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STAGES AND MAIN TASKS OF THE CENTURY-LONG THE CONTROL THEORY AND SYSTEM IDENTIFICATION DEVELOPMENT. Part I. STATE SPACE METHOD IN THE THEORY OF LINEAR AUTOMATIC CONTROL SYSTEMS

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This article opens a series of publications dedicated to the scientific and technical achievements of the century-long development of control theory and methods. Problems that could not be fully resolved based on one approach or another are also noted. The first of these articles is devoted to the state space method, which began to actively develop somewhere in the 70-80s of the last century within the framework of systems theory and general systems theory. A description of the method is given in relation to linear automatic control systems. A generally accepted description of controlled and observable dynamic processes in such systems is given. The problems of synthesis of stabilizing control in closed loop systems with feedback based on the theory of stability formulated by Lyapunov are mainly considered. In addition to stability, within the framework of the state space method, such important qualitative properties as controllability and observability are formulated. The most developed method is the synthesis of state-based linear controllers. Among them, a special pole placement by method of modal control synthesis occupies a special place which is considered within the framework of the Jordan implementation and various modifications of the method depending on the dimension of the control vector. The synthesis of an optimal controller with a quadratic criterion of a fairly general form is considered. In case of incomplete measurements, when the dimension of the output is less than the dimension of the state vector, in order to implement synthesized control laws based on the state, it becomes necessary to include a state estimator in the control system. In the deterministic case, you can use the Luenberger estimator described in the article, and in the presence of measurement errors in the form of zero-mean white noise, the widely used Kalman filter, also presented in the article. The output control method was not left unattended in the work. The conclusions give the advantages and disadvantages of control methods based on state-space descriptions.

Keywords: state-space method, control, feedback, stability, controllability, observability, pole placement, Kalman filter.

Introduction

The authors joined forces in writing a series of articles in which they attempted to analyze the century-long development of automatic control theory (ACT).

Over the 100 years, time interval ACT has gone through several stages of development which in generalization form can be presented in the following sequence:

1. State space methods in the theory of linear automatic control systems.
2. Methods for identifying the structure and parameters of mathematical models of controlled objects.
3. Methods for designing linear control systems based on mathematical models of «input-output» type processes.
4. Robust control methods.
5. Filtering and control methods with separable different tempo movements.
6. Prediction control systems.
7. Control of different nature complex systems based on cognitive maps models with impulse processes.
8. Trajectory control methods.

Control methods at these stages of ACT development differ in the formulation of problems, various initial conditions, mathematical models of controlled objects, optimality criteria for the synthesis of optimal controllers, description of internal and external disturbances, etc.

The authors considered it is worthwhile appropriate to analyze the main problems that were solved and are being solved at each stage of ACT development, to characterize new approaches to the design of control systems at the present stage, which were impossible or imperfect at the previous stages and appeared at subsequent stages, to generalize the classes controlled objects for which ones are developed control systems at every stage.

In this series of articles, only linear control systems are considered, assuming that possible nonlinearities of controlled objects were previously linearized befor when developing mathematical models of these objects.

1. State space methods

The state space method consists in using systems of differential equations of the first order as a mathematical model which describes dynamics of controlled objects. The system of differential equations has the form [1]

$$\begin{aligned} \dot{x}_1(t) &= a_{1_1}(t)x_1(t)+\ldots+a_{1_n}(t)x_n(t)+b_{1_1}u_1(t)+\ldots+b_{1_m}(t)u_m(t); \\ &\vdots \\ \dot{x}_n(t) &= a_{n_1}(t)x_1(t)+\ldots+a_{n_n}(t)x_n(t)+b_{n_1}u_1(t)+\ldots+b_{n_m}(t)u_m(t). \end{aligned} \qquad (1)$$

Such system is called a normal system. When entering designations

$$A(t)=\begin{bmatrix} a_{1_1}(t) & \ldots & a_{1_n}(t) \\ \vdots & \ddots & \vdots \\ a_{n_1}(t) & \ldots & a_{n_n}(t) \end{bmatrix};\ B(t)=\begin{bmatrix} b_{1_1}(t) & \ldots & b_{1_n}(t) \\ \vdots & \ddots & \vdots \\ b_{n_1}(t) & \ldots & b_{n_m}(t) \end{bmatrix},$$

the system is written in vector-matrix form

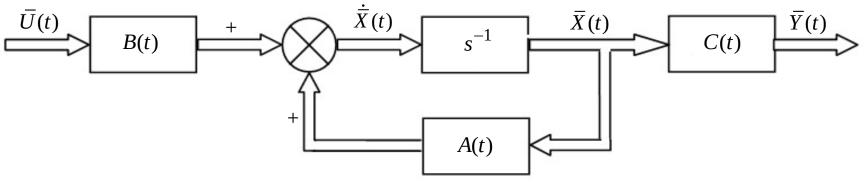
$$\dot{\bar{X}}(t)=A(t)\bar{X}(t)+B(t)\bar{U}(t)\ , \qquad (2)$$

which was called the state equation. If the matrices $A(t)$, $B(t)$ in model (2) depend on time t, then the system is multidimensional and non-stationary. When $A(t) = \text{const}$, $B(t) = \text{const}$ the system is called stationary. The vector $\bar{X}(t)$ was named the vector of state variables $x_1, x_2, \ldots, x_n$. A set of vectors $\bar{X}(t)$ is called a state space. The object is controlled using a vector $\bar{U}(t)$. The movement of the system is determined by the phase trajectory, which fully characterizes the state of the system in space R^n and time.

