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CALCULATING THE PRICE FOR DERIVATIVE FINANCIAL ASSETS OF BESSEL PROCESSES USING THE STURM-LIOUVILLE THEORY

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Calculating the Price for Derivative Financial Assets of Bessel Processes Using the Sturm-Liouville Theory

In the paper we apply the spectral theory to find the price for derivatives of financial assets assuming that the processes described are Markov processes and such that can be considered in the Hilbert space L^2 using the Sturm-Liouville theory. Bessel diffusion processes are used in studying Asian options. We consider the financial flows generated by the Bessel diffusions by expressing them in terms of the system of Bessel functions of the first kind, provided that they take into account the linear combination of the flow and its spatial derivative. Such expression enables calculating the size of the market portfolio and provides a measure of the amount of internal volatility in the market at any given moment, allows investigating the dynamics of the equity market. The expansion of the Green function in terms of the system of Bessel functions is expressed by an analytic formula that is convenient in calculating the volume of financial flows. All assumptions are natural, result in analytic formulas that are consistent with the empirical data and, when applied in practice, adequately reflect the processes in equity markets.

Keywords: *spectral theory, financial flows, Bessel diffusion process, Bessel functions, Green function, singular parabolic operator, infinitesimal operator.*

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Буртняк І. В., Малицька Г. П. Знаходження ціни похідних фінансових активів бесселівських процесів методом Штурма-Ліувілля

Застосовано спектральну теорію для знаходження ціни похідних фінансових активів, вважаючи, що процеси, які описуються, є марківськими та такими, що можуть бути розглянуті в гільбертових просторах L^2, застосовуючи теорію Штурма-Ліувілля. Процеси дифузії Бесселя використовуються при дослідженні азійських опціонів. Розглянуто фінансові потоки, що породжені процесами бесселівської дифузії подавши їх за системою функцій Бесселя першого роду за умови, яка враховує лінійну комбінацію потоку та його просторової похідної. Таке представлення дає можливість обчислити величину ринкового портфеля акцій і забезпечує вимір внутрішньої волатильності на ринку в будь-який момент часу, дозволяє дослідити динаміку фондового ринку. Розклад функції Гріна за системою бесселевих функцій дається аналітичною формулою, за допомогою якої зручно обчислювати величину фінансових потоків. Всі припущення є природними, приводять до аналітичних формул, які узгоджені з емпіричними даними і при практичному застосуванні адекватно відображають проходження процесів на фондових ринках.

Ключові слова: *спектральна теорія, фінансові потоки, дифузійний процес Бесселя, функції Бесселя, функція Гріна, сингулярний параболічний оператор, інфінітіземальний оператор.*

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Буртняк И. В., Малицкая А. П. Нахождение цены производных финансовых активов бесселевских процессов методом Штурма-Лиувилля

Применена спектральная теория для нахождения цены производных финансовых активов полагая, что описывающиеся процессы являются марковскими и такими, которые можно рассматривать в гильбертовом пространстве L^2, применяя теорию Штурма-Лиувилля. Процессы диффузии Бесселя используются при исследовании азиатских опционов. Рассмотрены финансовые потоки, порожденные процессами диффузии Бесселя, представив их по системе функций Бесселя первого рода при условии, учитывающем линейную комбинацию потока и его пространственной производной. Такое представление дает возможность вычислить величину рыночного портфеля акций и обеспечивает измерение внутренней волатильности на рынке в любой момент времени, позволяет исследовать динамику фондового рынка. Разложения функции Грина по системе функций Бесселя представлены аналитической формулой, с помощью которой удобно вычислять величину финансовых потоков. Все предположения являются естественными, приводят к аналитическим формулам, которые согласованы с эмпирическими данными и при практическом применении адекватно отражают прохождение процессов на фондовых рынках.

