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THE PROBLEM OF AN OBJECT EVALUATION AND OPTIMIZATION UNDER SEVERAL CRITERIA

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The problems of estimation and optimization of an object pursuing several goals are considered. In the estimation problem, the evaluation function is calculated with known parameters that determine the state of the object. In the optimization problem, there are optimization arguments that deliver the extremum of the objective function. Both the evaluation and objective functions are built on the basis of the concept of a nonlinear trade-off scheme, for which the principle «away from restrictions» is fulfilled. Both tasks are solved in a formalized manner, without the direct participation of the decision maker (DM). Model examples are given. The object O is considered, the state of which is determined by the set of values $x_1, x_2, ..., x_n$, that make up the vector $x = \{x_i\}_{i=1}^n \in X$. The object pursues several goals, the degree of achievement of each of them is expressed by the corresponding criterion $y_k(x), k \in [1, ... s]$. The criteria form a vector $y = \{y_k(x)\}_{k=1}^s \in M$. Restrictions are imposed on the criteria $y_{k\min}(x) \le$ $\le y_k(x) \le y_{k\max}(x)$. The problem of estimating the quality of the functioning of an object O is to determine the value of a certain function $Y[y(x)]$ with known parameters $x_1, x_2, ..., x_n$. The function $Y[y(x)]$ in this case is called the evaluation function. The optimization problem is to determine the values $x_1, x_2, ..., x_n$ by extremizing the function $Y[y(x)]$. In this case, the function $Y[y(x)]$ is the objective, and the parameters are called optimization arguments. Both tasks require the function $Y[y(x)]$. In fact, this function is a scalar convolution of the criteria vector $y(x)$, which reflects the utility function of the decision maker (DM) in solving a specific estimation or optimization problem. Scalar convolution is an act of composing criteria. The criterion $y_k(x)$ is a measure of the quality of the object O functioning in relation to the achievement of the k-th goal. If «more» means «better», then such a criterion should be maximized to improve the quality. Otherwise, the criterion is minimized. For definiteness, we consider the optimization problem under minimized performance criteria.

Keywords: multicriteria, utility function, scalar convolution, formalization, situation, nonlinear trade-off scheme, Harrington scale.

Optimization problem

The objective function $Y[y(x)]$ links the quality criteria vector with the optimization arguments. This function is a model of the decision maker’s utility function [1] when solving a specific synthesis problem. With some reservations, the optimization problem is formulated as finding such a combination of arguments from the domain of their definition, in which the objective function acquires an extreme value. If, without loss of generality, we assume that «better» means «less», then

$$x^* = \arg \min_{x \in X} Y[y(x)].$$

In the concept of optimality, in addition to criteria, restrictions play an equally important role, both in terms of optimization arguments $x \in X$ and in terms of decision efficiency criteria $y \in M$. Even small changes can significantly affect the solution [2]. Moreover, the very concept of a decision-making situation is evaluated by a measure of the dangerous approximation of individual criteria to their extremely permissible values (restrictions). It is logical to consider the difference between the current value of the criterion and its extremely allowable value as a measure of the intensity of the situation:

$$\rho_k(x) = y_{k\max} - y_k(x), \rho_k \in [0, \dots y_{k\max}], k \in [1, \dots s],$$

where $y_{\max} = \{y_{k\max}\}_{k=1}^{s}$ is the vector of maximum admissible minimized criteria.

If some criterion $y_p(x)$, $p \in [1, \dots s]$, dangerously approaches its limit $y_{p\max}$, that is $\rho_p(x) \to 0$, then we call such a situation tense. In a tense situation, the DM pays primary attention only to this, the most «unfavorable» criterion, trying to remove it from the dangerous border. In this case, under criteria of one dimension, the optimization problem is solved using the minimax (Chebyshev) model

$$x^* = \arg \min_{x \in X} Y[y(x)]_1 = \arg \min_{x \in X} \max_{k \in [1, \dots s]} y_k(x).$$

In less tense situations, it is necessary to return to the simultaneous satisfaction of other criteria, considering the contradictory unity of all the interests and goals of the system. At the same time, the decision maker varies his assessment of gain according to one criteria and loss according to others, depending on the situation.