The relationship between the measured output coordinates $y(t)$ and the vector of state variables is described using the algebraic measurement equation

$$\bar{Y}(t) = C(t)\bar{X}(t). \quad (3)$$

Based on (2), (3), the structural diagram of the dynamic controlled object will look like this



If a one-dimensional system is described by a scalar differential equation

$$\frac{dx^n(t)}{dt^n} + a_{n-1}\frac{dx^{n-1}(t)}{dt^{n-1}} + \ldots + a_1\frac{dx(t)}{dt} + a_0 x(t) = u(t),$$

then the system in matrix form (2) by the method of phase variables will have the form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t),$$

where is the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix}$$

proposed by Frobenius and is called the Frobenius matrix. This form was widely used.

With the selected discretization period T_0, system (2) can be presented in discrete form

$$\bar{X}(k+1) = F\bar{X}(k) + G\bar{U}(k), \quad (4)$$

where $F = e^{AT_0}$, $G = A^{-1}(e^{AT_0} - I)B$, and the measurement equation (3) will have the form

$$\bar{Y}(k) = C\bar{X}(k). \quad (5)$$

2. Problems of analysis of linear systems in state space

During the development of methods of the modern theory of analytical design of control systems, there was a need for a preliminary study of the dynamic properties of controlled objects.

2.1. Stability analysis of systems in state space. The definition of the stability concept for motion of a dynamic system was first formulated by A.M. Lyapunov in 1982 as follows [2].

Definition. Systems solutions $\bar{X}(t),\ t_0 \le t < \infty$

$$\dot{\bar{X}}(t) = A(t)\bar{X}(t) + f(t) \tag{6}$$

is called Lyapunov stable if for any $\varepsilon > 0$ one can specify a number $\delta > 0$ such that for any $\bar{X}_1(t_0)$, which satisfies the condition

$$\left\|\bar{X}_1(t_0) - \bar{X}(t_0)\right\| \le \delta$$

inequality holds

$$\left\|\bar{X}_1(t) - \bar{X}(t)\right\| < \varepsilon \text{ at all } t \in [t_0, \infty).$$

For linear stationary systems, the idea of stability is to ensure the trajectory of the state vector $\bar{X}_1(t)$ to zero without external disturbances. For the non-stationary system (2), (3), the concept of stability with bounded input and bounded output (BIBO) was introduced, in which the system will be stable if:

a) all its solutions exist for $t \ge t_0$ or for all $k = 0, 1, 2, \ldots$;

b) every bounded input vector $\bar{U}(t)$ at $t \ge t_0$ generates a bounded output vector $\bar{Y}(t)$ or any bounded sequence $\bar{U}(k)$ at $k = 0, 1, 2, \ldots$ generates a bounded sequence $\bar{Y}(k)$ at $k = 0, 1, 2, \ldots$.

The following theorems on BIBO stability were formulated and proved [3]:

Theorem 1. *Stationary continuous time system*

$$\begin{aligned} \dot{\bar{X}}(t) &= A\bar{X}(t) + B\bar{U}(t), \\ \bar{Y}(t) &= C\bar{X}(t) + D\bar{U}(t) \end{aligned} \tag{7}$$

is Lyapunov stable if all eigenvalues of the matrix A have negative real parts.

Theorem 2. *Stationary discrete time system*

$$\begin{aligned} \bar{X}(k+1) &= F\bar{X}(k) + G\bar{U}(k), \\ \bar{Y}(k) &= C\bar{X}(k) + D\bar{U}(k) \end{aligned} \tag{8}$$

is BIBO stable if all eigenvalues of the matrix F are inside a circle of unit radius.

Stability conditions of dynamic systems in the state space based on Lyapunov functions were formulated [4]. For a linear system, the Lyapunov function is chosen in the of a quadratic form

$$V(\bar{X}) = \bar{X}^{\mathrm{T}} Q \bar{X},\ Q > 0\ . \tag{9}$$

Consider the definition of stability for the discrete system (8). Due to the positive definiteness of the matrix Q, the second property of the Lyapunov function $V(\bar{X}) > 0$

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holds for all $\bar{X} > 0$. To satisfy the third property, we calculate $V(\bar{X}(k+1)) = \bar{X}^{\mathrm{T}}(k+1)Q\bar{X}(k+1)$ using the equation of state $\bar{X}(k+1) = F\bar{X}(k)$ in free motion

$$V(\bar{X}(k+1)) = \bar{X}^{\mathrm{T}}(k)F^{\mathrm{T}}QF\bar{X}(k) .$$

Then the first difference

$$\Delta V(\bar{X}(k)) = V(\bar{X}(k+1)) - V(\bar{X}(k)) = \bar{X}^{\mathrm{T}}(k)F^{\mathrm{T}}QF\bar{X}(k) - \bar{X}^{\mathrm{T}}(k)Q\bar{X}(k) =$$

$$= \bar{X}^{T}(k)[F^{T}QF - Q]\bar{X}(k) = -\bar{X}^{\mathrm{T}}(k)P\bar{X}(k).$$

Therefore, for the increment $\Delta V(\bar{X}(k))$ to be negative definite, it is necessary and sufficient to have a positive definite matrix Q satisfying the equation

$$F^{\mathrm{T}}QF - Q = -P , \tag{10}$$

where P is a positive definite matrix. This equation is called the discrete Lyapunov equation with respect to the matrix Q. It always has a solution if the linear discrete system (8) is stable. Indeed, if the matrix P is enough positive definite, then $\Delta V(\bar{X}(k)) = -\bar{X}(k)P\bar{X}(k) < 0$, when $V(\bar{X}(k+1)) > 0$. That is, the function $V(\bar{X}(k))$ decreases and the trajectory of the system move towards the origin.

