

# Excitation of magnetostatic spin waves in anisotropic ferromagnetic films, magnetized in arbitrary direction

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Exact analytical expressions for propagator of small-amplitude linear magnetostatic waves in ferromagnetic thin film between two antennae and their corresponding mutual impedance are obtained by solving the linearized torque equation of spin dynamics (Landau–Lifshitz equation) in magnetostatic approximation. This is done for the case of arbitrary orientation of uniform static magnetization of the film and full account for arbitrary magnetic anisotropy. The result also contains full description of the magnetostatic spin-wave spectrum.

PACS: 75.30.Ds Spin waves;

**76.50.+g** Ferromagnetic, antiferromagnetic, and ferromagnetic resonances; spin-wave resonance.

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## 1. Introduction

The terms “magnetostatic oscillations” or “magnetostatic waves” (MSW) [1–4] mean relatively long spin waves (SW), whose properties are dominated by the long-range quasi-static dipole interaction between “spins” and, thus, strongly influenced by geometry of ferromagnetic sample (its shape and dimensions). In contrast, properties of relatively short SW (spin waves in narrow sense) are dictated by the short-range exchange interaction [3–5].

In ferrite films MSW can be easily excited by microwave-frequency magnetic fields from wire or strip-like antennae [6–8] with amplitudes as high as to induce strongly nonlinear interactions of MSW between themselves and with SW. Other novel methods of creating small and high-amplitude spin-waves are based on spin-polarized current injection [9] or highly-focused femtosecond laser pulses [10]. In addition, their small velocity ensures compactness of MSW-based nonlinear electromagnetic transducers. When exploring these possibilities, it is important to know characteristics of linear MSW modes and their contribution to the mutual impedance between the antennae. There are widely used exact results by Damon and Eshbach [2] for MSWs in flat normally magnetized homogeneous films. Other authors [11,12] (see also paper by Kalinikos and coworkers in Ref. 6 and references therein) developed approximate approaches for the case of nonhomogeneous films having specific surface pinning effects. At the same time, to the best of our knowledge, the homogeneous case remains incompletely investigated. In this

paper we present new solutions for this case believing that they can be useful both in practice and for theoretical modeling of more complicated situations.

Our consideration is restricted to “magnetostatic approximation”, which neglects SW (in narrow sense) at all, as if the exchange stiffness  $C$  (and, consequently, the exchange length,  $r_e = (C / (4\pi M_s))^{1/2}$ , relating it to the saturation magnetization of the film's material,  $M_s$ ) is equal to zero. This formal trick seems reasonable when cross-sectional dimensions of all the antennae or/and their distances to film's surface are much greater than  $r_e$ . Indeed, under this condition, magnetic field, created by antennae, is smoothly distributed in film's interior (with spatial scales much greater than  $r_e$ ) that it interacts very poorly with the SW, whose length is comparable or smaller than  $r_e$ , leaving them mostly unexcited. We do not know of a rigorous proof of this statement, but it follows rather convincingly from results of the “magnetostatic approximation” itself (including those, presented below) which consistently demonstrate insignificance of “infinitely short” MSW in the sense that both their excited energy and total contribution to film's linear response to weak external field are negligible. On the other hand, in strongly nonlinear regime SW much shorter than  $r_e$  can be generated from a long MSW [6] (this behavior is shared by all the nonlinear models, collected in the Ref. 6). However, here we are interested in linear (small-amplitude) response only.

Additionally, we will assume that the film under consideration is thick — its thickness,  $D$ , is large, compared to  $r_e$ . This condition should bode well with applications of

our results to real films with surface-induced nonuniformity. Moreover, if ground (static) state of film's magnetization consists of domains and, hence, is strongly nonuniform, then this state itself must be understood and described before its MSW excitations. Of course, we avoid this extremely difficult problem, being satisfied by consideration of MSW on uniformly magnetized static background. Fortunately (e.g., in yttrium–iron garnet films), such state can be enforced by comparatively small bias field.