Ключевые слова: *спектральная теория, финансовые потоки, диффузный процесс Бесселя, функции Бесселя, функция Грина, сингулярный параболический оператор, инфинитиземальний оператор.*

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Introduction. Bessel processes are most often used in the theory of mass service, in particular in the queuing theory. There exist different types of stochastic Bessel process in relation to problems of eigenvalues and eigenfunctions. The cases when the spectrum of the Bessel operator is continuous or has a finite number of values and a continuous part, the representation of the density by means of Wittaker functions and Laguerre polynomials are considered in many papers, and most extensively covered in [1], which considers the boundary value problem on the interval $x \in (0, \infty)$ with the boundary conditions imposed on the first derivative of the density function at $x \to \infty$.

Bessel processes play an important role in financial mathematics. Since by their nature they have a strong relation to the models of geometric Brownian motion or Cox–Ingersoll–Ross (CIR) processes, they allow for an explicit representation of the density of transition, in particular of prices for bonds and options, which makes the statistical estimation of the process parameters much easier [2]. At certain characteristics, the diffusion process with the Bessel operator never hits zero, and a number of papers [3] are dedicated to these cases, while we consider those ones when the derivative of the financial flow of the Bessel process can hit zero. Under such conditions, the excess growth rate of the portfolio of stocks can be determined, and it can be explained how the excess growth rate of the market portfolio can provide a measure of the amount of internal volatility in the market at any given moment [4].

Work [5] describes using Bessel processes in financial markets as well as its connection with the integral of geometric Brownian motion. In [6] a review of the pricing of Asian options through Bessel processes is presented.

The diffusion with the Bessel operator was investigated in [7], but under other boundary conditions, and in the orthogonal systems of functions.

We consider one-dimensional diffusion of the Bessel process with zero drift (there are a number of processes of this type with non-zero drift, but their studying can be reduced to the processes with zero drift). Such processes are used in solving economic problems related to calculating short-term interest rates, credit spreads and stochastic volatility of derivatives.

The aim of the article is to develop a convenient approach to monitoring financial flows produced by two-barrier Bessel processes. We present density in the form of development through Bessel functions of the first and second kind, as well as their derivatives. Since Bessel series are well researched, and eigenvalues of the set boundary value problem are tabulated, then the decomposition of density in series in terms of Bessel functions is very convenient to be used in practice when calculating options prices with required precision.

The problem of this type is considered for the first time. Among problems of this type there remains unsolved the problem when the Bessel diffusion has a non-local volatility that is dependent on various factors, but this problem is only partially solved for ordinary diffusion processes generated by Brownian motion. Therefore, new approaches to solving problems of this type are required as well as analysis of the existing ones.

The approach we have developed can be applied to studying the pricing of Asian options generated by Bessel processes. For this purpose, we need to consider the financial flows generated by Bessel diffusion processes by expanding them in the system of Bessel functions of the first kind under condition of taking into account the linear combination of the flow and its spatial derivative. This expansion allows calculating the size of the market portfolio and determining the level of internal volatility in the market at any given time and investigating the dynamics of the equity market.

In the general theory there considered more widespread assumptions on stochastic processes, in particular considering them as martingales, but there is not always an analytic formula to represent the solution. Therefore, we assume that these processes are Markov processes.

The spectral method is applied to derivative financial instruments, in particular there presented the price for the derivative $u(t,x)$ through a function that is neutral to the risk of expecting the future value of the real-valued process *X*, that is as:

$$u(t,x)=\tilde{E}_X[H(X_t)]=\int H(y)p(t,x,y)dy,$$

where $p(t,x,y)$ – transition density of *X* with the probability *P*.

If the infinitesimal generator *L* of the real-valued process is self-ajoint in the Hilbert space with the increment of the measure $m(x)dx$, and the *L* –spectrum is discrete, then the transition density of *X* is developing with respect to its own functions [2]:

$$p(t,x,y)=m(y)\sum_n e^{(-\lambda nt)}\varphi_n(y)\varphi_n(x),$$

where $\{\lambda_n\}$ eigenvalues L i $\{\varphi_n\}$ – eigenfunctions: that is $L\varphi n=\lambda n$.

