(t, v)$ on the interval $I$. The elements $U$ from the class $P_B$ are called stroboscopic strategies of a pursuer. To each triple $\xi \in R^d$ , $U \in P_B$ , $v(\cdot) \in Q(I)$ the mapping $\Gamma$ assigns a function $z(\cdot) \in AC(I)$ defined by the formula

$$z(t) = \xi - \int_0^t w(t)dt, \text{ where } w(t) = U(t, v(t)).$$

We call the function $z(t) = z(t, \xi, U, v(\cdot))$ the trajectory of system (1) on the segment $I = [0, \tau]$ corresponding to the initial state $\xi \in R^d$, the pursuer's strategy $U \in P_B$ , and the admissible control of the evader $v(\cdot) \in Q(I)$.

*Definition 1.* We will say that it is possible to complete the pursuit from initial point $\xi \in R^d$ in time $\tau$ ( at the time instant $\tau$ ) in the class of strategies $P_B$, if there is a pursuer strategy $U \in P_B$ for every $v(\cdot) \in Q(I)$ for the trajectories $z(t, \xi, U, v(\cdot)) \in$ $\in \Gamma(\xi, U, v(\cdot))$ corresponding to the triple $\xi, U, v(\cdot)$ there is an inclusion $z(t_*, \xi, U, v(\cdot)) \in M$ for certain $t_* \in I$ (correspondingly $z(\tau) \in M$ ).

Let $W: I \times Q \to coK^d$ be measurable mapping with respect to $t$ fixed $v$, continuous (in Hausdorff metric) with respect to $v$ fixed $t$ , and the function $g: I \times R^d \to R^d$ be continuous and the function $\xi: I \to R^d$ be measurable. We consider the mapping

$$\Gamma(t, v) = \{w \in W(t, v) \mid g(t, w) = \xi(t)\}$$

**Lemma 1.** *There exists a zero measure subset $S, S \subset I$ such that the restriction of the mapping $\Gamma(t, v)$ to the set $G^0 = \{(t, v) \mid t \in I \setminus S, v \in Q\}$ has a Borelean measurable cross-section.*

*Proof.* Let the function $\xi(t)$ and mapping $W(t,v)$ be continuous. Then the mapping $\Gamma(t, v)$ will be upper semicontinuous with respect to the inclusion. Therefore, the upper semicontinuous mapping has a Borelean measurable cross-section [10].

Let the function $\xi(t)$ and mapping $W(t, v)$ be measurable with respect to $t$ fixed $v$». Then there exists a set $\Delta_n \subset I, \mu(I \setminus \Delta_n) < \frac{1}{n}$ for any $\varepsilon_n = \frac{1}{n}$ such that the function $\xi(t)$ and the mapping $W(t, v)$ are continuous on this set $\Delta_n$ . Therefore the mapping $\Gamma(t, v)$ has a Borelean measurable section $w_n(t, v)$ on the set $G^n = \{(t, v) \mid t \in \Delta_n, v \in Q\}$ . We define


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$$w(t,v)=\begin{cases} w(t,v), & (t,v) \in G \\ w(t,v), & (t,v) \in G \\ .. & .. \quad .. \quad .. \\ w(t,v), & (t,v) \in G \end{cases}$$

It is easy to verify that the constructed function $w(t, v)$ is Borelean measurable and is defined on the set $G^0 = \bigcup_{n=1}^{\infty} G^n$. We put $S = I \setminus \bigcup_{n=1}^{\infty} \Delta_n$, then the mapping $\Gamma(t, v)$ has a Borelean measurable cross-section $w(t, v)$ defined on the set $G^0 = \{(t, v) \mid t \in I \setminus S, v \in Q\}$ and $\mu(S) = 0.$

Lemma is proved.

