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The finite-size scaling study of four-dimensional Ising model in the presence of external magnetic field

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The four-dimensional ferromagnetic Ising model in external magnetic field is simulated on the Creutz cellular automaton algorithm using finite-size lattices with linear dimension $4 \le L \le 8$. The critical temperature value of infinite lattice, $T_c^{\chi}(\infty) = 6.680(1)$ obtained for $h = 0$ agrees well with the values $T_c(\infty) \approx 6.68$ obtained previously using different methods. Moreover, $h = 0.00025$ in our work also agrees with all the results obtained from $h = 0$ in the literature. However, there are no works for $h \ne 0$ in the literature. The value of the field critical exponent ($\delta = 3.0136(3)$) is in good agreement with $\delta = 3$ which is obtained from scaling law of Widom. In spite of the finite-size scaling relations of $|M_L(t)|$ and $\chi_L(t)$ for $0 \le h \le 0.001$ are verified; however, in the cases of $0.0025 \le h \le 0.1$ they are not verified.

PACS: **05.50.+q** Lattice theory and statistics (Ising, Potts, etc.);
64.60.Cn Order-disorder transformations;
75.40.Cx Static properties (order parameter, static susceptibility, heat capacities, critical exponents, etc.);
75.40.Mg Numerical simulation studies.

Keywords: Ising model, finite-size scaling, cellular automaton, external magnetic field.

1. Introduction

While the four-dimensional ferromagnetic Ising model is not directly applicable to real magnetic systems, it is useful to investigate the influence of dimensionality on phase transitions [1]. In fact, in Euclidean quantum field theory, the 4d ferromagnetic Ising model describes the physical dimension. As the dimensionality and/or the lattice size increases, the simulation of the ferromagnetic Ising model by the conventional Monte Carlo method becomes impractical and faster algorithms are needed. The Creutz cellular automaton algorithm [2] does not require high-quality random numbers, it is an order of magnitude faster than the conventional Monte Carlo method and compared to the Q2R cellular automaton [3], it has the advantage of fluctuating internal energy from which the specific heat can be computed.

The question of the four-dimensional ferromagnetic Ising model exact solution with an external magnetic field is not known. In two dimensions, the solution of ferromagnetic Ising model is investigated [4–7]. By considering different approximate methods, such as approximate two-dimensional ferromagnetic Ising model solutions are presented [8–12]. In addition, the four-dimensional ferromagnetic Ising model solution is approximated by using Creutz cellular automaton algorithm with nearest neighbor interactions and near the critical region [13–17]. The algorithm of approximating finite size behavior of ferromagnetic Ising model is extended to higher dimension [13–24]. It is established that the algorithm has been powerful in terms of providing the values of static critical exponents near the critical region in four and higher dimensions with nearest neighbor interactions [13–24].

In this paper, we simulated the four-dimensional ferromagnetic Ising model in the presence of an external magnetic field with the Creutz cellular automaton algorithm. The value of the field critical exponent (δ) is obtained. The finite-size scaling relations for $|M_L(t)|$ and $\chi_L(t)$ are verified for $0 \le h \le 0.001$, where $|M_L(t)|$ is the absolute value of the magnetization and $\chi_L(t)$ is the magnetic susceptibility.

The model is described in Sec. 2, the results are discussed in Sec. 3 and a conclusion is given in Sec. 4.

2. Model

Five binary bits are associated with each site of the lattice. The value for each site is determined from its value and those of its nearest neighbors at the previous time step. The updating rule, which defines a deterministic cellular automaton, is as follows: Of the five binary bits on each site, the first one is the Ising spin B_i. Its value may be "0" or "1". The Ising spin energy in the presence of an external magnetic field, H_I, is described by the Hamiltonian of the form

$$H_I = -J \sum_{<ij>} S_i S_j - h \sum_{<i>} S_i \tag{1}$$

taking into account of the interaction between the nearest neighbors and also interaction of the spins S_i with external magnetic field h, directed "up" ($S_i = +1$). The spins affected by the field are directed "up" and not changed during the simulation. Therefore, these spins play the role of the magnetic field. In the Hamiltonian, $S_i = 2B_i - 1$ and h is the ratio of the number of "up" spins to the number of all spins. The next three bits are for the momentum variable conjugate to the spin (the demon). These three bits form an integer which can take on the values within the interval (0,7). The kinetic energy (in units of J) associated with the demon can take four times these integer values. The total energy