Problem statement. In our work, we use the diffusion process to calculate the volume of financial flows, expressing them in terms of Bessel functions of the first kind, in particular we consider the Sturm-Liouville problem where the boundary conditions use the Bessel functions and their derivatives. In particular, we consider the operator:

$$L=\partial_{xx}^2+x^{-1}\partial_x-x^{-2}p^2, \qquad (1)$$

where p – constant value called an index. It generates a singular parabolic equation at $x > 0$. L is a singular parabolic operator, an infinitesimal one, to which a number of operators where $\sigma^2 = 2x^2$ is reduced, L is called the Bessel operator.

To study L for eigenvalues and eigenfunctions under certain boundary conditions, we consider the Bessel equation:

$$x^2v'' + xv' + (x^2 - p^2)v = 0. \qquad (2)$$

The solution of equation (2), except for the partial values of p, is not expressed in terms of elementary functions (in the finite form), these non-elementary functions are called Bessel functions, they are widely used in economics, technology and physics. Since the Euler-Bessel equation is a linear one, its total integral can be written in the form:

$$v = C_1v_1 + C_2v_2$$

Where v_1, v_2 are any two linearly independent partial solutions of the Euler-Bessel equation, and C_1, C_2 are arbitrary constants.

In the case of $p \geq 0$ we make a substitution $v = x^p w$ and obtain the following equation for the function:

$$w'' + \frac{2p+1}{x}w' + w = 0.$$

The solution of the resulting equation is a power series that is absolutely convergent for all $x \in (-\infty;\infty)$ and has the form:

$$v = x^p w = \frac{\left(\frac{x}{2}\right)^p}{\Gamma(p+1)} + \sum_{m=1}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{p+2m}}{1\,2\ldots m(p+1)\ldots(p+m)\Gamma(p+1)}$$

Where Γ — gamma function. By transforming (3) on the basis of the properties of the gamma function, we obtain a Bessel function of the first kind of the p-th order:

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{p+2m}}{\Gamma(m+1)\Gamma(p+m+1)}. \qquad (3)$$

Note. Since equation (2) contains p^2, the substitution of p for $(-p)$ does not influence the solution of the equation, thus there exists a solution for any value of p.

If p is not an integer, then the Bessel functions cannot be linearly dependent and the general integral of equation (2) has the form [8]:

$$J = C_1J_p(x) + C_2J_{-p}(x).$$

With an integer p we find one more partial solution:

$$Y_p(x) = \frac{J_p(x)\cos p\pi + J_{-p}(x)}{\sin p\pi},$$

which is expressed by a Bessel function of the second kind that is undefined at $x = 0$. Using L'Hôpital's rule we find the boundary for $x \to 0$ and by this value define the function at zero:

$$Y_0 = \frac{2}{\pi}J_0(x)\left(\ln\frac{x}{2} + C\right) - \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^m\left(\frac{x}{2}\right)^{2m}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}\right).$$

For any value of p the following formulas can be used:

$$\frac{d}{dx}(x^p J_p(x)) = x^p J_{p-1}(x),$$

$$\frac{d}{dx}(x^{-p} J_p(x)) = -x^{-p} J_{p+1}(x).$$

The Bessel functions $J_p(\lambda x)$, $J_p(\mu x)$ where λ and μare the roots of the equation $J_p(x)$ =0, are orthogonal on the interval [0, 1] with the weight x, that is

$$\int_0^1 xJ_p(\lambda x)J_p(\mu x)dx = 0, \ \lambda \neq \mu \quad ,$$

and if $\lambda = \mu$ the two cases are possible:

$$\int_0^1 xJ_p^2(\lambda x)dx = \begin{cases} \frac{1}{2}J_p'^2(\lambda), & J_p(\lambda) = 0, \\ \frac{1}{2}\left(1 - \frac{p^2}{\lambda^2}\right)J_p^2(\lambda), & J_p'(\lambda) = 0. \end{cases}$$

For all $\alpha, \beta \geq 0, \alpha + \beta > 0$ there exists a countable set of positive roots $\alpha v_k'(\mu) + \beta\mu v_k(\mu) = 0$, whose boundary point is at infinity.