**Theorem 1.** *If the following inclusion*

$$z_0 \in W_{\text{mod}}^{\tau} = \bigcup_{A(\cdot)\in cl\Phi} W[A(\cdot), \tau]$$

*holds. Then it is possible to complete pursuit in time* $\tau$ *in the game (1) in the class of stroboscopic strategies* $U_B$ *(see also [5–9]).*

*Proof.* If $z_0 \in W_{\text{mod}}^{\tau}$, then

$$z_0 \in \int_0^{\tau} \bigcap_{v\in Q} [A(t) + F(t, v)dt$$

for some $A(\cdot) \in cl\Phi.$ Hence we obtain $z_0 = \int_0^{\tau} w(t)dt$ for some $w(t) \in [A(t) + F(t, v)$ almost for all $t \in [0, \tau]$. Therefore, in virtue of Lemma 1, there exists a Borelean measurable cross-section $w(t, v) = a(t) + f(t, v), a(t) \in A(t), f(t, v) \in F(t, v)$ of mapping $A(t) + F(t, v)$ on a compact set $I \times Q$ such that

$$z_0 = \int_0^{\tau} w(t, v)dt = \int_0^{\tau} a(t)dt + \int_0^{\tau} f(t, v)dt \text{ for any } v \in Q .$$

Let $v(\cdot) \in Q(I)$ be chosen arbitrarily. We assume $f(t, v(t)) = U(z_0, v(t))$.

We define the trajectory of system (1) corresponding to the initial point $z_0 \in R^d$, the pursuer's strategy $f(t, v(t)) = U(z_0, v(t)) \in P_B$, and the admissible control $v(\cdot) \in Q(I)$ of the evader on the interval $I = [0, \tau]$, as follows

$$z(t) = z_0 - \int_0^{t} f(t, v(t))dt.$$

On the other hand,

$$z_0 = \int_0^{\tau} a(t)dt + \int_0^{\tau} f(t, v(t))dt.$$

Since $\int_0^{\tau} a(t)dt \in M$. It follows $z(\tau) \in M.$

Theorem is proved.

The set $W_{\text{mod}}^{\tau}$ is called the integral of the modified first direct method of pursuit.

We give an example demonstrating the advantages of the integral $W_{\text{mod}}^{\tau}$ over the integral $W^{\tau}$ of the first pursuit method [6].

**Example.** Let $\dot{z} \in F(t, v)$, where $z \in R^2$, $F(t, v) = e^{tC}[P - v]$, $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $P = \{(\bar{u}, 0) \in R^2 \,\|\bar{u}| \le l\}, t \in [0, \pi]$ and $v \in Q$, $Q = \left\{(0, \bar{v}) \in R^2 \,\middle|\, |\bar{v}| \le \frac{l}{2}\right\}$, $M = \{(z_1, z_2) \in$ $\in R^2 \,\big|\, z_1^2 + z_2^2 \le l^2\}$.

In this case, the integral of the first direct method [6] $W^{\tau} = \varnothing$ for any $\tau \ge 0$. Let us prove that the integral of the modified first direct method $W_{\text{mod}}^{\tau} \ne \varnothing$ for any $\tau \ge 0$. Let $\tau = \pi$. We put $A(t) = e^{tC}Q$. Then it follows from [11] $\int_0^{\pi} A(t)dt = lH$.

We calculate [11]

$$W[A(\cdot), \pi] = \int_0^{\pi} \bigcap_{v \in Q} [A(t) + F(t, v)]dt = \int_0^{\pi} e^{tC} P dt = 2lH.$$

It is easy to verify that $W_{\text{mod}}^{\pi}$.

The problem of computation $W_{\text{mod}}^{\tau}$ is important. However, in most cases the integral $W_{\text{mod}}^{\tau}$ cannot be calculated exactly. For computation $W_{\text{mod}}^{\tau}$ it is necessary to take the union over all possible measurable closed-valued mappings. Naturally, there is a question: is it possible to single out a subfamily of measurable closed-valued mappings, which would have a simpler structure and sufficient for an approximate calculation of the set $W_{\text{mod}}^{\tau}$. In the present article, by narrowing the union operations in the definition $W_{\text{mod}}^{\tau}$ to the class of compact-valued mappings to the posed question the answer is given

**Lemma 2.** *Let $A \subset C^d$, $B \subset \beta H$. Then the following inclusions hold*

$$\begin{aligned} (A + B) \cap \alpha H &\subset A \cap (\alpha + \beta)H + B, \\ (A \underset{-}{*} B) \cap \alpha H &\subset A \cap (\alpha + \beta)H \underset{-}{*} B. \end{aligned} \tag{4}$$

*Proof.* Let $x$ be an arbitrary element from the left side of the first inclusion (4). By definition $x \in A + B$ and $x \in \alpha H$. Moreover, $x = a + b, a \in A, b \in B$ and $|x| \le \alpha$. It follows $|a| \le \alpha + \beta$. On the other side $a \in A$. That's why $a \in A \cap (\alpha + \beta)H$. Therefore $x \in A \cap (\alpha + \beta)H + B$. The proof of the second inclusion (3) is similar to the proof of the first inclusion (3).