2.2. Controllability analysis of linear systems. A linear dynamic system, which is described by the state equation (2), is called controlled if there is a control vector $\bar{U}(t)$, which makes it possible to transfer the system from an arbitrary initial state $\bar{X}_0(t_0)$ to a final state $\bar{X}_1(t_1) \neq \bar{X}_0(t_0)$ in time $t_1 - t_0$.

Theorem 3. *For the linear nonstationary system (2) if and only if there exists a control vector $\bar{U}(t)$ that transfers the system from state $\bar{X}_0(t_0)$ to state $\bar{X}_1(t_1)$ when the vector $[\bar{X}_0(t_0) - \Phi(t_0, t_1)\bar{X}_1(t_1)]$ belongs to the range of values of the linear transformation*

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)B(t)B^{\mathrm{T}}(t)\Phi^{\mathrm{T}}(t_0, t)dt,$$

where $\Phi(t_0, t_1)$ is the transition matrix which in stationary case has the form $\Phi(t_0, t_1) = e^{A(t_1 - t_0)}$.

The proof of the theorem is given in [2].

Consider the controllability analysis of a discrete stationary system with a scalar input

$$\bar{X}(k+1) = F\bar{X}(k) + Gu(k). \tag{11}$$

Let us assume that an arbitrary initial state for it is given as $\bar{X}(0)$. Then the state of this system $\bar{X}(N)$ on a finite bounded time interval, equal to N the discretization periods, is determined based on the recurrent procedure to the equation when $k = 0, 1, 2, \ldots, N-1$ as

$$\bar{X}(N) = F^{N}\bar{X}(0) + F^{N-1}Gu(0) + F^{N-2}Gu(1) + \ldots + Gu(N-1) = F^{N}\bar{X}(0) +$$

$$+[G \;\; FG \;\; F^{2}G \;\; \ldots \;\; F^{N-1}G] \times [u(N-1) \;\; u(N-2) \;\; u(N-3) \;\; \ldots \;\; u(0)]^{\mathrm{T}}. \tag{12}$$

The composite matrix

$$R_c = [G \;\; FG \;\; F^2G \; \ldots \; F^{N-1}G] \tag{13}$$

is called the controllability matrix. When the matrix R_c has the rank $rank\, R_c = n,$ where n is the dimension of the state vector $\bar{X},$ then based on (12) it is possible to determine the composite vector of control actions $U_n = [u(n-1) \; u(n-2) \; \ldots \; u(0)]^{\mathrm{T}}$ as follows

$$U_n = R_c^{-1}[\bar{X}(n) - F^n \bar{X}(0)]. \tag{14}$$

When $\mathrm{rank}\, R_c = n,$ then from (14) we obtain the n equations for determining the sequence of control actions that transfers the system from the initial state $\bar{X}(0)$ to the final state $\bar{X}(n).$ If $k < n,$ then the solution of equation (14) does not exist. Also, when $\mathrm{rank}\, R_c < n,$ then its inversion is impossible and system (11) will be noncontrollable [5].

2.3. Analysis of observability of linear systems. The task of observability consists in estimating the state of the system at the instant of time t_0 with known control vectors $\bar{U}(t)$ and vectors of initial measurements $\bar{Y}(t)$ at $t \geq t_0.$ For a linear non-stationary system $\dot{\bar{X}}(t) = A(t)\bar{X}(t),$ $\bar{Y}(t) = C(t)\bar{X}(t)$ if the matrices $A(t),$ $C(t)$ are given on the interval $t_0 \leq t \leq t_1,$ then the initial state $\bar{X}(t_0)$ can be determined with accuracy up to the constant vector that is in the kernel of the operator [2]

$$M(t_0, t_1) = \int_{t_0}^{t_1} \Phi^{\mathrm{T}}(t, t_0) C^{\mathrm{T}}(t) C(t) \Phi(t, t_0) dt.$$

At the same time, the matrix $M(t_0, t_1)$ is symmetric and non-negatively defined for $t_1 \geq t_0.$

For a linear stationary system in discrete time (4), (5) with known measurements $\bar{Y}(0), \bar{Y}(1), \ldots, \bar{Y}(n-1)$, the system will be observable if the observability matrix $W = [C \;\; CF \;\; CF^2 \; \ldots \; CF^{n-1}]^{\mathrm{T}}$ will have full rank n where $n = \dim \bar{X}.$ Then the initial state vector can be determined based on the system of equations $\bar{Y}_n = W\bar{X}(0) + N\bar{U}_n,$ according to

$$\bar{X}(0) = W^{-1}[\bar{Y}_n - N\bar{U}_n]. \tag{15}$$

3. Problems of linear control systems synthesis

3.1. Synthesis of modal state regulators. The method of modal control began to develop rapidly in the 70s and 80s of the 20th century [6–8]. At the same time, a mathematical model of the controlled object in the state space in a deterministic environment was used. It is also known as a pole placement control.

Consider a linear stationary plant with several inputs and outputs in discrete time (4), (5), bearing in mind that the eigenvalues of the state matrix F are different.

