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ON ONE PROBLEM OF SEARCH FOR MOBILE TARGET

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At present the problems of control in conditions of conflict and uncertainty are especially relevant. This work deals with the problem of search for a mobile target in condition when the pursuer knows only probability distribution of the target initial state. In this paper we first address the auxiliary problem of controlled object convergence given terminal set. Herewith, its motion is described by a system of nonlinear differential equations and probability distribution of its initial state. We deduce formula for the probability of bringing the trajectory of controlled object to the terminal set at fixed moment of time. In so doing, the Fokker–Planck–Kolmogorov equation is used. In the case of the linear dynamics of the controlled object this formula takes an explicit form. In the paper, we apply this formula to study the problem of search in the case of linear dynamics of the pursuing controlled object (pursuer) and the mobile target. The pursuer starts moving from a given point and strives to get close to a given distance from the target. At the time it occurs the search is considered completed. Therewith, information on current state of the target is not available to the pursuer; however he is aware of probability distribution of its initial state. We derive sufficient conditions, under which at a certain time the pursuer can achieve its goal with certain probability and deduce formula for this probability. To this end, the idea of Pontryagin's first direct method, based on the condition of the same name, is employed. In so doing, we use Minkowski operation of geometric subtraction, the properties of the set-valued mappings, and the measurable choice theorem. It is shown that, in the case of simple motion of the target and the Gaussian probability distribution of its initial state this probability takes its maximal value at the above mentioned time. On the sake of geometric descriptiveness this is illustrated with the example of simple motions on the plain.

Keywords: conflict-controlled process, problem of search, density of probability distribution, Fokker–Planck–Kolmogorov equation, selection of set-valued mapping, Gaussian probability distribution.

Introduction

Most of the dynamic processes that occur in technical systems, proceed in conditions of conflict and uncertainty [1]. It should be noted that in the classical works of R. Isaacs [2], L.S. Pontryagin [3], N.N. Krasovskii [4] and their followers [5–7], in constructing controls, full information of the object state is used, or the decision is made on the basis of counter-strategies.

However, in real conditions of conflict interaction, information about phaze state becomes available with some delay, or in the form of its probabilistic distribution. In the first case, the equivalence of the approach problem with information delay to the approach problem with complete information, but with the changed motion equation and terminal set, is established for a wide class of dynamic systems [8]. To the second case refer the search problems [9], in which the Fokker–Planck–Kolmogorov equation [10], describing evolution of the probability distribution of current object state, plays the key role.

In this paper, we first deduce formula for the probability of bringing at fixed moment of time the trajectory of controlled object, whose motion is described by a system of nonlinear differential equations and known probability distribution of its initial state, to the given terminal set. In so doing, the Fokker–Planck–Kolmogorov equation is employed.

We apply this formula to solve the problem of search in the case of linear dynamics of the pursuer and the target. It should be emphasized that under this condition the above mentioned formula takes an explicit form. The pursuer starts moving from a given point, knowing only the initial distribution of the target position. His goal is to get close to a given distance from the target. In addition, during the search process, the current control of the target becomes known to the pursuer at each moment in time. We obtain sufficient conditions for the achievement of pursuer's goal at given instant of time with certain probability and deduce formula for this probability. It is shown that, in the case of «simple motion» of the target and the Gaussian distribution of its initial position, this probability takes maximal value at this moment of time. In so doing, an analog of the Pontryagin condition and the measurable choice theorem [11] are used. The results are illustrated on the example of «simple motions» on a plain.

Probability of bringing the trajectory of controlled process to the terminal set

Let motion of the controlled object y in the space R^n be described by the system of ordinary differential equations

$$\dot{y} = g(y, v), \quad g(y, v) = \mathrm{col}(g_i(y, v)), \quad y(t_0) = y_0, \quad i = 1, ..., n. \tag{1}$$

Here $y \in R^n$, $v \in V$, $V \subset K(R^n)$, where $K(R^n)$ is the set of all compacts from R^n.