On the basis of the inclusions (3), the following theorem is proved.

**Theorem 2**. *The equality holds*

$$W^{\tau} = \bigcup_{A(\cdot) \in \text{comp}\Phi} W[A(\cdot), \tau].$$

Note that the mappings $A(\cdot), A(\cdot) \in \text{comp}\,\Phi$ that participate in the union operations of the right-hand side of this equality are integrable. More precisely, there will be an integrable function $a(t) = |A(t)|$. Here $|A| = \sup\{|a| \,|\, a \in A\}$ for $A \in C^d$.

Let $\omega$ be an arbitrary partition from $\Omega$ and $A(\cdot) \in \text{comp}\Phi$. We put


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$$A_i = \int_i A(t)dt, \;\; F_i(v) = \int_i F(t, v)dt,$$

$$Y_i = \bigcap_{v \in Q} [A_i + F_i(v)], \;\; X^{\tau}[A(\cdot), \omega] = \sum_{i=1}^{n} Y_i.$$

We define $X^{\tau}[A(\cdot)] = \bigcap_{\omega \in \Omega} X^{\tau}[A(\cdot), \omega].$

Let $\gamma(\delta) = \min\{h[F(t_1, v_1), F(t_2, v_2)], |t_1 - t_2| < \delta, |v_1 - v_2| < \delta\}$ be modulus of continuity of the mapping $F(t, v)$.

**Theorem 3**. *Let* $A(\cdot) \in \mathrm{comp}\Phi$. *Then the following equality holds:*

$$W[A(\cdot), \tau] = X^{\tau}[A(\cdot)].$$

*Proof.* It is possible to verify easily the validity of the inclusion $W[A(\cdot), \tau] \subset$ $\subset X^{\tau}[A(\cdot)]$. Let us prove now inverse inclusion

$$X^{\tau}[A(\cdot)] \subset W[A(\cdot), \tau].$$

Let $\hat{A}_n(t)$ be a sequence of piecewise constant mappings converging to $A(t)$ almost everywhere on the interval $I$. Moreover, we can assume that $\hat{A}_n(t) \subset a(t)H$, where $a(t) = |A(t)|$. If $\hat{A}(t) \not\subset a(t)H$, then we replace it with $\frac{\hat{a}_n(t)}{|\hat{A}_n(t)|}\hat{A}_n(t)$, where $\hat{a}_n(t)$ is a sequence of piecewise constant functions that converges to $a(t)$ a.e. on $I$ and $\hat{a}_n(t) \le a(t)$ at $t \in I$. Let $\alpha_n(t) = h[A_n(t), A(t)]$ and $\varepsilon_n = \frac{1}{n}$. Then it is easy to verify that $\alpha_n(t) \to 0$ a.e. on $I$. Then by the Egorov theorem it follows that for any $\varepsilon_n = \frac{1}{n}$ there exists a set $e_n \subset I$ such that $\mu(e_n) < \varepsilon_n$ and the sequence $\alpha_n(t)$ uniformly converges to 0. Therefore, we can assume $\alpha_n(t) < \varepsilon_n$ on $I \setminus e_n$.

Let $\bar{\omega}_n = \{0 = \bar{\tau}_0^n < \bar{\tau}_1^n < \ldots < \bar{\tau}_{n-1}^n < \bar{\tau}_n^n = \tau\}$ be a sequence of refining (monotonously decreasing ) partitions of segments $I$, i.e. $\bar{\delta}_n = |\bar{\omega}_n| = \max_{1 \le i \le n} |\bar{\tau}_{i-1}^n - \bar{\tau}_i^n|$. Let us define a sequence of mappings $F_n(t, v) = F(\xi_i^n, v)$, $\xi_i^n \in \Delta_i^n = [\tau_{i-1}^n, \tau_i^n]$, $t \in \Delta_i^n$, $\xi_i^n$ — fixed point from $\Delta_i^n$. Note that the sequence $F_n(t, v)$ uniformly converges to $F(t, v)$ on $I$ and we will assume $\gamma(\bar{\delta}_n) < \varepsilon_n$. On the other hand, there is a sequence $\hat{\omega}_n = \{0 = \hat{\tau}_0^n < \hat{\tau}_1^n < \ldots < \hat{\tau}_{m-1}^n < \hat{\tau}_m^n = \tau\}$ of partitions of a segment $I$ such that $\hat{A}_n(t) = A_i$ for $t \in \hat{\Delta}_i^n = [\hat{\tau}_{i-1}^n, \hat{\tau}_i^n]$.