If $v(x)$ is the solution of (2), then the function $v(\lambda x)$ will also be the solution of the equation of the following form:

$$x^2v'' + xv' + (\lambda^2x^2 - p^2)v = 0. \qquad (4)$$

Equation (4) is a Bessel equation with the parameter λ.

Any solution of equation (2) expressed by a Bessel function has an infinite set of positive roots that are close to the roots of the function $\sin(x + \omega)$, which has the form: $k_n = n\pi - \omega$, $\omega = const$, n an integer (it is similar for negative roots, because they are symmetrical relative to the origin of coordinates), if $k_n \neq 0$ they are simple roots and form a countable set.

Since Bessel functions are alternating series, the calculation of values can be performed using the Leibniz lemma, which makes it possible to determine the accuracy of the approximation.

To find the eigenfunctions and eigenvalues, let us consider the following boundary value problem:

$$x^2v_k'' + xv_k' + (\lambda_k^2x^2 - p^2)v_k = 0, \qquad (5)$$

$$|v_k|_{x=0} < +\infty, \qquad (6)$$

$$\alpha v_k'(x_0) + \beta v_k(x_0) = 0. \qquad (7)$$

That is we are considering a Sturm-Liouville problem. The given problem has a unique solution. We impose condition (6) because $x = 0$ is a special point of equation (5) and the operator L. x_0 is a regular point of equation (5). The values of λ_k with which the boundary value problem (5)-(7) has a non-trivial solution v_k are called eigenvalues, and v_k — eigenfunctions of the problem. It is known that under conditions (6) the operator L has a countable number of eigenvalues, they are simple and not negative [9]. The multiplication of L by x^2 does not change either the eigenvalues, the eigenfunctions, or their quantity.

Let us consider (5), under the condition $\alpha = 1, \quad \beta = h > 0$ we have the boundary value problem:

$$\begin{cases} x^2 v_k'' + x v_k' + (\lambda_k^2 x^2 - p^2) v_k = 0, \\ |v_k|_{x=0} \le +\infty, \\ v'_k(x_0) + h v_k(x_0) = 0. \end{cases} \quad (8)$$

It is obvious that $\lambda = 0$ is not the eigenvalue of problem (8). The general integral has the following form:

$$v = C_1 J_p(\lambda x) + C_2 Y_p(\lambda x).$$

By substituting it in the boundary conditions we obtain:

$$\lambda J_p'(\lambda x_0) + h J_p(\lambda x_0) = 0$$

with the substitution $\lambda x_0 = \mu$ it can be put as the equation:

$$\mu J_p'(\mu) + h x_0 J_p(\mu) = 0 \quad (9)$$

Equation (9) has a countable set of positive roots:

$\mu_k, \ 0 < \mu_1 < \mu_2 < \ldots < \mu_k < \ldots$

Thus $\lambda_k = \dfrac{\mu_k}{x_0}$, where μ_k the root of (9), and

$$v_k(x) = J_p\left(\frac{\mu_k}{x_0} x\right), \ k = 1,2,$$

Let us find the norm of $v_k(x)$:

$$\|v_k(x)\|^2 = \frac{1}{2}\left(x_0^2 (J_p'(\mu_k))^2 + \left(x_0^2 - \frac{v^2}{\mu_k^2} \right)(J_p(\mu_k))^2 \right), \ k = 1,2,\ldots$$

Since from (9) with $\mu = \mu_k$ we have $J_p'(\mu_k) = -\dfrac{h x_0}{\mu_k} J_p(\mu_k)$, the norm is equal to:

$$v_k(x)^2 = \frac{x_0^2}{2}\left(1 + \frac{h x_0^4 - v^2}{\mu_k^2 x_0^2} \right)() J_p(\mu_k)^2, \ k = 1,2,\ldots$$