To provide existence of Caratheodory (absolutely continuous) solution to this equation we suppose that the velocity vector, namely vector-function $g(y, v)$, is continuous in y. As a whole, we assume that this function is jointly continuous in its variables. Also we suppose that function $g(y, v)$ meets the local Lipshitz condition. This yields uniqueness of the system (1) solution for any measurable control v. Fulfillment of the Filippov condition: $|g(y, v)| \le C(1 + \|y\|)$ for all $v \in V$, C is a constant, ensures the solution extendibility to the whole infinite half-interval of time.

We assume that control v is chosen in the form of Lebesgue measurable function with values in compact V.

Also, the terminal set M, $M \subset R^n$, is given. We suppose that M is a closed set.

Let us denote $p(t, y)$ the density of object probability distribution at time t and by $P(t, M)$ the probability of hitting set M by the trajectory of controlled process (1) at the time t. Evidently, the following formula is true:

$$P(t, M) = \int_M p(t, y) dy. \tag{2}$$

It is known [10] that the density $p(t, y)$ satisfies the following differential equation in partial derivatives (Fokker–Plank–Kolmogorov)

$$\partial p(t, y)/\partial t = -(\nabla, g(y, v) p(t, y)), \tag{3}$$

under the initial condition $p(t_0, y) = p_0(y)$. Here the formal notation was used: $\nabla = (\partial/y_1, ..., \partial/y_n)$.

Equation (3) is valid for all admissible controls $v(\cdot)$. Its integration is equivalent to integration of the system of ordinary differential equations (1) and the following equation [12]

$$dp(t, y)/dt = -(\nabla, g(y, v) p(t, y)). \tag{4}$$

Let us denote by $y(t, z)$ the solution of equation (1), which at the moment t is passing through the point z, i.e. $y(t, z) = z$. We substitute this solution into the equation (4) and solve it under the initial condition

$$p(t_0, y(t_0, z)) = p_0(y(t_0, z)). \tag{5}$$

Then, upon substitution the solution of problem (4), (5) into formula (2) we obtain

$$P(t, M) = \int_M p_0 y(t_0, z))\exp\left\{ -\int_{t_0}^{t} (\nabla, g(y(\theta, z), v(\theta))d\theta \right\}dz. \tag{6}$$

If the right-hand side of equation (1) does not depend on variable y formula (6) is essentially simplified (the exponent turns into unit).

In the case of the linear dynamics of the object:

$$\dot{y} = Ay + v \tag{7}$$

(A is a square matrix of order $n \times n$) we have

$$(\nabla, g(y, v)) = \sum_{i=1}^{n} a_{ii} = trA.$$

Here by trA is denoted the sum of diagonal elements of the matrix A.

Let $t_0 = 0$. Then, in view of the Cauchy formula, expression (6) takes the form

$$P(t, M) = \exp(-ttrA) \int_M p_0 \left(\exp(-At)y - \int_0^t \exp(-A\theta)u(\theta)d\theta \right) dy. \tag{8}$$

One problem of search for mobile target

The problem of search of moving object by another controlled object frequently arises in engineering. Various assumptions concerning motion nature of the former can be made. We will consider that it is controlled as well.

Let motions of the pursuer and the mobile target are described by the systems of linear differential equations, respectively:

$$\dot{x} = A_1 x + u, \tag{9}$$

$$\dot{y} = A_2 y + v. \tag{10}$$

Here $x, y \in R^n$, A_1 and A_2 are square matrices of order n, u and v are control parameters of the objects, $u \in U$, $v \in V$, U and V are convex compacts from R^n.

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The pursuer set out in search for the target from the point x_0 at the time $t=0$. Together with the point x moves its neighborhood, having the form of closed n-dimensional ball of radius ε centered at the origin. Let us denote this ball by $S_0(\varepsilon)$. The pursuer strives to achieve at some finite instant of time the inclusion:

$$y(t) \in x(t) + S_0(\varepsilon).$$

Its fulfillment means completion of the search. Denote the probability of this inclusion by $P(t)$.

Suppose that the pursuer is aware of the density probability distribution of the target initial state $p_0(y)$. Let $p_0(y)$ be a continuous function. In addition, during the search process, the current control of the target becomes known to the pursuer at each moment in time. The pursuer and the target are allowed to use as controls measurable functions with values in U and V respectively. It is required their realizations in time be measurable functions. Such controls will be referred to as admissible.

