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OPTIMAL ALLOCATION OF A LIMITED RESOURCE

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In various subject areas, the problem arises of such a distribution of a limited resource between the elements (objects) of the system, in which the system as a whole functions in the best possible way. Often this task is solved subjectively, based on the experience and professional qualifications of the decision maker (DM). In simple cases, this approach may be justified. However, with a large number of objects and in critical cases, the price of an error in a management decision increases sharply. It becomes necessary to develop formalized decision support methods for the competent distribution of resources between objects, taking into account all given circumstances. Many of such circumstances are usually limited resources. The most common case is that the total (global) resource of the system, which is to be distributed among individual objects, is limited from above. In practical cases, restrictions are imposed not only on the global resource, but also on the partial resources allocated to individual objects. In this case, restrictions can be imposed both from below and from above. Such restrictions are either known in advance or determined by technical and economic calculations or expert assessment methods. It is necessary to distinguish between conditional restrictions (when violation of the limits is undesirable) and unconditional restrictions (when their violation is physically impossible). It is easy to see that the sum of the lower constraints for all partial resources is the lower constraint for the global resource, and the sum of the upper constraints limits the global resource from above. Considering the given set of restrictions, it is required to distribute the global resource of the system between objects in such a way that the most efficient operation of the entire system as a whole is ensured. The problem lies in the construction of an adequate objective function to optimize the process of resources allocation in conditions of their limitation. A simple uniform distribution in this case is not suitable, as it can put some objects on the verge of impossibility of their functioning, while other objects will receive an unreasonably large resource.

Keywords: constraints, optimality criteria, vector optimization, resource allocation, compromise schemes.

Formalization of the problem

Given a global resource R to be distributed, as well as $n \geq 2$ system elements (objects), to each of which is allocated a partial resource r_i, their totality is a vector $r = \{r_i\}_{i=1}^n$. It is clear that the requirement must be fulfilled

$$\sum_{i=1}^{n} r_i = R. \quad (1)$$

For each object, the minimum value of the allocated resource $r_{i\,\min}$ is known (or determined by the method of expert evaluations), below which this object cannot function. This is how the system of restrictions is set from below

$$r_i \geq r_{i\,\min}, \sum_{i=1}^{n} r_{i\,\min} \leq R, i \in [1,..., n]. \quad (2)$$

On the other hand, for each object, a value $r_{i\,\max}$ is known, which the resource of the object cannot or should not exceed. The system of constraints from above has the form

$$r_i \leq r_{i\,\max}, \sum_{i=1}^{n} r_{i\,\max} \geq R, \ i \in [1,..., n]. \quad (3)$$

The formula for the domain of the vector r has the form

$$r \in X_r = \{r \,|\, r_{i\,\max} \geq r_i \geq r_{i\,\min}, i \in [1,..., n]\}. \quad (4)$$

From (2) and (3) it follows that

$$\sum_{i=1}^{n} r_{i\,\max} \geq R \geq \sum_{i=1}^{n} r_{i\,\min}. \quad (5)$$

Polar (degenerate) cases of inequality (5) lead to obvious solutions. So if $R = \sum_{i=1}^{n} r_{i\,\min}$, then the analyzed problem is reduced to such a distribution of the global resource, in which each object receives its minimum permissible partial resource: $r_i^* = r_{i\,\min}, i \in [1,..., n]$. If the global resource allows to satisfy fully the needs of the objects, i.e. $R = \sum_{i=1}^{n} r_{i\,\max}$, then the problem is solved as $r_i^* = r_{i\,\max}, i \in [1,..., n]$.

And only if expression (5) becomes a strict inequality

$$\sum_{i=1}^{n} r_{i\,\max} > R > \sum_{i=1}^{n} r_{i\,\min}, \quad (6)$$

the problem of optimizing the distribution of limited resources becomes meaningful.

The optimization problem assumes the existence of an objective function $f(r)$, the extremization of which provides a solution to the problem under consideration:

$$r^* = \arg \underset{r \in X_r}{\operatorname{extr}} f(r). \quad (7)$$

The problem is set: under conditions (6), to determine such partial resources $r^* \in X_r$, in which requirement (1) is fulfilled and some objective function $f(r)$, acquires extreme value, the type of which should be chosen and justified.

Solution method

When constructing the objective function, one should remember a specific feature of the problem — upper and lower restrictions on partial resources are unequal. If the upper bound is usually taken as a simple optimization constraint, the lower bound is deterministic. Indeed, when a partial resource approaches its minimum allowable value, it threatens the very possibility of functioning of this object.