In the case when $p = 0$ we have the following problem:

$$\begin{cases} (x v')' + \lambda^2 x v = 0, \\ |v|_{x=0} < +\infty, \\ v_k'(x_0) + h v_k(x_0) = 0, \ h > 0. \end{cases} \quad (10)$$

It is easy to check out that $\lambda = 0$ is not the eigenvalue of the problem, and at $\lambda > 0$ from the condition of boundary values it follows that $v = C_1 J_0(\lambda x), \ C_2 = 0$,

$$C_1(\lambda J_0'(\lambda x_0) + h J_0(\lambda x)) = 0,$$

With $C_1 = 1$ we have the following equation:

$$\mu J_0'(\mu) + h x_0 J_0(\mu) = 0.$$

As in case (9) all the roots are positive and form a countable set $0 < \mu_1 < \mu_2 < \ldots < \mu_k < \ldots \ \lambda_k = \dfrac{\mu_k}{x_0}$, $v_k(x) = J_0\left(\dfrac{\mu_k}{x_0} x\right)$, $k = 1,2,\ldots$ with the norm of $v_k(x)$ equal to:

$$\|v_k(x)\|^2 = \frac{x_0^2}{2}\left(1 + \frac{h^2 x_0^2}{ì_k^2}\right)(J_0(\mu_k))^2, \ k = 1,2,\ldots$$

We can note that the functions 1, $J_p\left(\dfrac{\mu_k}{x_0} x\right)$, $k = 1,2,\ldots$ are orthogonal to each other on $[0, x_0]$ with the weight x [10].

Results of the research. Let us consider the Bessel process described by the equation:

$$\frac{\partial v(t,x)}{\partial t} = \frac{\partial^2 v(t,x)}{\partial x^2} + x^{-1}\frac{\partial v(t,x)}{\partial x} - p^2 x^{-2} v(t,x), \quad 0 < x < x_0 \quad (11)$$

and the boundary condition:

$$v(0,x) = K(e^x - 1)^+, \ |v(t,0)| < +\infty, \ \ v_x'(t,x_0) + h v(t,x_0) = 0, \ \ h > 0 \quad (12)$$

Where K is a strike value. The process is homogeneous, therefore, $v(t, x) = \varphi(t) v(x)$.

From the Sturm-Liouville theory we have:

$$v(t,x) = \sum_{n=1}^{\infty} c_{np} e^{-\frac{\mu_n^2}{x_0^2} t} J_p\left(\frac{\mu_n}{x_0} x\right), \ p \ge 0$$

where $\mu_n \ \ 0 < \mu_1 < \mu_2 < \ldots < \mu_n < \ldots$, the positive roots of the equation are:

$$J_p'(\mu_k)\mu_k + h J_p(\mu_k) = 0,$$

$$c_{np} = \frac{K \int_0^{x_0} x(e^x - 1) J_p\left(\frac{\mu_n}{x_0} x\right) dx}{\int_0^{x_0} x J_p^2\left(\frac{\mu_n}{x_0} x\right) dx}.$$

To calculate $\displaystyle\int_0^{x_0} x(e^x - 1) J_p\left(\frac{\mu_n}{x_0} x\right) dx$, we will use the expression of $J_p\left(\dfrac{\mu_n}{x_0} x\right)$ and e^x through the power series:

$$x(e^x - 1) = x\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots\right) = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n!},$$

$$\forall x \in (-\infty, +\infty)$$

$$J_p\left(\frac{\mu_n}{x_0} x\right) = \sum_{l=0}^{\infty} \frac{(-1)^l}{Ã(l + p + 1) Ã(l + 1)} \left(\frac{\mu_n}{2 x_0} x\right)^{2l + p}.$$