We put $\omega_n = \bar{\omega}_n \cup \hat{\omega}_n$. We extend the sequences $\hat{A}_n(t)$ and $F_n(t, v)$ on $\omega_n$ and we can assume that $\hat{A}_n(t)$ and $F_n(t, v)$ converge uniformly to $\hat{A}(t)$ and $F(t, v)$ on the set $I \setminus e_n$, respectively.

Let us consider the partition $\tilde{\omega}_n = \{\Delta_1^n \setminus e_n, \Delta_2^n \setminus e_n \ldots, \Delta_l^n \setminus e_n, e_n\}$.Then

$$W[A(\cdot), \tau] \subset X^{\tau}[A(\cdot)] \subset X^{\tau}[A(\cdot), \tilde{\omega}_n] = \sum_{i=1}^{l} \bigcap_{v \in Q} [\int_{\Delta_i^n \setminus e_n} A(t)dt +$$

$$+\int_{\Delta_i^n \setminus e_n} F(t,v)dt] + \bigcap_{v\in Q}\left[\int_{e_n} A(t)dt + \int_{e_n} F(t,v)dt\right] \subset$$

$$\subset \sum_{i=1}^{l}\bigcap_{v\in Q}\left[\int_{\Delta_i^n\setminus e_n}(\hat{A}_n(t)+\varepsilon_n H)dtn + \int_{\Delta_i^n\setminus e_n}(F_n(t,v)+\varepsilon_n H)dt\right] + \rho\mu(e_n)H \subset$$

$$\subset \sum_{i=1}^{l}\bigcap_{v\in Q}(\hat{A}_n^i + 2\varepsilon_n H + F_n^i)\,\mu(\Delta_i^n \setminus e_n) + \rho\varepsilon_n H. \tag{5}$$

On the other hand,

$$\int_I \bigcap_{v\in Q}[A(t) + 4\varepsilon_n H + F(t,v)]dt + (4\varepsilon_n + \rho)\varepsilon_n H =$$

$$= \int_{I\setminus e_n}\bigcap_{v\in Q}[A(t)+4\varepsilon_n H + F(t,v)]dt + \int_{e_n}\bigcap_{v\in Q}[A(t)+4\varepsilon_n H + F(t,v)]dt +$$

$$+(4\varepsilon_n + \rho)\varepsilon_n H \supset \int_{I\setminus e_n}\bigcap_{v\in Q}[A(t)+4\varepsilon_n H + F(t,v)]dt. \tag{6}$$

Now, by the following inclusions $A(t)\subset \hat{A}_n(t)+\varepsilon_n H$, $F(t,v)\subset F_n(t,v)+\varepsilon_n H$ on $I\setminus e_n$ we have

$$\int_{I\setminus e_n}\bigcap_{v\in Q}[A(t)+4\varepsilon_n nH + F(t,v)]dt \supset \int_{I\setminus e_n}\bigcap_{v\in Q}[\hat{A}_n(t)+2\varepsilon_n H + F_n(t,v)]dt =$$

$$= \sum_{i=1}^{l}\bigcap_{v\in Q}(\hat{A}_n^i + 2\varepsilon_n H + F_n^i)\mu(\Delta_i^n \setminus e_n). \tag{7}$$

In virtue of (4), (5) and (6) we obtain

$$W[A(\cdot),\tau] \subset X^\tau[A(\cdot)] \subset \sum_{i=1}^{l}\bigcap_{v\in Q}(\hat{A}_n^i + 2\varepsilon_n H + F_n^i)\mu(\Delta_i^n\setminus e_n) + \rho\varepsilon_n H \subset$$

$$\subset \int_I \bigcap_{v\in Q}[A(t)+4\varepsilon_n H + F(t,v)]dt + (4\varepsilon_n + 2\rho)\varepsilon_n H.$$

This implies

$$W[A(\cdot),\tau]\subset X^\tau[A(\cdot)] \subset \bigcap_{\varepsilon_n>0}\int_I\bigcap_{v\in Q}[A(t)+4\varepsilon_n H + F(t,v)]dt +$$

$$+(4\varepsilon_n + 2\rho)\varepsilon_n H = W[A(\cdot),\tau].$$

This concludes $W[A(\cdot),\tau] = X^\tau[A(\cdot)]$.