At $\rho_k(x) \to 1, k \in [1, \dots s]$, the situation is so calm that the criteria are small and there is no threat of violation of the constraints. In a calm situation, the DM believes that a unit of deterioration in any of the criteria is fully compensated by an equivalent unit of improvement in any of the others. Here, the optimization problem is solved by applying the integral optimality model

$$x^* = \arg \min_{x \in X} Y[y(x)]_2 = \arg \min_{x \in X} \sum_{k=1}^{s} y_k(x).$$

So, as a rule, the DM varies his choice from the integral optimality model in calm situations to the minimax model in stressful situations. In intermediate cases, the DM chooses compromise schemes that give different degrees of satisfaction of individual criteria, in accordance with the given situation.

From the standpoint of a systematic approach, it is expedient to replace the problem of choosing a compromise scheme with an equivalent problem of synthesizing a certain unified scalar convolution of criteria, which in various situations would automat-

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ically express adequate principles of optimality. Separate models of trade-off schemes are combined into a single integral model, the structure of which is adapted to the situation of making a multicriteria decision.

Requirements for the synthesized function $Y^*[y(x)]$:

— it must be smooth and differentiable;

— in tense situations, it should express the minimax principle $Y[y(x)]_1$;

— in calm conditions – the principle of integral optimality $Y[y(x)]_2$;

— in intermediate cases should lead to Pareto-optimal solutions, giving different measures of partial satisfaction of the criteria.

In order to express the desire «away from restrictions» in any situation, it is necessary to explicitly include in the expression for the desired scalar convolution the characteristic of the tension of the situation $\rho_k(x) = y_{k\max} - y_k(x),\ k \in [1, \dots s]$.

Several functions can be considered that satisfy the above requirements. The simplest of them in the case of minimized criteria is the scalar convolution

$$Y^*[y(x)] = \sum_{k=1}^{s} y_{k\max}[y_{k\max} - y_k(x)]^{-1}.$$

Thus, a nonlinear trade-off scheme (NTS) is proposed, which corresponds to the vector optimization model, which explicitly depends on the characteristics of the intensity of the situation:

$$x^* = \arg\min_{x \in X} \sum_{k=1}^{s} y_{k\max}[y_{k\max} - y_k(x)]^{-1}.$$

It can be seen from this expression that if any of the criteria, for example $y_i(x)$, starts to come close to its limit $y_{i\max}$, i.e. the situation becomes tense, then the corresponding term $Y_i = \dfrac{y_{i\max}}{y_{i\max} - y_i(x)}$ in the sum being minimized will increase so much that the problem of minimizing the entire sum will be reduced to minimizing only the given worst term, i.e., ultimately, the criterion $y_i(x)$. This is equivalent to the action of the minimax model $Y[y(x)]_1$.

If all the criteria are far from their limits, i.e. the situation is calm, then the model $Y^*[y(x)]$ acts equivalent to the model of integral optimality $Y[y(x)]_2$. In intermediate situations, various degrees of partial alignment of the criteria are obtained.

Note that the scalar convolution construction $Y^*[y(x)]$ allows solving the optimization problem even in the case when the criteria have different dimensions. The solution of a multicriteria optimization problem according to a nonlinear trade-off scheme is carried out in a formalized manner, without the direct participation of the decision maker.

If «better» means «more», then for the maximizing criteria, the scalar convolution according to the nonlinear trade-off scheme has the form

$$Y^*[y(x)] = \sum_{k=1}^{s} y_{k\min}[y_k(x) - y_{k\min}]^{-1},$$

where $y_{k\min}$ are the minimum allowable values of the criteria to be maximized.