Since these series converge absolutely $\forall x \in R^1$ the product of this series is absolutely convergent $\forall x \in R^1$ and can be determined by any of the rules: either by Cauchy or Dirichleti rule:

$$\sum_{l=0}^{\infty} a_n \sum_{l=0}^{\infty} b_n = \sum_{l=0}^{\infty} b_n \sum_{l=0}^{\infty} a_n = \sum_{l=0}^{\infty}(a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + ... + a_n b_0) =$$
$$= \sum_{l=0}^{\infty}(a_0 b_n + a_1 b_n + ... + a_n b_n + a_n b_{n-1} + a_n b_{n-2} + ... + a_n b_0).$$

therefore, c_n are calculated by the formula with the corresponding μ_n:

$$c_n = \frac{K_1 \int_0^{x_0} x\left(e^x - 1\right) J_p\left(\frac{\mu_n}{x_0} x\right) dx}{\int_0^{x_0} x J_p^2\left(\frac{\mu_n}{x_0} x\right) dx} =$$

$$= K_1 \sum_{n=1}^{\infty}\left[\frac{x_0^{n+3}}{(n+1)!\Gamma(p+1)\Gamma(1)(p+1)}\left(\frac{\mu_n}{2}\right)^p - \right.$$

$$- \frac{x_0^{n+2}}{n!\Gamma(p+2)\Gamma(2)(p+3)}\left(\frac{\mu_n}{2}\right)^{p+2} +$$

$$+ \frac{x_0^{n+1}}{(n-1)!\Gamma(p+3)\Gamma(3)(p+5)}\left(\frac{\mu_n}{2}\right)^{p+4} + \ldots$$

$$+ \frac{(-1)^k x_0^{n-k+1}}{(n-k-1)!\Gamma(k+p+1)\Gamma(p+1)(2k+p+1)}\left(\frac{\mu_n}{2}\right)^{2k+p} +$$

$$\left. + \frac{(-1)^n x_0^3}{\Gamma(2n+p+1)\Gamma(p+1)(2n+p+1)}\left(\frac{\mu_n}{2}\right)^{2n+p}\right] / \int_0^{x_0} x J_p^2\left(\frac{\mu_n}{x_0} x\right) dx.$$

The financial flows have the following form:

$$u(t,x) = \sum_{n=1}^{\infty} K c_{n\,p} e^{-\left(\frac{\mu_n}{\ln K}\right)^2 (T-t)} J_p\left(\mu_n \ln\frac{x}{K}\right).$$

In the case when the process is completed at time T, when $X_T = K$,:

$$u(t,x) = \sum_{n=0}^{\infty} K c_n e^{-\left(\frac{\mu_n}{\ln\frac{R}{L}}\right)^2 (T-t)} J_p\left(\frac{\mu_n(\ln\frac{x}{L})}{\ln\frac{R}{L}}\right),$$

Where $L < x < R$, L, R – barriers, K – strike value, and $c_{n\,p}$ are calculated as follows:

$$c_{n\,p} = 2K\frac{\int_0^1 t(e^{Kt} - 1) J_p(\mu_n t) dt}{\frac{x_0^2}{2}\left(1 + \frac{h x_0^4 - v^2}{\mu_n^2 x_0^2}\right)\left(J_p(\mu_n)\right)^2}.$$

We have calculated the expansion of the financial flow in terms of the system of Bessel functions J_p of the first kind, while the distribution of the flows is set by the Green function of the corresponding problem. Therefore, for the calculations it is convenient to expand the Green function in terms of the system of Bessel functions. The process that we consider corresponds to an inhomogeneous boundary value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x^{-1}\frac{\partial u}{\partial x} - \frac{p^2 u(t,x)}{x^2} + f(t,x), \quad x > 0, \qquad (13)$$

Where $f(t,x)$ is twice continuously differentiable in x and continuously differentiable in t, absolutely integrable with the derivatives, $(t, x) \in [0, +\infty)$, and is expressed as follows:

$$f(t,x) = \sum_{n=0}^{\infty} f_n(t) J_p\left(\frac{\mu_n x}{x_0}\right), \ 0 < x < x_0 < +\infty, 0 < t < T,$$

μ_n are the roots of the equation $J_p(\mu_n) = 0$.