The control law was used to synthesize the regulator

$$\bar{U}(k) = -K_p \bar{X}(k) \tag{16}$$

provided that the state vector $\bar{X}(k)$ is fully measurable. Equations of closed-loop control system

$$\bar{X}(k+1) = [F - GK_p]\bar{X}(k) \tag{17}$$

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in the general case is multiconnected, that is, the state matrix $[F - GK_p]$ is not diagonal. This led to the interdependence of the control loops of the multidimensional system (17), that is, the closed-loop control system was not autonomous. In [8] the method of diagonalization of the matrix F by

$$\bar{X}_t(k) = T\bar{X}(k), \quad (18)$$

which leads to the transformation of model (4), (5) to this form

$$\begin{aligned} \bar{X}_t(k+1) &= F_t \bar{X}_t(k) + G_t \bar{U}(k), \\ \bar{Y}(k) &= C_t \bar{X}_t(k). \end{aligned} \quad (19)$$

Now the system state matrix

$$F_t = TFT^{-1} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = \bar{\lambda} \quad (20)$$

is presented in a diagonal form, where the diagonal elements λ_i of the matrix F_t are equal to the eigenvalues of the state matrix F of the model (4), and the matrix G_t, C_t are equal to

$$G_t = TG, \quad (21)$$

$$C_t = CT^{-1}. \quad (22)$$

When the model (4), (5) is transformed, its characteristic equation $\det[zI - F] = 0$ will be equal to the characteristic equation $\det[zI - F_t] = 0$ of the transformed model (19).

The control vector of the modal controller in model (19) can be denoted as $\bar{U}_t = G_t \bar{U}(k)$ formed on the basis of the new control law

$$\bar{U}_t(k) = -K_{p_t} \bar{X}_t(k), \quad (23)$$

as a result of which the closed-loop control system equation will be

$$\bar{X}_t(k+1) = (\bar{\lambda} - K_{p_t}]\bar{X}_t(k), \quad (24)$$

where the matrix K_{p_t} is diagonal, i.e. individual eigenvalues (modes) of the state matrix of the closed system (24) can be changed independently by choosing the coefficients of the matrix K_{p_t}.

Thus, the closed control system (24) consists of independent control loops, and its characteristic equation has the form

$$\begin{aligned} \det[zI - (\bar{\lambda} - K_{p_t})] &= [z - (\lambda_1 - K_{p_{t_1}})] \cdot \\ \cdot [z - (\lambda_2 - K_{p_{t_2}})] &\cdot \ldots \cdot [z - (\lambda_n - K_{p_{t_n}})] = 0. \end{aligned} \quad (25)$$

The design of the coefficients $K_{p_{t_i}}$ of the modal regulator can be performed by the method of the desired pole placement of the characteristic equation of the closed system $z_i = \lambda_i - K_{p_{t_i}}$, from which we determine $K_{p_{t_i}} = \lambda_i - z_i$. At the same time, the desired roots of the characteristic equation are chosen in the middle of a circle of unit radius $|z_i| < 1$.

The disadvantage of this method of designing a modal controller is the formation of a real control vector $\bar{U}(t) = -G_t^{-1}\bar{U}_t(k) = G^{-1}T^{-1}\bar{U}_t(k)$, which is determined on the basis of (21), (23) and applied to the controlled object (4), (5). For its implementation, it is necessary that the dimensions of the control vector $\bar{U}(k)$ and the state vector $\bar{X}(k)$ coincide. This introduces a significant limitation of the above design methodology, because in most real control systems $\dim\bar{U} < \dim\bar{X}$.

If the dimension of the control vector $\dim\bar{U} = m > 1$ is $m < n$, then several algorithms were developed for matrix K_p synthesis [7].

The most successful algorithm, which has found wide application in practice, is the canonical transformation of closed systems [7, 9]. At the same time, it is assumed that system (4) is fully controlled. According to the algorithm, the design sequence K_p will be as follows:

a) desired eigenvalues $\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*$ of the closed control system (17) at $|\lambda_i^*| < 1$ are chosen. At the same time, the condition is fulfilled, under which λ_i^* there will be different, $i = 1, 2, \ldots, n$;

b) right eigenvectors of the closed system $R_1, R_2, \ldots, R_n$ are introduced for consideration, for which the equality holds

$$(F - GK_p)R_i = \lambda_i^* R_i \tag{26}$$

or

$$(F - \lambda_j^* I)R_j = GK_p R_j = GP_j,\ j = \overline{1, n}, \tag{27}$$

where

$$P_j = K_p R_j \tag{28}$$

dimensional column $m \times 1$;

c) a dimensional $(m \times n)$ matrix $P = [P_1, \ldots, P_n]$ is formed, where $P_j \neq 0$ are nonzero columns, so that the matrix P has full rank ($rankP = m$);

d) under the condition that $rank\, G = m$, the eigenvectors are determined based on expression (27).

$$R_j = (F - \lambda_j^* I)^{-1} GP_j,\ j = \overline{1, n}. \tag{29}$$

Moreover, the desired eigenvalues λ_j^* should not coincide with the eigenvalues of the matrix F, that is $\det[F - \lambda_j^* I] \neq 0$;

e) the complete set of eigenvectors $R = [R_1, R_2, \ldots, R_n]$ is determined on the basis of (29) and with the selected matrix $P = [P_1, \ldots, P_n]$ according to (28) the matrix of the modal controller is calculated by

$$K_p = PR^{-1}. \tag{30}$$

The choice of the matrix P affects the nature of the control actions, but the spectrum $\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*$ of the closed system will be invariant with respect to the choice of P.

3.2. Synthesis of linear-quadratic state regulators. This problem in the Soviet scientific literature was called analytical design of regulators.