$$H = H_I + H_K \tag{2}$$

is conserved, where H_I is the Ising spin energy and H_K is the kinetic energy of the lattice. For a given total energy the system temperature T (in units of J / k_B where k_B is the Boltzmann constant) is obtained from the average value of the kinetic energy. The fifth bit provides a checkerboard style updating, and so it allows the simulation of the ferromagnetic Ising model on a cellular automaton. The black sites of the checkerboard are updated and then their color is changed into white. White sites are changed into black without being updated. The updating rules for the spin and the momentum variables are as follows: For a site to be updated, its spin is flipped and the change in the Ising spin energy (internal energy), H_I, is calculated. If this energy change is transferable to or from the momentum variable associated with this site, such that the total energy H is conserved, then this change is done and the momentum is appropriately changed. Otherwise the spin and the momentum are not changed.

As the initial configuration all the spins are taken to be ordered (up or down). The initial kinetic energy is given to the lattice via the first and the third bits of the momentum variables in the white sites randomly, such that the value of the initial kinetic energy for such a demon is 20 (in units of J), which is just the amount needed to flip a spin at its initial configuration.

Simulations are carried out on simple hypercubic lattices L^4 of linear dimensions $4 \le L \le 8$ with periodic boundary conditions. The cellular automaton develops $9.6 \cdot 10^5$ (L = 4, 6, 8) sweeps for each run, with 7 runs for each total energy.

3. Results and discussion

The temperature dependence of the order parameter and the magnetic susceptibility for several values of h are illustrated in Fig. 1 for the lattice with L = 8. These functions exhibit a similar behavior for a lattice of linear dimensions L = 4 and 6. The temperature dependence of the order parameter is in an agreement with the expected behavior in the presence of the field (Fig. 1(a)). As the external magnetic field is increased, the peaks of the magnetic susceptibility decrease with respect to the peak values for $h = 0$. For $h = 0$, the magnetic susceptibility peak is very sharp. As the external magnetic field is increased, the temperature dependence of the magnetic susceptibility becomes smooth (Fig. 1(b)). The critical temperatures of the finite-size lattices obtained from the magnetic susceptibility maxima $T_c^{\chi}(L)$ for $0 \le h \le 0.1$ are listed in Table 1.

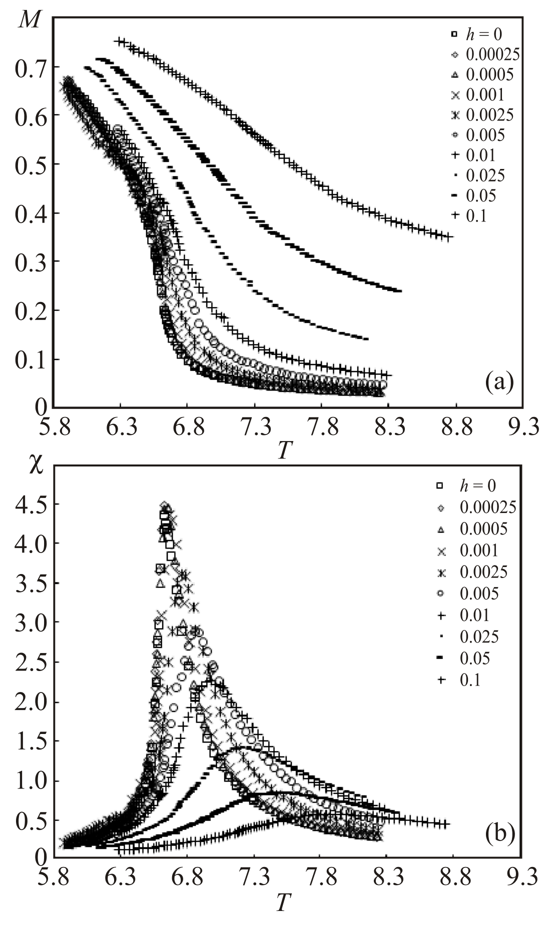

Fig. 1. The temperature dependence of the order parameter and the magnetic susceptibility for several values of h for L = 8.