In this case, the optimization problem is solved as

$$x^* = \arg\min_{x \in X} \sum_{k=1}^{s} y_{k\min}[y_k(x) - y_{k\min}]^{-1}.$$

The analytical solution of the optimization problem is represented as a solution to the system of equations

$$\frac{\partial Y^*[y(x)]}{\partial x_k} = 0,\ k \in [1, ..., s].$$

If the analytical solution turns out to be difficult, then numerical methods or a computer program for multicriteria optimization are used [3].

Example 1. Let us consider the problem of optimizing the distribution of a limited volume P of water for irrigation of n fields. There are known lower limits for allocated water resources for each of the fields: $p_i \geq p_{i\min},\ i \in [1, n].$ It is pointless to allocate less water, the plants will simply dry out. We assume that the upper constraints in our problem are satisfied, so they are not considered here.

Considering the given set of restrictions, it is required to distribute the global water resource between the fields in such a way that the most efficient operation of the entire irrigation system as a whole is ensured.

We will solve this problem within the framework of the concept of a nonlinear trade-off scheme. We represent the objective function in the form

$$Y^*[y(p)] = \sum_{i=1}^{n} p_{i\min}(p_i - p_{i\min})^{-1},$$

where $p = \{p_i\}_{i=1}^{n}$ is the vector of partial water resources allocated to individual fields, $p \in X_p = [0, ..., P].$ It is clear that $\sum_{i=1}^{n} p_i = P,$ where P is the global resource to be distributed.

The presented objective function is nothing more than an expression of the scalar convolution of the vector of maximized criteria $p = \{p_i\}_{i=1}^{n}$ according to the nonlinear trade-off scheme (NTS) in the multi-objective optimization problem [3]. Indeed, in this problem the resources $p_i,\ i \in [1, ..., n],$ have a dual nature.

On the one hand, they can be considered as independent variables, optimization arguments of objective function $Y^*[y(p)].$ On the other hand, it is logical for each of the fields to «strive» to maximize its partial water resource for irrigation, to move as far as possible from a dangerous limitation $p_{i\min}$ in order to increase the efficiency of its functioning.

This gives grounds to consider resources $p_i,\ i \in [1, ..., n],$ as criteria for the quality of the functioning of the corresponding fields. These criteria are subject to maximization, they are limited from below, nonnegative and contradictory (an increase in one resource is possible only at the expense of a decrease in others).

Based on the above, the problem of vector optimization of the distribution of limited resources, taking into account the isoperimetric constraint for the arguments $\sum_{i=1}^{n} p_i = P,$ takes the form

$$p^* = \arg\min_{p \in X_p} Y^*[y(p)] = \arg\min_{p \in X_p} \sum_{i=1}^{n} p_{i\min}(p_i - p_{i\min})^{-1},\ \sum_{i=1}^{n} p_i = P.$$

This problem can be solved both analytically, using the method of indefinite Lagrange multipliers, and numerically, if the analytical solution turns out to be difficult.

The analytical solution provides for the construction of the Lagrange function in the form

$$L(p,\lambda) = f(p) + \lambda\left(\sum_{i=1}^{n} p_i - P\right),$$

where λ is the indefinite Lagrange multiplier, and the solution of the system of equations

$$\frac{\partial L(p,\lambda)}{\partial p_i} = 0,\ i \in [1, n],$$

$$\frac{\partial L(p,\lambda)}{\partial \lambda} = \sum_{i=1}^{n} p_i - P = 0.$$

Let's go back to our example. For irrigation of two $(n = 2)$ fields, the farm has a water reserve with a total volume of $P = 12$ tons (conditional figures). The minimum irrigation requirement for the first field (sunflower) is $p_1 \geq p_{1\min} = 2$ tons, the second field (cabbage) — $p_2 \geq p_{2\min} = 5$ tons. These are the lower bounds for partial resources.

The task is set: to obtain a solution for a compromise-optimal distribution of water between fields.

We solve the problem of vector optimization of the distribution of limited resources analytically, using the NTS and the Lagrange method of indefinite multipliers.