Dipole-exchange theory of spin-wave spectrum [13,14], additionally to including the arbitrary anisotropy and magnetization direction, is valid in a wider range of wavelengths and covers both magnetostatic and exchange spin waves. For technical applications, however, it is convenient to express the resonant properties of magnetic medium in terms of directly measurable quantities, such as impedances. This is why we derive here the “MSW propagator”: function, describing mutual influence of two antennae due to MSW propagating in the film, while restricting ourselves to a simpler magnetostatic approximation.

## 2. Basic equations

The classical dynamics of magnetization [4,5,15] is based on the Landau–Lifshitz equation (the torque equation)

$$\frac{d\mathbf{S}}{dt} = \mathbf{F} \times \mathbf{S} + \gamma \{ \mathbf{F} - \mathbf{S}(\mathbf{S} \cdot \mathbf{F}) \}. \quad (1)$$

Here  $\mathbf{S}$  is the direction vector of the local magnetization ( $|\mathbf{S}|=1$ );  $\mathbf{F}$  is the thermodynamic force, or effective internal magnetic field, expressed in units of the saturation magnetization,  $M_s$ ; time  $t$  is expressed in units of  $\tau_0 = (2\pi g M_s)^{-1}$  ( $g \approx 2.8$  MHz/Oe is the gyromagnetic ratio);  $\gamma$  is the phenomenological friction (dissipation) parameter.

The internal field,  $\mathbf{F}$ , is composed at least from [4,5,15] (i) the external bias field,  $\mathbf{H}_e$ ; (ii) the magnetic anisotropy field,  $\mathbf{H}_a$ ; (iii) “demagnetizing” magnetic field, created by the magnetization itself,  $\mathbf{H}_s$ ; and (iv) the exchange field,  $\mathbf{H}_{\text{exch}}$ . We do not consider magnetic field due to eddy currents, assuming that the medium is a good insulator. Usually small dimensions of real samples and typical frequencies of MSW allow to neglect retardation effects in  $\mathbf{H}_s$ , permitting to express it as a quasi-static solution of the Maxwell's equations. In the absence of conductors and other ferromagnets in the vicinity of the sample, this solution can be written as

$$\mathbf{H}_s = -\widehat{G}\mathbf{S} \equiv \nabla \left( \nabla \cdot \int \frac{\mathbf{S}(r') dr'}{|r-r'|} \right). \quad (2)$$

The usual simplest model for the exchange interaction corresponds to  $\mathbf{H}_{\text{exch}} = r_e^2 \nabla^2 \mathbf{S}$  [4,5,15]. It should be noted that the exchange interaction ensures local smoothness of the magnetization distribution,  $\mathbf{S}$ , while the dipole–dipole interaction operator,  $\widehat{G}$ , is bounded when acting on smooth

distributions:  $\|\widehat{G}\| \leq 4\pi$ . If we denote the anisotropy energy density by  $A(\mathbf{S})$ , then

$$\mathbf{H}_a = -A'(\mathbf{S}) \equiv -\partial A(\mathbf{S}) / \partial \mathbf{S}.$$

In principle, all the consideration in this and the next sections can be easily generalized to the case of nonuniform anisotropy, when  $A(\mathbf{S})$  explicitly depends on spatial coordinates.

Let the subscript “0” mark the attributes of background equilibrium static magnetization state at constant bias field,  $\mathbf{H}_e = \mathbf{H}_0 = \text{const}$ . According to Eq. (1) the vectors  $\mathbf{S}_0$  and  $\mathbf{F}_0$  in such state are parallel: that is  $\mathbf{F}_0 = W_0 \mathbf{S}_0$ , where scalar field  $W_0$  (absolute value of static internal magnetic field) is defined by the requirement  $|\mathbf{S}| = 1$ .