Table 1. The maximum values and the critical temperatures of the magnetic susceptibility for $0 \le h \le 0.1$

L	T_c^{χ}	$\chi_{\max}$	h
4	6.513(21)	0.912(20)	
6	6.611(23)	2.340(34)	0
8	6.635(11)	4.467(32)	
4	6.514(1)	0.908(12)	
6	6.629(3)	2.330(43)	0.00025
8	6.640(9)	4.452(2)	
4	6.515(1)	0.889(4)	
6	6.631(1)	2.325(11)	0.00050
8	6.659(1)	4.440(9)	
4	6.524(1)	0.865(1)	
6	6.637(56)	2.321(7)	0.001
8	6.672(63)	4.363(16)	
4	6.586(81)	0.862(9)	
6	6.680(5)	2.262(1)	0.0025
8	6.759(32)	4.166(1)	
4	6.592(12)	0.857(1)	
6	6.786(11)	2.000(2)	0.0050
8	6.859(23)	3.882(3)	
4	6.891(13)	0.817(5)	
6	6.933(67)	1.617(3)	0.01
8	6.982(56)	3.260(54)	
4	7.168(1)	0.703(21)	
6	7.246(3)	1.097(31)	0.025
8	7.249(1)	2.811(18)	
4	7.741(12)	0.473(25)	
6	7.766(31)	0.642(42)	0.050
8	7.788(14)	1.906(12)	
4	8.455(9)	0.271(1)	
6	8.495(12)	0.377(4)	0.1
8	8.524(15)	1.024(12)	

The dependence of the critical temperatures $T_c^{\chi}(L)$ obtained from the magnetic susceptibility maxima of the finite-size lattices on linear dimension L is given by the following expression [13–17,25,26]:

$$T_c^{\chi}(\infty) - T_c^{\chi}(L) \propto L^{-2} \log^{-1/6} L \, . \qquad (3)$$

The values of the infinite-lattice critical temperature for the four-dimensional ferromagnetic Ising model, $6.680(1) \le T_c(\infty) \le 8.539(3)$ are obtained from the straight line fit of the magnetic susceptibility maxima for $0 \le h \le 0.1$ (Fig. 2). The value obtained of infinite lattice critical temperature $T_c^{\chi}(\infty) = 6.680(1)$ for $h = 0$ agrees well with the value $T_c(\infty) \approx 6.68$ obtained previously using different methods [13–17,27–29]. However, there are no