Building the Lagrange function

$$L(p,\lambda) = p_{1\min}(p_1 - p_{1\min})^{-1} + p_{2\min}(p_2 - p_{2\min})^{-1} + \lambda(p_1 + p_2 - P).$$

We get a system of equations

$$\frac{\partial L(p,\lambda)}{\partial p_1} = -p_{1\min}(p_1 - p_{1\min})^{-2} + \lambda = 0,$$

$$\frac{\partial L(p,\lambda)}{\partial p_2} = -p_{2\min}(p_2 - p_{2\min})^{-2} + \lambda = 0, .$$

$$p_1 + p_2 - P = 0.$$

Substituting numeric data

$$-2(p_1 - 2)^{-2} + \lambda = 0,$$

$$-5(p_2 - 5)^{-2} + \lambda = 0,$$

$$p_1 + p_2 - 12 = 0$$

and solving this system by the Gauss method (successive elimination of variables), we obtain

$$p_1^* = 3{,}94 \text{ tons},\ \ p_2^* = 8{,}06 \text{ tons}.$$

In more complex cases, numerical methods or a computer program for multi-objective optimization are used [3].

Assessment problem

Unlike optimization problems, multicriteria estimation belongs to the class of analysis problems. Here, the convolution $Y[y(x)]$ is not an objective, but an evaluation function, and its value quantitatively expresses a measure of the quality of a multicriteria object for given values of the *x* arguments.

In the multicriteria evaluation of objects, it often becomes necessary to obtain not only an analytical, but also a qualitative assessment. To do this, the scalar convolution expression $Y[y(x)]$ should be normalized and the resulting value Y_0 correlated with the qualitative gradations of some normalized scale.

To determine Y_0, we use the expression for the normalized scalar convolution of the minimized criteria, obtained in [3], which in the case of the same weight coefficients (or their absence) has the form:

$$Y_0 = 1 - \frac{s}{Y[y(x)]},$$

where s is the number of criteria.

Table

Quality category	Intervals of the reversed normalized rating scale Y_0
Unacceptable	1,0 – 0,7
Low	0,7 – 0,5
Satisfactory	0,5 – 0,4
Good	0,4 – 0,2
High	0,2 – 0,0

Harrington’s verbal-numerical scale [4, 5] for minimized criteria is presented in Table. It shows the relationship between the qualitative gradations of the properties of objects and the corresponding normalized quantitative estimates Y_0.

A qualitative (linguistic) assessment of an object is obtained by comparing the analytical assessment Y_0 with Harrington’s verbal-numerical scale. This scale is a characteristic of the severity of the criterion property and has a universal character. The numerical values of the gradations are obtained on the basis of the analysis and processing of a large array of statistical data.

It can be said that in terms of the theory of fuzzy sets, the verbal-numerical scale acts as a universal membership function for the transition from a number to the corresponding qualitative gradation and vice versa. A transition is made from a linguistic variable (average, high score, etc.) to the corresponding quantitative scores on a scale of points, i.e. transition from fuzzy qualitative gradations to numbers and vice versa.

Evaluation of objects according to a unified verbal-numeric Harrington scale makes it possible to solve multicriteria tasks, in addition to traditional formulations, and in the case when it is required to choose an alternative from a variety of heterogeneous alternatives for which it is impossible to formulate a single set of quantitative evaluation criteria, as well as for evaluating a single (unique) alternative.

Example 2. Let us consider the problem of assessing the quality of the glide path process by several criteria when landing an aircraft. During the time $t \in [0, ..., T]$, the aircraft descends along the glide path. At time $t = T$, the aircraft touches the runway (RW) at a point located at a distance ΔlT from the calculated point.

To assess the quality of the landing of the aircraft, we will use two terminal (at $t = T$) quality criteria, namely y_1 and y_2, as well as one integral criterion y_3:

$y_1 = |\Delta l_T| \le l_{\min}$ — module of deviation from the calculated point of contact in the longitudinal plane;

$y_2 = V_T \le V_{\min}$ — vertical speed at the terminal point;

$y_3 = \frac{1}{T}\int_0^T |\Delta h|(t)\,dt \le \Delta h_{\min}$ — the average deviation from the glide path in the vertical plane.