Theorem is also proved.

*Definition 2***.** Let the mapping $U: I\times Q \to R^d$ for every partition $\omega\in\Omega$ and for every function $v_i(\cdot)\in Q(\Delta_i)$ associate with the Borelean measurable cross-section $f_i(t, v_i(t))$ of the mappings $F(t, v_i(t))$ on $\Delta_i$ and hold the conditions

$$U(\cdot, v_1(\cdot))\Big\|_{\Delta_i} = U(\cdot, v_2(\cdot))\Big|_{\Delta_i} \quad \text{for every } v_1(\cdot)\Big|_{\Delta_i} = v_2(\cdot)\Big|_{\Delta_i}.$$

We call such mapping nearly stroboscopic pursuer strategy. Denote the family of all such pursuer strategies by $\hat{P}_B$.

We put $X^\tau[M] = \bigcup_{A(\cdot)\in \operatorname{comp}\Phi} X^\tau[A(\cdot)]$.

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**Theorem 4.** *In order to complete pursuit in game (1) at time instant* $\tau$ *in the class* $\hat{P}_B$ *of nearly stroboscopic strategies,* $z_0 \in X^{\tau}[M]$ *is necessary and sufficient* **.**

*Proof.* Let us prove initially the necessity. Let the strategy $U \in \hat{P}_B$ completes a game at time instant $\tau$, i.e. $z(\tau, U, v(\cdot)) = M$ for every $v(\cdot) \in Q(I)$.

Let $\omega \in \Omega$ and $v(\cdot) \in Q(I)$ be arbitrary. Then $z_0 - \int_0^{\tau} U(t, v(t))dt \in M$ . Otherwise,

$$z_0 - \sum_{i=1}^{n} \int_i U(t, v_i(t))dt \in M, \tag{8}$$

where $v_i(t) = v(t)$ for $t \in \Delta_i$ . We can rewrite the inequality (8) as

$$z_0 - \sum_{i=1}^{n-1} \int_i U(t, v(t))dt \in M + \int_n U(t, v(t))dt$$

Due to the convexity and closedness of the set, we have [13]

$$\sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} \frac{1}{\tau} M dt = M.$$

Taking this into account and arbitrariness of the function $v(\cdot) \in Q(I)$, and substitution of its values by $t \in \Delta_n$, we obtain

$$z_0 - \sum_{i=1}^{n-1} \int_i U(t, v(t))dt \in \bigcap_{v(\cdot) \in Q(\Delta_n)} \int_n \left[ \frac{1}{\tau} M + F(t, v(t) \right] dt + \sum_{i=1}^{n-1} \int_i \frac{1}{\tau} M dt.$$

Repeating this process ($n-1$) times, we pass to the relation $z_0 \in X^{\tau}[A(\cdot), \omega]$, where $A(t) = \frac{1}{\tau} M \in cl\Phi$ . In virtue of the arbitrariness of the partition $\omega \in \Omega$, we have

$$z_0 \in \bigcap_{\omega \in \Omega} X^{\tau}[A(\cdot), \omega] = X^{\tau}[A(\cdot)].$$

Let us prove now the sufficiency of the condition $z_0 \in X^{\tau}[M]$ for the completion of the game in the class $\hat{P}_B$. This implies that there exists $A(\cdot) \in \mathrm{comp}\Phi$ such that $z_0 \in X^{\tau}[A(\cdot), \omega]$ for arbitrary $\omega \in \Omega$. We have $z_0 \in \sum_{i=1}^{n} z_i$, where $z_i \in$ $\in \bigcap_{v(\cdot) \in Q(\Delta_i)} \int_i [A(t) + F(t, v(t))]\, dt$ .

Thus, $z_i = i\int_i [A(t) + F(t, v(t))]dt$ for every $v(\cdot) \in Q(\Delta_i)$ .

Let $v(\cdot)$ be an arbitrary admissible control of the evader on $I = [0, \tau]$, i.e. $v(\cdot) \in Q(I)$. Then by virtue of Lemma 1 it follows that there is $a_i(t) + f_i(t, v(t))$ Borelean measurable cross-section of mapping $A(t) + F(t, v(t))$ on $\Delta_i$, such that $z_i = \int_i [a_i(t) + f_i(t, v(t))]dt$ .