We consider the model of the controll object in the state space (4), where the state vector has the dimension $\dim \bar{X} = n,$ and the control vector $\dim \bar{U} = m$ satisfy $m \le n.$ The initial value is $\bar{X}(0) \ne 0.$ It is assumed that the state variables $\bar{X}(k)$ can be measured accurately. The task is to design a state controller (16), which forms a control vector $\bar{U}(k)$ to transfer the control object from the initial state $\bar{X}(0) \ne 0$ to the final state $\bar{X}(n) = 0$ based on the minimization of the quadratic optimality criterion

$$J = \left\{ \bar{X}^{\mathrm{T}}(N) Q_1 \bar{X}(N) + \sum_{k=0}^{N-1} \left[\bar{X}^{\mathrm{T}}(k) Q_2 \bar{X}(k) + \bar{U}^{\mathrm{T}}(k) Q_3 \bar{U}(k) \right] \right\},$$

$$k = 0, 1, 2, \ldots, N-1, \tag{31}$$

where are the weight matrices $Q_1 \ge 0,$ $Q_2 \ge 0,$ $Q_3 > 0.$

The following theorem was formulated and proved for the synthesis of the linear quadratic regulator according to criterion (31).

Theorem 4. Let $\bar{U}(k)$ *be a function of state vectors* $\bar{X}(k)$, $\bar{X}(k-1)$, ... *If the model of the controll object in the state space (4), (5) is controllable and observable, then there is a unique admissible vector* $\bar{U}(k)$ *that minimizes the criterion (31) provided that the controller matrix (16) is determined by a recurrent procedure*

$$K_p(k) = [Q_3 + G^{\mathrm{T}} L(k+1) G]^{-1} G^{\mathrm{T}} L(k+1) F\,, \tag{32}$$

where is the matrix

$$L(k) = F^{\mathrm{T}} L(k+1) F + Q_2 - K_p^{\mathrm{T}}(k) [Q_3 + G^{\mathrm{T}} L(k+1) G] K_p(k) =$$

$$= F^{\mathrm{T}} L(k+1) F + Q_2 - F^{\mathrm{T}} L(k+1) G [Q_3 + G^{\mathrm{T}} L(k+1) G]^{-1} \cdot G^{\mathrm{T}} L(k+1) F, \tag{33}$$

when $L(N) = Q_1.$

This problem belongs to optimal control problems and is solved using the dynamic programming method or the Pontryagin maximum principle.

3.3. State observers (identifiers). For most linear system controlled objects, not all variables of vector state $\bar{X}$ can be directly measured.

Therefore, to implement the state regulators (16), it is necessary to estimate (identify) the state vector $\bar{X}(t)$ based on known input $\bar{U}(t)$ and output $\bar{Y}(t)$ coordinates. The task of estimating (identifying) the state vector is to implement the estimate $\bar{X}(t)$ at a given moment in time t_0 with known input and output signals measured at $t \le t_0$.

Consider the controlled plant model (4), (5) in the state space for discrete time. We assume that all state variables of the vector $\bar{X}(k)$ are observed according to (15), and the vectors $\bar{U}(t)$ and $\bar{Y}(k)$ are precisely measured. Then the estimation of the state vector $\hat{\bar{X}}(k)$ is implemented according to the following recursive procedure

$$\hat{\bar{X}}(k+1 \mid k) = F \hat{\bar{X}}(k \mid k-1) + G \bar{U}(k) + L[\bar{Y}(k) - C \hat{\bar{X}}(k \mid k-1)], \tag{34}$$

where the matrix L should be chosen in such a way that the state matrix of the closed system $(F - LC)$ in terms of the estimation error $\tilde{\bar{X}}(k+1) = \bar{X}(k+1) - \hat{\bar{X}}(k+1)$ has eigenvalues inside a circle of unit radius. Then $\lim\limits_{k \to \infty} \tilde{\bar{X}}(k) = 0.$

The recurrent procedure (34) is called the Lewinberger observer.

3.4. The problem of linear-quadratic Gaussian optimal control. A linear discrete model of the controll object in the state space under the action of random disturbances is considered

$$\bar{X}(k+1) = F\bar{X}(k) + G\bar{U}(k) + \bar{v}_1(k)\,, \tag{35}$$

$$\bar{Y}(k) = C\bar{X}(k) + \bar{v}_2(k)\,, \tag{36}$$

where F is matrix $(n \times n)$, G — matrix $(n \times m)$, $\bar{X}(k)$ — state vector $(n \times 1)$, $\bar{U}(k)$ — control vector $(m \times 1)$; $\bar{Y}(k)$ is the vector of initial measurements $(p \times 1)$. Signals $\{\bar{v}_1(k), \bar{v}_2(k)\}$ are sequences of «white noise» with zero mathematical expectations $E\{\bar{v}_1(k)\} = 0$; $E\{\bar{v}_2(k)\} = 0$ and covariance matrices

$$E\left\{\begin{bmatrix}\bar{v}_1(k)\\ \bar{v}_2(k)\end{bmatrix}\begin{bmatrix}\bar{v}_1^{\mathrm{T}}(k) & \bar{v}_2^{\mathrm{T}}(k)\end{bmatrix}\right\} = \begin{bmatrix}V & R\\ R^T & N\end{bmatrix}\delta_{kn}, V \geq 0, N > 0. \tag{37}$$

It is also assumed that

$$E\{\bar{X}(0)\bar{v}_1^{\mathrm{T}}(k)\} = 0,\ \ E\{\bar{X}(0)\bar{v}_2^{\mathrm{T}}(k)\} = 0\,.$$

Moreover, all state variables of the vector $\bar{X}$ in model (35) are not measured. Thus, to implement the state regulator (16), the state vector in the model (35), (36) must be evaluated. Minimization of the mean squared error is used as an evaluation criterion

$$E\{[\hat{\bar{X}}(k) - \bar{X}(k)]^{\mathrm{T}}[\hat{\bar{X}}(k) - \bar{X}(k)]\} = Sp(P)\,,$$

where P is the covariance matrix of the estimation error

$$P(k) \triangleq E\{[\hat{\bar{X}}(k) - \bar{X}(k)][\hat{\bar{X}}(k) - \bar{X}(k)]^{\mathrm{T}}\,\big|\,\bar{Y}(k-1),\ \bar{Y}(k-2), \ldots, \bar{Y}(0)\}. \tag{38}$$

The problem of estimating a non-measurable state vector in a stochastic environment is called a filtering problem, which can be formulated in the form of the following theorem.