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The following values of restrictions by criteria are set (conditional numbers):

$$l_{\min} = 15 \text{ m}; \; V_{\min} = 1 \text{ м/sec}; \; \Delta h_{\min} = 30 \text{ m}.$$

Next, using the nonlinear trade-off scheme, we will evaluate the quality of the aircraft landing with the numerical values of the criteria obtained during a specific landing:

$$y_1 = 6 \text{ m}; \; y_2 = 0{,}2 \text{ м/sec}; \; y_3 = 10 \text{ m}.$$

We calculate the scalar convolution of the criteria according to the nonlinear trade-off scheme

$$Y(y) = \frac{l_{\min}}{l_{\min} - y_1} + \frac{V_{\min}}{V_{\min} - y_2} + \frac{\Delta h_{\min}}{\Delta h_{\min} - y_3}.$$

Substituting numerical data, we get

$$Y(y) = \frac{15}{15-6} + \frac{1}{1-0.2} + \frac{30}{30-10} = 4{,}42.$$

We calculate the normalized scalar convolution by the formula

$$Y_0 = 1 - \frac{s}{Y(y)} = 1 - \frac{3}{4{,}42} = 0{,}3.$$

Comparison of this value with the qualitative gradations of the inverted normalized scale (Table) allows us to conclude that this landing can be assessed as good.

The described procedure for multicriteria evaluation is applicable, in particular, to the education and training of pilots and in similar cases in other subject areas.

The solution of multicriteria problems according to the nonlinear scheme of compromises is carried out in a formalized manner, without the direct participation of the DM. This solution is basic and intended for general use. If such a task is solved in the interests of a particular person, then the basic solution can only be adjusted by introducing weight coefficients in accordance with the informal preferences of the DM.

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ПРОБЛЕМА ОЦІНКИ ТА ОПТИМІЗАЦІЇ ОБ'ЄКТА ЗА КІЛЬКОМА КРИТЕРІЯМИ

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Розглядається об'єкт O, стан якого визначається сукупністю величин $x_1, x_2, ..., x_n$, що складають вектор $x = \{x_i\}_{i=1}^n \in X$. Об'єкт переслідує кілька цілей, ступінь досягнення кожної виражається відповідним критерієм $y_k(x)$, $k \in [1, ... s]$. Критерії утворюють вектор $y = \{y_k(x)\}_{k=1}^s \in M$. На критерії накладаються обмеження $y_{k\min}(x) \le y_k(x) \le y_{k\max}(x)$. Задача оцінювання якості функціонування об'єкта O полягає у визначенні величини деякої функції $Y[y(x)]$ за відомими параметрами $x_1, x_2, ..., x_n$. Функція $Y[y(x)]$ у цьому разі називається оцінною. Задача оптимізації полягає у визначенні величин $x_1, x_2, ..., x_n$ за допомогою екстремізації функції $Y[y(x)]$. Тут функція $Y[y(x)]$ є цільовою, а параметри $x_1, x_2, ..., x_n$ називаються аргументами оптимізації. Обидві задачі передбачають наявність функції $Y[y(x)]$. По суті, ця функція є скалярною згорткою вектора критеріїв $y(x)$, яка відображає функцію корисності особи, що приймає рішення (ОПР) при вирішенні конкретної задачі оцінювання або оптимізації. Скалярна згортка є актом композиції критеріїв. Критерій $y_k(x)$ — це міра якості функціонування об'єкта O щодо досягнення *к*-ї мети. Якщо «більше» означає «краще», то для підвищення якості такий критерій підлягає максимізації. В іншому разі критерій мінімізується. Для визначеності розглядається задача оптимізації за мінімізованими критеріями якості.

Ключові слова: багатокритеріальність, функція корисності, скалярна згортка, формалізація, ситуація, нелінійна схема компромісів, шкала Харрінгтона.

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