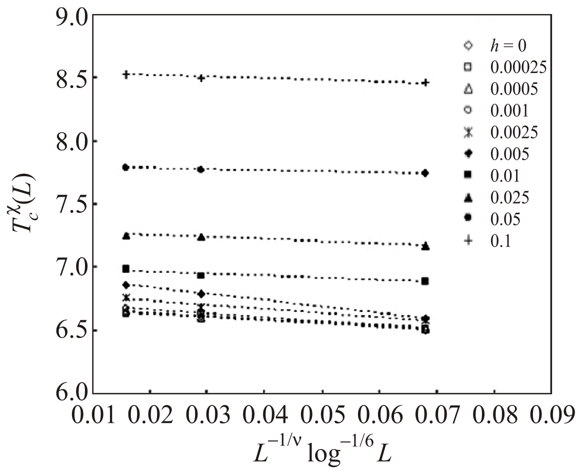

Fig. 2. The values of the infinite-lattice critical temperature for the four-dimensional ferromagnetic Ising model with $\nu = 1/2$, $T_c^{\chi}(\infty) =$ 6.680(1) for $h = 0$ ($R^2 = 0.995$, marked ◊), $T_c^{\chi}(\infty) =$ = 6.680(3) for $h = 0.00025$ ($R^2 = 0.946$, □), $T_c^{\chi}(\infty) =$ 6.691(28) for $h = 0.00050$ ($R^2 = 0.995$, Δ), $T_c^{\chi}(\infty) =$ 6.723(15) for $h =$ = 0.001 ($R^2 = 0.999$, ○), $T_c^{\chi}(\infty) =$ 6.793(52) for $h = 0.0025$ ($R^2 =$ = 0.948, ∗), $T_c^{\chi}(\infty) =$ 6.938(85) for $h = 0.0050$ ($R^2 = 0.999$, ◆), $T_c^{\chi}(\infty) =$ 6.997(62) for $h = 0.01$ ($R^2 = 0.898$, ■), $T_c^{\chi}(\infty) =$ = 7.284(45) for $h = 0.025$ ($R^2 = 0.955$, ▲), $T_c^{\chi}(\infty) =$ 7.797(51) for $h = 0.050$ ($R^2 = 0.948$, ●) and $T_c^{\chi}(\infty) =$ 8.539(3) for $h = 0.01$ ($R^2 = 0.967$, +), obtained by extrapolating temperatures of the lattice with the linear dimension $4 \le L \le 8$ as $L \to \infty$.

works for $h \ne 0$ in the literature; moreover, $h = 0.00025$ in our work agrees with all the results obtained from $h = 0$ in the literature. In this work, T_c are obtained for $h \ne 0$ and given in Table 2. In our opinion, symmetry is conserved for $h = 0$ and 0.00025. That is why the results agree with the values in the literature. However, broken symmetry occurs for the values of $h > 0.00025$.

Table 2. The values of the infinite-lattice critical temperature for $0 \le h \le 0.1$

T_c^{χ}	Method
6.6802(2) [27]	Series expansion
6.6803(1) [27]	Dynamic Monte Carlo
6.680(1) [28,29]	Cluster Monte Carlo
6.680, 6.6802 [13–17]	Creutz cellular automaton
6.680(1) for $h = 0$, this work	Creutz cellular automaton
6.680(3) for $h = 0.00025$, this work	Creutz cellular automaton
6.691(28) for $h = 0.00050$, this work	Creutz cellular automaton
6.723(15) for $h = 0.001$, this work	Creutz cellular automaton
6.793(52) for $h = 0.0025$, this work	Creutz cellular automaton
6.938(85) for $h = 0.0050$, this work	Creutz cellular automaton
6.997(62) for $h = 0.01$, this work	Creutz cellular automaton
7.284(45) for $h = 0.025$, this work	Creutz cellular automaton
7.797(51) for $h = 0.050$, this work	Creutz cellular automaton
8.539(3) for $h = 0.1$, this work	Creutz cellular automaton

For a lattice linear dimension L and very small h at $T = T_c(L)$, the order parameter is given by

$$M(L) \propto h^{1/\delta(L)} , \quad (4)$$

where $\delta(L)$ is the field critical exponent (Fig. 3). Scaling law of Widom is following:

$$\gamma = \beta(\delta - 1) , \quad (5)$$

where $\gamma = 1$, $\beta = 1/2$ and $\delta = 3$ for $d = 4$ [30,31]. Since the magnetic susceptibility in the presence of the external magnetic field is smooth, the finite-size lattice critical temperatures for the values of each h are obtained from the locus of points of maximum slope in the magnetization results as Monte Carlo calculations [8].

The log-log plots of $M(L)$ at $T = T_c(L,h)$ versus h for h in the interval $0.00025 \le h \le 0.1$ yields to $1/\delta(L)$ (Fig. 4). The straight line which fits to the plot of $\delta(L)$ against $1/L$ results in the infinite-lattice critical exponents $\delta = 3.0136(3)$ (Fig. 5). The result for the $\delta(\infty)$ is compared with $\delta = 3$ which is obtained from scaling law of Widom [30,31].

Privman–Fisher hypothesis for the singular part of the free-energy density $f_L^{(S)}(t,h)$ of a hypercubic finite system L^d with periodic boundary conditions is adapted for the Ising model in $d = 4$ dimensions, by proposing the finite-size scaling function $Y(x,y)$, correct to leading logarithms as below:

$$f_L^{(S)}(t,h) = L^{-4} Y(tL^2 \log^{1/6} L, hL^3 \log^{1/4} L), \quad (6)$$
$$t \to 0,\ h \to 0,\ L \to \infty ,$$

where $t = (T - T_c)/T_c$ is the reduced temperature and h is the external magnetic field [13]. From Eq. (6) the finite-size scaling expressions for the magnetization $M_L(t,h)$, the magnetic susceptibility $\chi_L(t,h)$ can be derived as below:

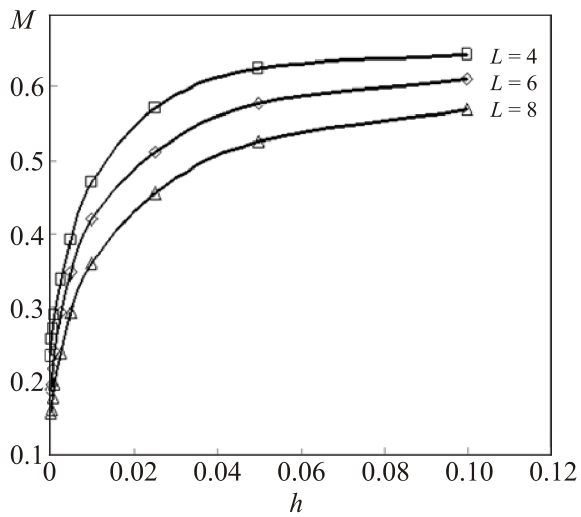

Fig. 3. The dependence of M against h for the lattices with the linear dimension $4 \le L \le 8$ ($T_c = 6.6802(2)$).

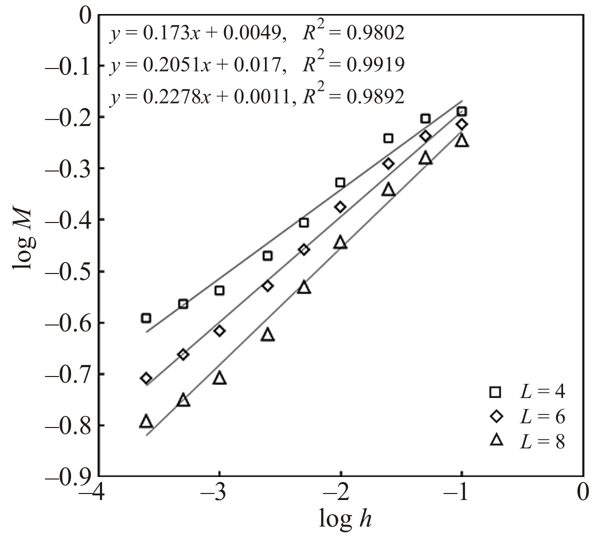

Fig. 4. The log-log plots of $M(L,T_c(L,h))$ against h with the slope giving the value of $1/\delta = 0.187$ for $L = 4$, 6 and 8 at $T_c = 6.6802(2)$.

$$M_L(t,h) = -\frac{\partial f_L}{\partial h} = L^{-1} \log^{1/4}(L) U(tL^2 \log^{1/6} L, hL^3 \log^{1/4} L), \quad (7)$$

$$\chi_L(t,h) = -\frac{\partial^2 f_L}{\partial h^2} = L^2 \log^{1/2}(L) V(tL^2 \log^{1/6} L, hL^3 \log^{1/4} L), \quad (8)$$

where U, V are the corresponding finite-size scaling functions. For $h = 0$ they reduced to the following equations [13]:

$$|M_L(t)| = L^{-\beta/\nu} \log^{1/4}(L) U(tL^2 \log^{1/6} L) , \quad (9)$$

$$\chi_L(t) = L^{\gamma/\nu} \log^{1/2}(L) V(tL^2 \log^{1/6} L) , \quad (10)$$

where β, γ and ν are critical exponents for the magnetization, the magnetic susceptibility and the correlation length

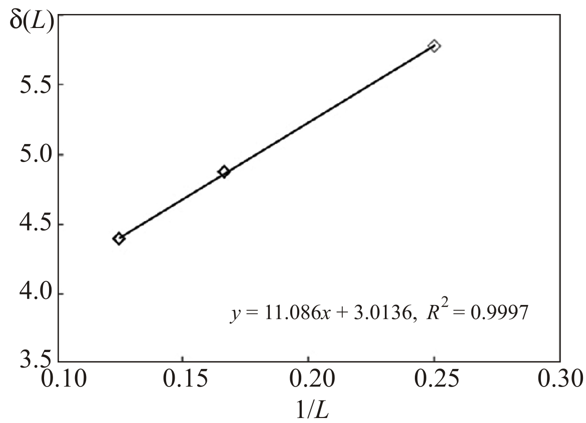

Fig. 5. The plot of $\delta(L)$ against $1/L$. The extrapolation of the fit lines to $1/L \to \infty$ gives $\delta = 3.0136(3)$.