Theorem 5. *If for the control plant model of (35), (36) with covariance matrices (37) we assume that the initial state vectors* $\bar{X}(0)$ *and disturbance vectors* $\bar{v}_1(k)$, $\bar{v}_2(k)$ *will be Gaussian, and also denote by* $\hat{\bar{X}}(k+1)$ *the conditional mathematical expectation of the state vector* $\bar{X}(k+1)$ *in the presence of previous measurements* $\{\bar{Y}(k), \bar{Y}(k-1), \ldots, \bar{Y}(k_0)\}$ *of the vector* $\bar{Y}$, *then the definition procedure of the conditional mathematical expectation* $\hat{\bar{X}}(k+1)$ *(state estimation) is implemented in the form of the following recurrent relations (Kalman filter):*

$$\hat{\bar{X}}(k+1) = F\hat{\bar{X}}(k) + G\bar{U}(k) + K_{\Phi}(k)[\bar{Y}(k) - C\hat{\bar{X}}(k)],\ \hat{\bar{X}}(k_0) = \bar{\bar{X}}(0)\,; \tag{39}$$

$$K_{\Phi}(k) = [FP(k)C^{\mathrm{T}} + R][CP(k)C^{\mathrm{T}} + N]^{-1}\,; \tag{40}$$

$$P(k+1) = FP(k)F^{\mathrm{T}} + V - K_{\Phi}(k)[CP(k)C^{\mathrm{T}} + N]K_{\Phi}^{\mathrm{T}}(k)\,; \tag{41}$$

$$P(k_0) = P(0).$$

The proof of the theorem is given in the monograph [10].

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The main properties of the Kalman filter are formulated as follows:

a) in the presence of Gaussian noise, the estimate $\hat{\bar{X}}(k)$ is a conditional mathematical expectation of the state vector $\bar{X}(k)$, i.e.

$$\hat{\bar{X}}(k) = E\{\bar{X}(k) \big| Y(k-1)\}, \tag{42}$$

where $Y(k-1) \triangleq [\bar{Y}(k-1), \bar{Y}(k-2), \ldots, \bar{Y}(0)]$.

b) it follows from the previous property that

$$\begin{gathered} E\{\hat{\bar{X}}(k) - \bar{X}(k) \big| Y(k-1)\} = 0, \\ P(k) \triangleq E\{[\hat{\bar{X}}(k) - \bar{X}(k)][\hat{\bar{X}}(k) - \bar{X}(k)]^{\mathrm{T}} \big| \bar{Y}(k-1)\} = \min, \end{gathered} \tag{43}$$

that is, the recurrent procedure (39)–(41) ensures the minimization of the covariance matrix of the estimation error $P(k)$.

For the first time, the Kalman filter was published in works [11–13].

Based on the Kalman filter, the implementation of the state controller is performed according to the control law

$$\bar{U}(k) = -K_p(k)\hat{\bar{X}}(k). \tag{44}$$

The synthesis of the control vector (44) is performed by on the minimization of the optimality criterion

$$J = E\left\{\bar{X}^{\mathrm{T}}(n)Q_1\bar{X}(n) + \sum_{k=0}^{n-1}[\bar{X}^{\mathrm{T}}(k)Q_2\bar{X}(k) + \bar{U}^{\mathrm{T}}(k)Q_3\bar{U}(k)]\right\}, \tag{45}$$

where the weight matrices satisfy $Q_1 > 0,\ Q_2 \geq 0,\ Q_3 > 0$.

The solution to the controller synthesis problem (44) is based on the separation theorem, the idea of which is that the problem of linear-quadratic Gaussian control is divided into two parts:

a) implementation of the state vector estimation procedure based on the filter (39)–(41);

b) synthesis of the linear optimal control law (44) based on the minimization of the criterion (45).

Theorem 6. *If the control vector* $\bar{U}(k)$ *is a cascade function*

$$\bar{\Psi}_{t-1} \triangleq \left\{\bar{Y}(k-1), \bar{Y}(k-2), \ldots, \bar{Y}(0), \bar{U}(k-1), \bar{U}(k-2), \ldots, \bar{U}(0)\right\}$$

then the estimation of the state variables $\hat{\bar{X}}(k)$ *is performed independently on the basis of the Kalman filter (39)–(41), and the synthesis of the linear optimal control law (44) is implemented according to the recurrent procedure*

$$K_p(k) = [G^{\mathrm{T}}S(k+1)G + Q_3]^{-1}G^{\mathrm{T}}S(k+1)F, \tag{46}$$

where

$$S(k) = K_p^{\mathrm{T}}(k)Q_3K_p(k) + Q_2 + [F - GK_p(k)]^{\mathrm{T}}S(k+1)[F - GK_p(k)] \tag{47}$$

$$S(n) = Q_1.$$

The proof of the separation theorem is given in [14].

The application of the Kalman filter in real control systems was associated with significant difficulties, which mainly consist in the fact that the optimal filtering (39)–(43) is possible under the condition that the disturbances $\bar{v}_1(k)$, $\bar{v}_2(k)$ in the model (35), (36) represent « white» noise. This condition, as a rule, is fulfilled on an infinite segment of time, and on short time intervals $E\{\bar{v}_1(k)\} \neq 0$, $E\{\bar{v}_2(k)\} \neq 0$. Therefore, in further modifications of the Kalman filter, algorithms were developed for the simultaneous estimation of the state vector $\hat{\bar{X}}(k)$ and the vector $\hat{\bar{v}}_1(k)$ with further consideration of the estimation $\hat{\bar{v}}_1(k)$ in the control law.