We define the mapping $U(t, v(t))$ on the interval $t \in I$ as $U(t, v(t)) = f_i(t, v(t))$ on every interval $t \in \Delta_i$ . It is evident that $U(\cdot, v(\cdot)) \in \hat{P}_B$ .

This implies

$$z_0 \in \int_0^{\tau} A(t)dt + \int_0^{\tau} U(t, v(t))dt \tag{9}$$

Let us determine the trajectory $z(t, z_0, U, v(\cdot))$ of system (1) corresponding to the initial point $z_0$, the admissible control $v(\cdot) \in Q(I)$ of an evader and the strategy of the pursuer on the segment , is defined as follows:

$$z(t) = z_0 - \int_0^t U(s, v(s))ds.$$

In virtue of condition (8), we get $z(\tau) \in M$. This means that on game (1) from the initial position $z_0$ it is possible to complete the pursuit at time instant $\tau$ in the class of the strategies $\hat{P}_B$.

Theorem is proved.

It should be noted that an analogue of Theorem 4 for linear differential games is proved in [12].

**Theorem 5.** *To complete pursuit at time instant $\tau$ in the class $P_B$ of stroboscopic strategies in game (1) from the point $z_0$ it is necessary and sufficient that*

$$z_0 \in W_{\text{mod}}^{\tau}. \tag{10}$$

*Proof.* The sufficiency of condition (9) follows from Theorem 1. Let us prove the necessity of condition (9) for pursuit completion in the class strategies $P_B$. Let for every $v(\cdot) \in Q(I)$ there be a strategy of pursuer $U \in P_B$ from the point $z_0$ completes pursuit at time instant $\tau$. Since every stroboscopic strategy is nearly stroboscopic it follows from Theorem 4 that $\pi z_0 \in X^{\tau}[M]$. In virtue of the Theorem 3 we obtain $z_0 \in W_{\text{mod}}^{\tau}$.

Theorem is proved.

*І.М. Іскаnаджієв*

# УЗАГАЛЬНЕННЯ ПЕРШОГО ПРЯМОГО МЕТОДУ ПЕРЕСЛІДУВАННЯ ДИФЕРЕНЦІАЛЬНИХ ВКЛЮЧЕНЬ

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Для вирішення задачі переслідування в лінійних диференціальних іграх Л.С. Понтрягін запропонував два прямі методи, які мають велике значення в розвитку теорії диференціальних ігор і теорії керування в умовах невизначеності. Це виявилося корисним і при вирішенні задачі синтезу керування. Прямі методи Понтрягіна зарекомендували себе як ефективний засіб вирішення проблем переслідування-ухилення та контролю. У них використовуються інтеграли, що мають низку істотних відмінностей від класичного інтеграла. Однією з відмінностей є використання багатозначного відображення. Прямий метод Понтрягіна, заснований на понятті знакозмінного інтеграла, не має аналогів в інтегруванні дійсної функції. Для визначення змінного інтеграла використовується інтегрування багатозначних відображень і геометрична різниця множини (різниця Мінковського). Ці операції ускладнюють обчислення змінного інтеграла. З цієї точки зору інтеграл, який використовується першим прямим методом, має більш просту конст-

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рукцію. Тому закономірно постає питання про узагальнення першого прямого способу переслідування. У статті досліджується узагальнення першого прямого методу для ігор переслідування, що описується диференціальними включеннями, де $F$ є неперервним багатозначним відображенням. Цей метод будемо називати модифікованим першим прямим методом переслідування диференціальних включень. Зокрема, визначено клас стробоскопічних стратегій, траєкторію руху системи. Для цих класів ігор доведено, якщо вихідна точка належить модифікованому першому інтегралу (інтегралу з багатозначного відображення, який присутній у визначенні модифікованого першого прямого методу), то це є необхідною і достатньою умовою для завершення гри в фіксований момент часу в класі стробоскопічних стратегій. Проблема обчислення цього інтеграла є важливою. У цій статті також доведено, що операції об'єднання у визначенні модифікованого першого інтеграла можна звузити до класу компактнозначних відображень.

**Ключові слова:** диференціальне включення, диференціальні ігри, перетин, стробоскопічна стратегія, допустимий контроль, утікач, переслідувач, розбиття переслідування, майже стробоскопічна стратегія.

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