Therefore, the expression of the desired objective function must: 1) include constraints from below in an explicit form, 2) penalize the system for the approximation of partial resources to these constraints, and 3) be differentiable by its arguments. The simplest objective function satisfying the specified requirements is

$$f(r) = \sum_{i=1}^{n} r_{i\,\min}(r_i - r_{i\,\min})^{-1}. \tag{8}$$

This formula expresses the scalar convolution of partial criteria $r_i, i \in [1,..., n]$ that are maximized by the nonlinear scheme of compromises (NSC) in the problem of multicriteria optimization [1]. Indeed, in the task under consideration, resources $r_i, i \in [1,..., n]$ have a dual nature. On the one hand, they can be considered as independent variables, arguments for optimizing the objective function $f(r)$.

On the other hand, it is logical for each of the objects to strive to maximize their partial resource, to get as far as possible from a dangerous limitation $r_{i\,\min}$ in order to increase the efficiency of their functioning. From this point of view, resources can be considered as partial criteria $r_i \geq r_{i\,\min}, i \in [1,..., n]$ for the quality of functioning of the respective facilities. These criteria are subject to maximization, they are limited from below, nonnegative and contradictory (an increase in one resource is possible only at the expense of a decrease in others).

On the basis of the above, the problem of vector optimization of the distribution of limited resources taking into account the isoperimetric constraint for arguments takes the form

$$r^* = \arg\min_{r \in X_r} f(r) = \arg\min_{r \in X_r} \sum_{i=1}^{n} r_{i\,\min}(r_i - r_{i\,\min})^{-1}, \ \sum_{i=1}^{n} r_i = R. \tag{9}$$

This is an isoperimetric problem that can be solved both analytically, using the method of undetermined Lagrange multipliers, and numerical methods, if the analytical solution turns out to be difficult.

The analytical solution involves the construction of the Lagrange function as

$$L(r, \lambda) = f(r) + \lambda(\sum_{i=1}^{n} r_i - R)$$

where λ is the undetermined Lagrange multiplier, and the solution of the system of equations

$$\frac{\partial L(r, \lambda)}{\partial r_i} = 0, \ i \in [1,..., n]$$

$$\frac{\partial L(r, \lambda)}{\partial \lambda} = \sum_{i=1}^{n} r_i - R = 0$$

Algorithms were developed and the TURBO-OPTIM computer program [2] was developed to solve multicriteria problems by numerical methods using the NSC concept and with restrictions on arguments and criteria.

Illustrative examples

Example 1. To keep animals (wolves and tigers), the zoo has at its disposal a supply of fodder with a total weight of R=12 tons (numbers are conditional). The minimum need to maintain an enclosure with wolves, below which animal depletion begins, is $r_1 \ge r_{1\min}$ = 2 tons. Similarly for tigers $r_2 \ge r_{2\min}$ = 5 tons. This is a lower bound for partial resources.

On the other hand, if wolves are allocated a resource of $r_{1\max} \ge 7$ tons, the animals will become obese, which is undesirable. For tigers — $r_{2\max} \ge 10$ tons. This is an upper bound.

Condition (6) in the form of a strict inequality (dimensions omitted)

$$r_{1\min} + r_{2\min} = 7 < R = 12 < r_{1\max} + r_{2\max} = 17$$

adheres to. Therefore, the problem of optimizing the distribution of limited resources can be set and the solution will be non-trivial.

The problem is to obtain an analytical solution for a compromise-optimal distribution of fodder between enclosures.

We construct the Lagrange function

$$L(r, \lambda) = r_{1\min}(r_1 - r_{1\min})^{-1} + r_{2\min}(r_2 - r_{2\min})^{-1} + \lambda(r_1 + r_2 - R).$$

We get a system of equations

$$\frac{\partial L(r, \lambda)}{\partial r_1} = -r_{1\min}(r_1 - r_{1\min})^{-2} + \lambda = 0$$

$$\frac{\partial L(r, \lambda)}{\partial r_2} = -r_{2\min}(r_2 - r_{2\min})^{-2} + \lambda = 0.$$

$$r_1 + r_2 - R = 0$$

Substituting numerical data

$$-2(r_1 - 2)^{-2} + \lambda = 0$$

$$-5(r_2 - 5)^{-2} + \lambda = 0$$

$$r_1 + r_2 - 12 = 0$$

and solving this system by the Gauss method (sequential elimination of variables), we get

$$r_1{}^* = 3{,}94 \text{ tons}, \ r_2{}^* = 8{,}06 \text{ tons}.$$

The problem is solved under the assumption that the relative importance of the supply of both enclosures for decision-maker (DM) is the same. If not, then the weighting coefficients α_1 and α_2, which reflect the individual advantages of DM, are introduced into the objective function. These coefficients must be normalized and determined on the simplex:

$$\alpha_1, \alpha_2 \in X_\alpha = \left\{ \alpha_i \,\middle|\, \alpha_i \ge 0, \sum_{i=1}^{n=2} \alpha_i = 1, i \in [1; 2] \right\}.$$

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Example 2. The city administration placed an order for the design and construction of three (n = 3) sports facilities: 1) a stadium, 2) a gym and 3) a swimming pool.