The second difficulty in applying the Kalman filter is the requirement to have complete information about the covariance matrices of disturbances V, R, N in (37), (40), (41). The variances of individual component vectors $\bar{v}_1(k)$ and $\bar{v}_2(k)$, as a rule, are unknown and change over time during operation of the controlled object. Therefore, many modifications of the Kalman filter have been developed, in which the covariance matrices in the expanded state vector are evaluated and specified. The main problem in the implementation of the extended Kalman filter is to ensure the stability of the recurrent evaluation procedure. These extended evaluation algorithms are fully reflected in the work [15]. A significant part of works on Kalman filtering is devoted to the development of filtering algorithms for controlling processes with distributed parameters, as, for example, in the monograph [16].

In total, more than 10,000 scientific works were published on Kalman filtering methods in international scientific publications during 1960–2000.

3.5. Feedback systems stabilization on the output variable. The dynamics of the controlled object is described by an equation in the discrete-time state space in a stochastic environment

$$\bar{X}(k+1) = F\bar{X}(k) + G\bar{U}(k) + \bar{v}(k), \quad (48)$$

$$\bar{Y}(k) = C\bar{X}(k). \quad (49)$$

A random sequence $\bar{v}(k)$ is characterized as follows:

$$E\{\bar{v}(k)\} = 0;\ E\{\bar{v}(k)\bar{v}^{\mathrm{T}}(k)\} = V. \quad (50)$$

The task is to synthesize the controller

$$\bar{U}(t) = K_p\bar{Y}(t), \quad (51)$$

that implements feedback on the output variable. All options for solving this problem are considered in articles [17, 18]. The most common option consists in the design of the controller matrix K_p (51), which ensures the stability of the close-loop system ($|\lambda(F + GK_pC)| < 1$) based on the minimization of the optimality criterion

$$J = E\left\{\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}[\bar{X}^{\mathrm{T}}(k)Q\bar{X}(k) + \bar{U}^{\mathrm{T}}(k)R\bar{U}(k)]\right\}, \quad (52)$$

which can be submitted in the form

$$J = tr(QP_X) + tr(RP_U), \quad (53)$$

where the matrices P_X and P_U are defined as follows:

$$P_X = \bar{F}P_X\bar{F}^{\mathrm{T}} + V;\quad \bar{F} = F + GK_pC;\quad P_U = K_pCP_XC^{\mathrm{T}}K_p^{\mathrm{T}}. \quad (54)$$

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To minimize criterion (53) by the controller matrix K_p based on (53), (54), the gradient of the objective function is determined

$$\frac{\partial J}{\partial K_p} = 2G^T L\bar{F}P_X C^{\mathrm{T}} + 2RK_p CP_X C^{\mathrm{T}},$$
$$L = \bar{F}^{\mathrm{T}} L\bar{F} + Q + C^{\mathrm{T}} K_p^{\mathrm{T}} RK_p C;\ \bar{F} = (1-\mu)F + GK_p C. \qquad (55)$$

To solve this problem, it is necessary to have an initial approximation for the matrix K_p, that is, such a value of this matrix that ensures finding the spectrum of the matrix $\bar{F}$ inside of a circle of unit radius.

To select the initial approximation of the matrix K_p, the matrix F in (48) and the optimality criterion in (52) are modified. In this case, the matrix F is replaced by the matrix $F_\mu = (1-\mu)F$, where the parameter μ is chosen in such a way that the spectrum of the matrix F_μ is inside a circle of unit radius, i.e. ($\left|\lambda(F_\mu)\right| < 1$). Criterion (52) is supplemented by the term $r\mu^2$. Then the modified variant of problem (48)–(52) will take the form

$$\min_{K_p,\,\mu} J;\ J = tr(QP_X) + tr(RP_U) + r\mu^2; \qquad (56)$$

$$\bar{X}(k+1) = F_\mu \bar{X}(k) + G\bar{U}(k) + \bar{v}(k);\ \ E\{\bar{v}(k)\} = 0;\ \ E\{\bar{v}(k)\bar{v}^{\mathrm{T}}(k)\} = V;$$

$$\bar{Y}(k) = C\bar{X}(k);\ \bar{U}(k) = K_p \bar{Y}(k).$$

If we enter the matrices M, N, Y, U, which depend on the parameter μ

$$M = \bar{F},\ \ N = V,\ \ Y = Q + K_p^{\mathrm{T}} C^{\mathrm{T}} RK_p C,\ \ U = L$$

and enter the scalar

$$tr(XY), \qquad (57)$$

where $X = P_X$, then according to [19] the derivative of the scalar (57) by μ is determined as follows:

$$\frac{\partial\left(tr(XY)\right)}{\partial\mu} = tr\left\{\frac{\partial Y}{\partial\mu}X + \frac{\partial N}{\partial\mu}U + 2U\frac{\partial M}{\partial\mu}XM^{\mathrm{T}}\right\}, \qquad (58)$$

where the matrices X, U are determined based on the solution of discrete Lyapunov equations

$$X = MXM^{\mathrm{T}} + N,$$
$$U = M^{\mathrm{T}} UM + Y. \qquad (59)$$

The first two components in (56) can be written down

$$tr(QP_X) + tr(RP_U) = tr[(Q + K_p^{\mathrm{T}} C^{\mathrm{T}} RK_p C)P_X].$$

Using relation (58), we get the gradient

$$\frac{\partial J}{\partial\mu} = 2[r\mu - tr(\bar{F}^{\mathrm{T}} LFP_X)]. \qquad (60)$$

Thus, the objective function in this problem is determined by criterion (56), and its gradients are determined by expressions (55), (60). The initial conditions for the minimization procedure (52) are chosen as follows: $K_{p_{ini}} = 0$; $|\lambda(1-\mu)F| < 1$.