The problem can be solved using the following equation:

$$u(t,x) = \sum_{n=0}^{\infty} T_n(t) J_p\left(\frac{\mu_n x}{x_0}\right).$$

By substituting (13) we obtain:

$$\sum_{n=0}^{\infty} T_n'(t) J_p\left(\frac{\mu_n x}{x_0}\right) = \sum_{n=0}^{\infty}\left\{\left[\left(J_p\left(\frac{\mu_n x}{x_0}\right)\right)''_{x^2} + \frac{\left(J_p\left(\frac{\mu_n x}{x_0}\right)\right)'_x}{x} - \right.\right.$$

$$\left.\left. - \frac{p^2 J_p\left(\frac{\mu_n x}{x_0}\right)}{x^2} + \lambda_n^2 J_p\left(\frac{\mu_n x}{x_0}\right)\right] - \lambda_n^2 J_p\left(\frac{\mu_n x}{x_0}\right)\right\} T_n(t) + \sum_{n=0}^{\infty} f_n(t) J_p\left(\frac{\mu_n x}{x_0}\right),$$

Then

$$\sum_{n=0}^{\infty}\left[T_n'(t) + \lambda_n^2 T_n(t) - f_n(t)\right] J_p\left(\frac{\mu_n x}{x_0}\right) \equiv 0,$$

therefore, $T_n'(t) + \lambda_n^2 T_n(t) - f_n(t) = 0, \quad \lambda_n = \frac{\mu_n}{x_0}, \ n \in N$ with the initial condition of $T_n(0) = 0$.

The inhomogeneous differential equation of the first order is solved by the method of constant variation. Since $T_n'(t) + \lambda_n^2 T_n(t) = 0$ has the first integral $T_{(n)}(t) = Ce^{(-\lambda_n^2 t)}$ (the solution of a inhomogeneous equation), then $T_n(t) = C(t)e^{-\lambda_n^2 t}$,

thus $C'^{(t)} = f_n(t)e^{\lambda_n^2 t}$, $C(t) = \int_0^t e^{\lambda_n^2 \beta} f_n(\beta) d\beta + C_1$.

$$T_n(t) = \int_0^t e^{\lambda_n^2 \beta} f_n(\beta) d\beta e^{-\lambda_n^2 t} + C_1 e^{-\lambda_n^2 t}$$

with $t = 0, C_1 = 0$ $T_n(t) = \int_0^t e^{-\lambda_n^2 (t-\beta)} f_n(\beta) d\beta$, therefore,

$$u(t,x) = \sum_{n=0}^{\infty}\int_0^t e^{-\lambda_n^2 (t-\beta)} f_n(\beta) d\beta \, J_p\left(\frac{\mu_n x}{x_0}\right).$$

Taking into account that:

$$f_n(t) = \int_0^{x_0} \xi f_n(\xi, t) J_p\left(\frac{\mu_n \xi}{x_0}\right) d\xi \left(\int_0^{x_0} x J_p^2\left(\frac{\mu_n x}{x_0}\right) dx\right)^{-1},$$

we have:

$$u(t,x)=$$

$$=\sum_{n=0}^{\infty}\int_0^t e^{-\lambda_n^2(t-\beta)}\int_0^{x_0}\xi f(\xi,t)J_p\left(\frac{\mu_n\xi}{x_0}\right)d\beta d\xi J_p\left(\frac{\mu_n x}{x_0}\right)\left(\int_0^{x_0} yJ_p^2\left(\frac{\mu_n y}{x_0}\right)dy\right)^{-1}=$$

$$=\int_0^{x_0}\int_0^t\sum_{n=0}^{\infty}\left(yJ_p^2\left(\frac{\mu_n y}{x_0}\right)dy\right)^{-2}e^{-\lambda_n^2(t-\beta)}\xi J_p\left(\frac{\mu_n\xi}{x_0}\right)J_p\left(\frac{\mu_n x}{x_0}\right)f(\xi,t)d\xi d\beta,$$

therefore,

$$G(t-\beta,x,\xi)=\sum_{n=0}^{\infty}\xi J_p\left(\frac{\mu_n\xi}{x_0}\right)J_p\left(\frac{\mu_n x}{x_0}\right)e^{-\frac{\mu_n^2}{x_0}(t-\beta)}$$