In what follows, we use the idea of Pontryagin's first direct method [3], based on the condition of the same name. At the heart of this condition lies the assumption that at each instant of time information on current control of the evader becomes available to the pursuer. We use the modified version of this condition.

Pontryagin's condition. $W(t) = e^{A_1 t}U \overset{*}{\_} e^{A_2 t}V \neq \varnothing, \ \forall t \geq 0.$

Here the Minkowski operation of geometric subtraction of sets is used [13]:

$$X \overset{*}{\_} Y = \{z,\, z+Y \subset X,\, z \in R^n\}\,,\ X \subset R^n, Y \subset R^n\,.$$

Here $e^{A_i t}$, $i=1,2,$ denote fundamental matrices of homogeneous systems $\dot{x} = A_1 x$ and $\dot{y} = A_2 y,$ respectively.

Theorem. *Suppose that the probability distribution of the evader initial state is Gaussian:*

$$p_0(y) = p_N(y-\mu) = \frac{1}{(2\pi)^{n/2}\sigma_1\sigma_2\cdot\ldots\sigma_n}\exp\left(-\frac{\|y-\mu\|^2}{2\sigma_1^2\sigma_2^2\cdot\ldots\sigma_n^2}\right),$$

$$\mu = (\mu_1, \ldots, \mu_n)\,,\ \mu_i \geq 0,\ \ \sigma_i \geq 0,\ \ i = 1, \ldots, n\,.$$

Let Pontryagin's condition hold and the set-valued mapping $W(t)$ have nonempty interior. Also, let there exist the least time t^*, at which the following inclusion is fulfilled:

$$e^{A_1 t^*}x_0 - e^{A_2 t^*}\mu \in \int_0^{t^*}(e^{A_1\theta}U \overset{*}{\_} e^{A_2\theta}V)d\theta. \tag{11}$$

Then at the time t^* the pursuer can achieve his goal with the probability

$$P(t_*) = \exp(-t^* trA_2)\int_{S_0(\varepsilon)} p_0(e^{-A_2 t^*}y)\,dy, \tag{12}$$

for any admissible control of the evader.

Proof. By formula (8), the probability of detection of the target at time t has the form:

$$P(t) = \exp(-t\,trA_2)\int_{x(t)+S_0(\varepsilon)} p_0\left(e^{-A_2 t}y - \int_0^t e^{-A_2\theta}v(\theta)\,d\theta - \mu\right)dy.$$

After the change of variable $y = y_1 + x(t)$ and subsequent replacement of y_1 by y, this formula takes the form:

$$P(t) = \exp P(t) = (-ttrA_2) \int_{S_0(\varepsilon)} p_0\left(e^{-A_2 t} y + e^{-A_2 t} x(t) - \int_0^t e^{-A_2\theta} v(\theta) d\theta - \mu \right) dy.$$

Using the Cauchy formula for the solution $x(t)$ of system (9) we obtain:

$$P(t) = \exp(-ttrA_2) \times$$

$$\times \int_{S_0(\varepsilon)} p_0\left(e^{-A_2 t}\left(y + e^{A_1 t} x_0 + \int_0^t e^{A_1(t-\theta)} u(\theta) d\theta - \int_0^t e^{A_2(t-\theta)} v(\theta) d\theta - e^{A_2 t}\mu \right)\right) dy. \quad (13)$$

By virtue of the properties of geometric difference of sets and the closeness of the sets $e^{At}U$ and $e^{At}V$, the set-valued mapping $W(t)$, $W(t) = e^{A_1\theta}U \overset{*}{-} e^{A_2\theta}V$, is measurable and closed-valued. Also, it has non-empty interior. Therefore, in view of relation (11), there exists time t^* and a measurable selection $\omega^*(\theta)$, $\omega^*(\theta) \in W(\theta)$, $\theta \in [0, t^*]$, such that

$$e^{A_1 t^*} x_0 - e^{A_2 t^*}\mu = \int_0^{t^*} \omega^*(\theta) d\theta. \quad (14)$$