To fulfill the order, financing in the total amount R = 10 million UAH has been provided (here and in the future, the figures are conditional). Calculated full financial estimate for each project (top limit):

$$r_1 \le r_{1\max} = 7 \text{ mln UAH}; \; r_2 \le r_{2\max} = 5 \text{ mln UAH}; \; r_3 \le r_{3\max} = 4 \text{ mln UAH}.$$

Economic calculations determine the minimum amount of funding for individual projects, below which design is impossible (lower limit):

$$r_1 \ge r_{1\min} = 2 \text{ mln UAH}; \; r_2 \ge r_{2\min} = 1 \text{ mln UAH}; \; r_3 \ge r_{3\min} = 0{,}5 \text{ mln UAH}.$$

Condition (6) is a strict inequality (dimensions omitted)

$$\sum_{i=1}^{n} r_{i\max} = 16 > R = 10 > \sum_{i=1}^{n} r_{i\min} = 3{,}5$$

therefore, the described technique can be used for non-trivial optimization of the distribution of limited resources.

The problem is set: using the TURBO-OPTIM vector optimization program, to find compromise-optimal values of partial amounts of financing r_1^*, r_2^* and r_3^* for the design and construction of a stadium, a gym and a swimming pool, respectively.

According to the stages of working with the program, we set: the «analytics» mode, the «simplex planning» optimization method (by default) and then enter numerical data (the dimensions are omitted):

$$r_{1\min} = 2; \; r_{1start} = 3; \; r_{1\max} = 7,$$

$$r_{2\min} = 1; \; r_{2start} = 3; \; r_{2\max} = 5,$$

$$r_{3\min} = 0{,}5; \; r_{3start} = 3; \; r_{3\max} = 4,$$

$$r_1 + r_2 + r_3 - 10 = 0,$$

$$y_1 = 1/r_1; \; y_2 = 1/r_2; \; y_3 = 1/r_3,$$

$$y_{1\max} = A_1 = \frac{1}{r_{1\min}} = 0{,}5; \; y_{2\max} = A_2 = \frac{1}{r_{2\min}} = 1;$$

$$y_{3\max} = A_3 = \frac{1}{r_{3\min}} = 2.$$

After that, we give the command «execute» and the program determines the necessary values of the partial amounts of funding for projects:

$$r_1^* = 4{,}945 \text{ mln UAH}; \; r_2^* = 3{,}083 \text{ mln UAH}; \; r_3^* = 1{,}972 \text{ mln UAH}.$$

The obtained result corresponds to the unified version of the convolution by the nonlinear trade-off scheme, which is widely used. If it is desirable to take into account the individual preferences of DM, the program contains a corresponding option.

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ОПТИМАЛЬНИЙ РОЗПОДІЛ ОБМЕЖЕНОГО РЕСУРСУ

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У різних предметних областях виникає проблема такого розподілу обмеженого ресурсу між елементами (об’єктами) системи, за яким система в цілому функціонує якнайкраще. Часто ця проблема вирішується суб’єктивно, на основі досвіду та професійної кваліфікації особи, яка приймає рішення (ОПР). У найпростіших випадках такий підхід може бути виправданим. Однак за великої кількості об’єктів і у відповідальних випадках різко зростає ціна помилки управлінського рішення. Стає необхідною розробка формалізованих методів підтримки прийняття рішень для грамотного розподілу ресурсів між об’єктами з урахуванням усіх обставин. Багато з таких обставин — зазвичай обмежені ресурси. Найбільш поширений випадок обмеженості зверху сумарного (глобального) ресурсу системи, що підлягає розподілу між окремими об’єктами. На практиці обмеження накладаються як на глобальний ресурс, так і на парціальні ресурси, виділені окремим об’єктам. При цьому обмеження можуть бути накладені як знизу, так і зверху. Такі обмеження або відомі заздалегідь, або визначаються техніко-економічними розрахунками чи методами експертних оцінок. Слід розрізняти умовні обмеження (коли порушення меж небажано) і обмеження безумовні (коли їх порушення фізично неможливе). Нескладно бачити, що сума обмежень знизу всіх парціальних ресурсів є обмеження знизу для глобального ресурсу, а сума обмежень зверху обмежує глобальний ресурс зверху. З огляду на заданий комплекс обмежень потрібно так розподілити глобальний ресурс системи між об’єктами, щоб забезпечувалася найбільш ефективна робота всієї системи в цілому. Проблема полягає у побудові адекватної цільової функції для оптимізації процесу розподілу ресурсів в умовах їхньої обмеженості. Простий рівномірний розподіл у цьому разі не підходить, оскільки може поставити деякі об’єкти на межу неможливості їх функціонування, тоді як інші об’єкти отримають невиправдано великий ресурс.

Ключові слова: обмеження, критерії оптимальності, векторна оптимізація, розподіл ресурсу, компромісні схеми.

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