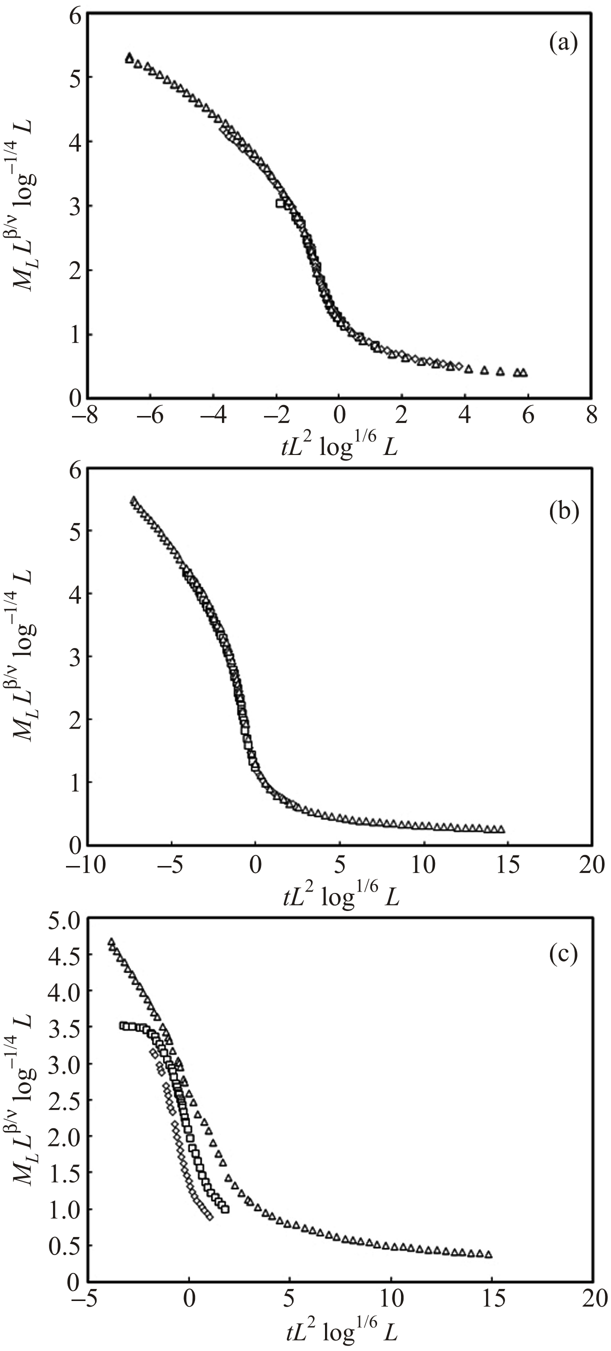

Fig. 6. Finite-size scaling plot of $|M_L|$ with $\beta/\nu = 1$ and $T_c = 6.6802(2)$ for $h = 0$ (a), 0.00050 (b), 0.0050 (c).