Let us construct control of the pursuer in accordance with the formula

$$e^{A(t^*-\theta)} u(\theta) = e^{A(t^*-\theta)} v(\theta) - \omega^*(\theta). \quad (15)$$

Measurable solution of this equation exists by virtue of the theorems on measurable choice, in particular, in the form of lexicographic minimum by the Filippov-Castaing theorem [11]. Upon substitution of formulas (14) and (15) into expression (13) we have:

$$P(t^*) = \exp(-t^* trA_2) \times \int_{S_0(\varepsilon)} p_0\left(e^{-A_2 t^*}\left(y - e^{A_2 t^*}\mu + e^{A_1 t^*} x_0 + \right.\right.$$

$$\left.\left. + \int_0^{t^*} e^{-A_2(t^*-\theta)} v(\theta) d\theta - \int_0^{t^*} \omega^*(\theta) d\theta - \int_0^{t^*} e^{-A_2(t^*-\theta)} v(\theta) d\theta \right)\right) dy.$$

Finally, we come to the formula

$$P(t^*) = \exp(-t^* trA_2) \cdot \int_{S_0(\varepsilon)} p_0(e^{-A_2 t^*} y) dy.$$

It should be noted that at the initial moment $t = 0$ the probability of detection of the evader by the pursuer is $P(0) = \int_{x_0 + S_0(\varepsilon)} p_0(y) dy$. Whether the value of $P(t^*)$ is greater than $P(0)$ depends on the matrix A_2. In this paper we do not touch this point.

Below we provide the case where the probability $P(t)$ takes maximal value at the time t^*.

Corollary. Suppose that motion of the target is «simple»: $\dot{y} = v$, $y \in R^n$.

Then, under the control, defined by formula (15), at the time t^* the probability $P(t)$ (13) achieves its maximal value:

$$P(t^*) = \int_{S_0(\varepsilon)} p_0(y) dy.$$

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Example. For the sake of geometric descriptiveness, let us consider «simple motions» on the plain:

$$\dot{x} = u, \quad \dot{y} = v, \quad x, y \in R^2, \quad x(0) = x_0, \quad \|u\| \le a, \quad \|v\| \le 1, \quad a > 1. \tag{16}$$

Suppose that probability distribution of the target initial state is Gaussian:

$$p_0(y) = \frac{1}{2\pi} \exp\left(-\frac{(y_1 - \mu)^2 + (y_2 - \mu)^2}{2} \right).$$

From formula (14), in view of the target dynamics (16), we have:

$$\omega^* = (a-1)\frac{x_0 - \mu}{\|x_0 - \mu\|}, \quad t^* = \frac{\|x_0 - \mu\|}{a-1}.$$

By formula (15), the control $u(\theta), \theta \in [0, t^*]$, has the form:

$$u(\theta) = v(\theta) - (a-1)\frac{x_0 - \mu}{\|x_0 - \mu\|}. \tag{17}$$

Under this control, the probability $P(t)$ (13) achieves its maximum at the time t^* since the point μ, $\mu \in R^n$, is the point where the probability density $p_0(y)$ takes its maximal value. Here

$$P(t) = \frac{1}{2\pi} \int\limits_{S_0(\varepsilon)} \exp\left\{ -\frac{1}{2} \left\| y + x_0 - t(u - v) \right\| \right\} dy .$$

This integral equals the volume of a body, cut off the surface $z = p_0(y)$ by the cylinder with a circle of radius ε as a base. By using control (17) the pursuer is pushing this body towards the origin and at the time t^*, $t^* = \frac{\|x_0\|}{a-1}$, the body takes its maximum volume

$$\frac{1}{2\pi} \int\limits_{S_0(\varepsilon)} \exp\left\{ -\frac{1}{2}\|y\|^2 \right\} dy.$$

Conclusion

We deduced the formula for the probability of hitting the terminal set by the trajectory of controlled process at fixed moment of time. Herewith, the dynamics of the process is described by a system of nonlinear differential equations and the probability distribution of its initial state. This was used to analyze the problem of search for mobile target. We deduced sufficient conditions for detection of mobile target at a certain time with some probability and deduced formula for this probability. The case is indicated when at this moment of time the mentioned probability achives its maximal value.