Conclusion

Of course, in one article it is impossible to generalize all scientific directions and methods of modern control theory based on the state space method. This is done most comprehensively in the handbook [20]. But our task is to analyze the possibilities of the entire theory for the design of control systems with many complexities of the functioning of controlled object, to show its positive aspects and to emphasize those difficult tasks where the modern theory has found limited application.

It should be noted that the beginning of the creation of modern control theory coincides with the beginning of the wide using digital computers for the design of multidimensional control systems. At the same time, it turned out that the programming of digital machines in the complex plane is very difficult and sometimes led to ambiguous results when using models of controlled plants in the form of frequency transfer functions. The state space method made it possible to describe and study the dynamics of complex systems under non-zero initial conditions and always provided unambiguous results when applied. The second important feature of the state space method is the ability to take into account all state variables of the system over time, which most fully reflect its important coordinates, but are not completely measured as initial measured variables.

A very valuable property of system models in state space is the possibility of their application to describe controlled plants with different dimensions of state, control, and measurement vectors, which was impossible when using multidimensional transfer functions in classical control theory.

At the same time, state-space method models are very difficult to apply to systems with delays, especially when the delays vary with time.

The main limitations in the application of models in the state space are the description of the dynamics of controlled plants in a stochastic environment, because in modern control theory it is assumed that disturbances have the character of «white noise», the mathematical expectation of which is zero. At the same time, most problems require information about the covariance matrices of the indicated disturbances. This leads to the solution of additional problems of estimating covariance matrices of disturbances and fluctuations of their average values. These problems are solved in practice for systems with constant covariance matrices when Kalman filtering is implemented. But when the variances of individual disturbances change all the time, their estimation becomes practically impossible.

The state space method, on the one hand, makes it possible to solve control problems for complex systems of large dimensions, and on the other hand, it leads in many cases to certain problems that still remain unsolved. Control feedback synthesis on the base of pole placement approach requires knowledge of the current value of the state vector. Therefore, for incomplete measurements, when the dimension of the output vector is less than the dimension of the state vector, its estimator is used. Since measurements almost always contain errors, its construction is fraught with certain problems. How to represent the existing uncertainty when constructing an estimator so that the estimation error is acceptable. Existing methods of estimation in conditions of inaccurate data based on the Kalman filter, methods of guaranteed ellipsoidal estimation and a number of others often turn out to be ineffective. This is due to a number of reasons, among which we highlight the following: error distributions differ from «white noise» or other acceptable conditions of the methods; the presence of poor observability or controllability, which can lead to incorrectly set tasks. In addition, the poles placement method for feedback synthesis, with the above features, does not allow design highly-precision control systems and, what is especially important, ensuring the robustness of the solutions to be obtained.

Existing methods are effective in synthesizing state-based control, including the use of estimators when the need arises. However, they become significantly more complicated when it is necessary to construct an output control system. In this case, matrix inequality methods are used, which in certain cases give acceptable results. However, there is no deep study of how to effectively solve such synthesis problems using state space methods.

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ЕТАПИ І ОСНОВНІ ЗАДАЧІ СТОЛІТНЬОГО РОЗВИТКУ ТЕОРІЇ СИСТЕМ КЕРУВАННЯ ТА ІДЕНТИФІКАЦІЇ. Частина 1. МЕТОД ПРОСТОРУ СТАНІВ В ТЕОРІЇ ЛІНІЙНИХ СИСТЕМ АВТОМАТИЧНОГО КЕРУВАННЯ

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Ця стаття відкриває серію публікацій, присвячених науково-технічним досягненням сторічного періоду розвитку теорії та методів керування. Описано проблеми, які вдалося повністю вирішити з урахуванням того чи іншого підходу. Перша з цих статей присвячена методу простору станів, який почав активно розвиватися десь із 70–80-х років минулого століття у рамках теорії систем та загальної теорії систем. Дається опис методу щодо лінійних систем автоматичного керування та загальноприйнятий опис керованих та спостережуваних динамічних процесів у таких системах. Розглядаються задачі синтезу стабілізуючого керування в замкнутих системах із зворотним зв'язком на основі теорії стійкості, сформульованої Ляпуновим. Крім стійкості, у рамках методу простору станів формулюються такі важливі якісні властивості, як керованість і спостережуваність. Найбільш розроблений метод синтезу лінійних регуляторів за станом. Серед них особливе місце займає метод синтезу модальних регуляторів, який розглянуто у рамках жорданової реалізації та різних модифікацій методу залежно від розмірності вектора керування. Розглянуто синтез оптимального регулятора з квадратичним критерієм досить загального виду. При неповних вимірах, коли розмірність виходу менша за розмірність вектора стану, для реалізації синтезованих законів керування за станом виникає необхідність включати в систему керування оцінювач стану. У детермінованому випадку можна використовувати описаний у статті оцінювач Люенбергера, а за наявності похибок вимірювань у вигляді білого шуму з нульовим середнім широко використовується фільтр Калмана, також представлений у статті. Не залишений у роботі поза увагою спосіб керування за виходом. У висновках даються переваги та недоліки методів керування на основі опису у просторі станів.

Ключові слова: метод простору станів, керування, зворотний зв'язок, стійкість, керованість, спостережуваність, стабілізація, модальне керування, фільтр Калмана.

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