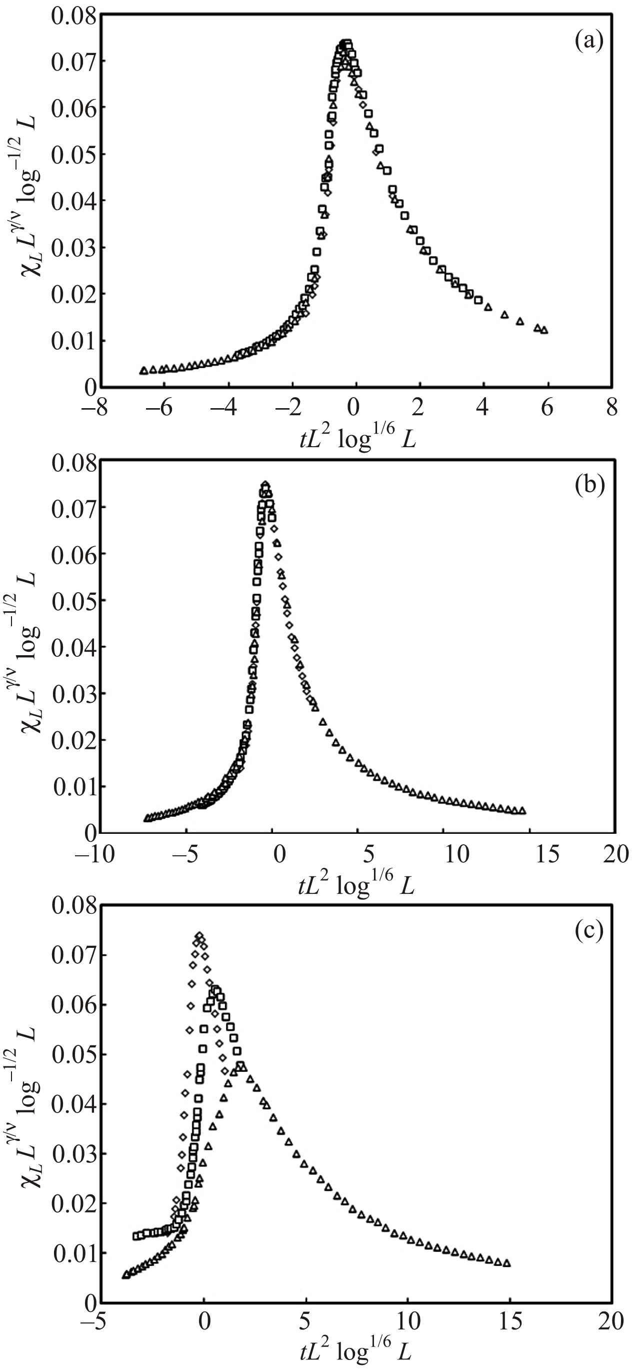

Fig. 7. Finite-size scaling plot of χ_L with $\gamma/\nu = 2$ and $T_c = 6.6802(2)$ for $h = 0$ (a), 0.00050 (b), 0.0050 (c).

of the infinite lattice, respectively. The relations for $0 \le h \le 0.1$ can be tested by simulations directly. The finite-size scaling plots for $|M_L(t)|$ and $\chi_L(t)$ are given in Figs. 6, 7, respectively. The overlap of the plots of the scaled quantities for different L (Figs. 6(a),(b) and 7(a),(b)) verifies the finite-size scaling relations given in Eqs. (9) and (10) at $T_c = 6.6802(2)$. Owing to the overlap of the plots of the scaled quantities for different L (Figs. 6(c) and 7(c)) doesn't verify the finite-size scaling relations given in Eqs. (9) and (10) at $T_c = 6.6802(2)$.

4. Conclusions

The four-dimensional ferromagnetic Ising model in external magnetic field is simulated on the Creutz cellular automaton algorithm by using the finite-size lattices with the linear dimensions $L = 4$, 6, and 8. In our work, the critical temperature value of infinite lattice for $h = 0$ is in agreement with the other simulation results. Moreover, $h = 0.00025$ in our work also agrees with all the results obtained from $h = 0$ in the other simulation results in literature. The values $h > 0.00025$ do not agree with T_c values in the literature because broken symmetry occurs. In this study, the value of the field critical exponent ($\delta = 3.0136(3)$) is satisfied by scaling law of Widom. Although the finite-size scaling relations of $|M_L(t)|$ and $\chi_L(t)$ for $0 \leq h \leq 0.001$ are verified; however, in the cases of $0.0025 \leq h \leq 0.1$ are not verified. The values of h ($0.0025 \leq h \leq 0.1$) aren't verified because of broken symmetry. Finally, we note that the inconsistency in temperature, the absolute value of the magnetization and $\chi_L(t)$ is the magnetic susceptibility are the result of broken symmetry and this affects the phase transition in finite systems.

The computer used to run the model is an Intel(R) Core Duo CPU at 1830 MHz. Since the CPU time was invested 11200 h for all the simulations, the case of $L \geq 10$ will be considered later.

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