$$\left(\frac{x_0^2}{2}\left(1+\frac{hx_0^4-v^2}{\mu_k^2x_0^2}\right)\left(J_p(\mu_k)\right)^2\right)^{-1},$$

$$u(t,x)=\int_0^t G(t-\tau,x,\xi)f(\tau,\xi)d\xi.$$

Since the problem of evaluating and studying two-dimensional barrier options is reduced to considering and solving boundary value problem [11–12]:

$$\frac{\partial u(t,x)}{\partial t}=\frac{\partial^2u(t,x)}{\partial x^2}+x^{-1}\frac{\partial u(t,x)}{\partial x}-\frac{p^2u(t,x)}{x^2},$$

$$x\in[L,H],\ t\in[0,T],$$

$$u'_x(t,L)+hu(t,L)=0,\ u'_x(t,H)+hu(t,H)=0,\ h>0,$$

$$u(T,x)=max(\pm(x(T)-K),0)\|_{(L<x(t)<H;t\in[0,T])}$$

this problem is reduced to solving the boundary value problem for the singular parabolic equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2u}{\partial y^2}+y^{-1}\frac{\partial u}{\partial y}-\frac{p^2u(t,x)}{y^2},$$

$$y=\ln x,\ y\in[A,B],t\in[0,T],\ A=\ln L,\ B=\ln H,$$

$$u'_x(t,A)+hu(t,A)=0,\ u'_x(t,B)+hut,B)=0,\ h>0,$$

$$u(0,y)=\psi(e^{(y(}T)=max(\pm(x(T)-K),0)\|_{(L<x(t)<H;t\in[0,T])}.$$

Taking into account all the considerations as to the solution of classical boundary value problems for the singular parabolic operator L, we have:

$$u(T,x)=\int_0^{\ln\frac{H}{L}}(e^{\xi}L-K)\mathbb{I}_{(L<x(t)<H;t\in[0,T])}G(x,\xi)d\xi=\int_0^{\ln\frac{H}{L}}(e^{\xi}L-K)\ \mathbb{I}_{(L<x(t)<H;t\in[0,T])}$$

$$2\sum_{n=0}^{\infty}e^{-\left(\frac{\mu_n}{\ln\frac{H}{L}}\right)^2t}J_p\left(\frac{\mu_n\xi}{\ln\frac{H}{L}}\right)J_p\left(\frac{\mu_n\ln\frac{x}{L}}{\ln\frac{H}{L}}\right)\left(\ln\frac{H}{L}\right)^{-2}\left(\left(1+\frac{h\left(\ln\frac{H}{L}\right)^4-v^2}{\mu_k^2\left(\ln\frac{H}{L}\right)^2}\right)\left(J_p(\mu_k)\right)^2\right)^{-1},$$

Where $\mathbb{I}_{(L<x(t)<H;t\in[0,T])}$ is the Heaviside step function.

Note. Since the roots of the Bessel function of the first kind and its derivative are simple, there is a relation between the derivative and Bessel functions of neighboring orders, therefore, we can prove that the equation $J'_p(x)+hJ_p(x)=0$ has a countable set of positive roots, so at large values of n the squares of these roots behave as n^2. Thus for the Green function and its first derivative the correct evaluation is:

$$C\sum_{n=1}^{\infty}c_0\ln^2t<+\infty,\ \forall x\in[L,H],$$

$$0<t<T,\ C>0, c_0>0.$$

Approximate calculations do not require a large number of coefficients in a row because of the rapid convergence.

Conclusions. There considered the Sturm-Liouville problem for the Bessel diffusion with the boundary conditions under which the derivative of the financial flow with respect to the price variable in combination with the flow volume hits zero. For this problem there built a Green function, which is expanded in terms of the system of Bessel functions of the first kind. An analytic form for the Green function enables calculating the volume of the financial flow, the rate of growth of the portfolio and investigating the volatility in the market at any given time.

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