Г.Ц. Чикрій

ПРО ОДНУ ЗАДАЧУ ПОШУКУ РУХОМОЇ ЦІЛІ

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Сьогодні задачі керування в умовах конфлікту та невизначеності є особливо актуальними. У цій статті розглядається задача пошуку рухомої цілі, коли шукачу відомий лише розподіл ймовірності початкового стану цілі. Спочатку досліджуємо допоміжну задачу про приведення траєкторії керованого об'єкта, рух якого описується системою нелінійних диференційних рівнянь із заданим розподілом ймовірності його початкового стану, до заданої термінальної множини. Далі виводимо формулу ймовірності цієї події у фіксований момент часу при заданому наперед керуванні. При цьому використовується рівняння Фоккера–Планка–Колмогорова. При лінійній динаміці керованого об'єкта формула набуває наявного вигляду. У роботі отримана формула застосовується для дослідження задачі пошуку при лінійній динаміці переслідувача і цілі. Переслідувач починає свій рух із заданої точки і прагне наблизитися на задану відстань від цілі. При цьому інформація про поточний стан втікача недоступна для переслідувача, проте йому відомий розподіл ймовірностей його початкового стану. Одержані достатні умови, за яких у певний час переслідувач зможе досягти своєї мети з деякою ймовірністю, і виведена формула для цієї ймовірності. Для цього використовується ідея першого прямого методу Понтрягіна, що базується на однойменній умові. Це потребувало застосування геометричної різниці Мінковського, деяких властивостей багатозначних відображень, а також теореми про вимірний вибір. Показано, що у разі «простих рухів» і гауссового розподілу ймовірностей початкового стану втікача у вказаний вище час ця ймовірность досягає максимального значення. З метою геометричної наочності це проілюстровано на прикладі «простих рухів» на площині.

Ключові слова: конфліктно-керований процес, задача пошуку, щільність розподілу ймовірності, рівняння Фоккера–Планка–Колмогорова, селектор багатозначного відображення, гауссовий розподіл ймовірності.

REFERENCES

1. Siouris G. Missile guidance and control systems. San Diego : Springer Science and Business Media, 2004. 653 p.
2. Isaacs R.F. Differential games. New York-London-Sydney : Wiley Interscience, 1965. 479 p.
3. Pontryagin L.S. Selected scientific papers. Vol 2. M. : Nauka, 1988. 576 p.
4. Krasovskii N.N., Subbotin A.I. Positional differential games. M. : Nauka, 1974. 455 p.
5. Pshenichnyi B.N. Ostapenko V.V. Differential games. Kyiv : Naukova dumka, 1992. 264 p.
6. Nikolskii M.S. First direct method of L.S. Pontryagin in differential games. M. : MGU, 1984. 65 p.
7. Chikrii A.A. Conflict controlled processes. Dordrecht; Boston; London : Springer Science and Business Media, 2013. 424 p.
8. Chikrii G.Ts. Principle of time stretching in evolutionary games of approach. *Journal of Automation and Information Sciences.* 2016. Vol. 48, N 5. 2016. P. 12–26. DOI: https://doi.org/10.1615/JAuto matInfScien.v48.i5.20
9. Chikrii A.A., Dzyubenko K.G. Bilinear Markov processes of searching for moving targets. *Journal of Automation and Information Sciences*. 2001. Vol. 33, N 8. P. 62–74. DOI: https://doi.org/10.1615/JAutomatInfScien.v33.i5-8.30
10. Doob D. Stochastic processes. New York : John Wiley & Sons, 1956. 605 p.
11. Filippov A.F. Differential equations with discontinuous right-hand sides. Dordrecht; Boston : Kluwer Publishers, 1988. 258 p. https://doi.org/10.1007/978-94-015-7793-9
12. Courant R. On partial differential equations. M. : Mir, 1964. 843 p.
13. Georges Matheron. Random sets and integral geometry. New York : John Wiley and Sons, 1974. 318 p.

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