When additional time-varying magnetic field is applied,  $\mathbf{H}_e = \mathbf{H}_0 + \mathbf{h}(t)$ , the magnetization will deviate from its static value. This deviation,  $\mathbf{s} = \mathbf{S} - \mathbf{S}_0$ , can be represented as

$$\mathbf{s} = \mathbf{S}_\perp + (S_\parallel - 1)\mathbf{S}_0, \quad \mathbf{S}_\perp = \widehat{\Pi} \mathbf{S}_\perp \mathbf{S}_0, \quad (3)$$

$$\widehat{\Pi} \equiv 1 - \mathbf{S}_0 \otimes \mathbf{S}_0, \quad S_\parallel = \pm \sqrt{1 - |\mathbf{S}_\perp|^2},$$

where  $\otimes$  stands for tensor product, and  $\widehat{\Pi}$  is a projection operator (matrix), projecting vectors onto the plane, perpendicular to  $\mathbf{S}_0$ . If the perturbation is small enough not to completely overturn any spins, the quantity  $S_\parallel$  is positive everywhere. In terms of  $\mathbf{s}$  and  $\mathbf{S}_\perp$ , Eq. (1) transforms into

$$\frac{d\mathbf{S}_\perp}{dt} = S_\parallel [\mathbf{F}_\perp \times \mathbf{S}_0] + \gamma (1 - \mathbf{S}_\perp \otimes \mathbf{S}_\perp) \mathbf{F}_\perp, \quad (4)$$

where  $\mathbf{F}_\perp = -\delta E_\perp / \delta \mathbf{S}_\perp$ , and  $E_\perp$  is the excess energy (energy of excitation) implied by the perturbation,

$$E_\perp = \int \left[ \frac{1}{2} W_0 s^2 + \widetilde{A}(\mathbf{s}) + \frac{\mathbf{s} \cdot \widehat{G}\mathbf{s}}{2} - \mathbf{h} \cdot \mathbf{s} + C(\nabla \otimes \mathbf{s}) \right] dr, \quad (5)$$

where function  $\widetilde{A}$  is defined by

$$\widetilde{A}(\mathbf{s}) = A(\mathbf{S}_0 + \mathbf{s}) - A(\mathbf{S}_0) - \mathbf{s} \cdot \nabla_{\mathbf{S}_0} A(\mathbf{S}_0),$$

subscript in  $\nabla_{\mathbf{S}_0}$  implies differentiation over components of  $\mathbf{S}_0$  and function  $C$  represents the exchange contribution to the excess energy. In the mentioned model,

$$C(\nabla \mathbf{s}) = \frac{1}{2} r_e^2 \sum_{\alpha\beta} (\nabla_\alpha s_\beta)^2.$$

The functional derivative  $\mathbf{F}_\perp$  in Eq. (4) should be evaluated carefully taking into account full dependence of  $\mathbf{s}$  on  $\mathbf{S}_\perp$  according to Eq. (3).

One may check directly that the frictionless ( $\gamma = 0$ ) version of Eq. (4) follows from the variational principle

$$\delta \int \left\{ \int \left( \mathbf{S}_0 \cdot \left[ \frac{d\mathbf{S}_\perp}{dt} \times \mathbf{S}_\perp \right] \right) \frac{dr}{1 + S_\parallel} + E_\perp \right\} dt = 0, \quad (6)$$

while the dissipation, as it is usual in the Lagrange formalism, has to be added externally.

Of course, in general,  $\mathbf{S}_0$  is a complicated function of spatial coordinates, hence all the related values ( $W_0$ ,  $\hat{\Pi}$ ,  $A(\mathbf{s})$ , and so on) depend on spatial coordinate.

### 3. Linear waves

If the static magnetization pattern  $\mathbf{S}_0$  is stable with respect to small perturbations, the functional (5) must represent a positively defined quadratic form, which allows us to speak about linear eigenmodes of the excitation. In this case, let us introduce the spin precession operator,  $\hat{R}$ , the anisotropy matrix,  $\hat{A}$ , and the exchange operator,  $\hat{C}$ , by the relations

$$\begin{aligned}\hat{R}\mathbf{V} &\equiv \mathbf{S}_0 \times \mathbf{V}, & \hat{C}\mathbf{V} &\equiv -r_e^2 \nabla^2 \mathbf{V}, \\ \hat{A}_{\alpha\beta} &= \partial^2 A(\mathbf{S}_0) / \partial \mathbf{S}_{0\alpha} \partial \mathbf{S}_{0\beta}.\end{aligned}$$

For the linear regime Eq. (4) yields

$$\frac{d\mathbf{S}_\perp}{dt} = -(\hat{R} - \gamma\hat{\Pi})(\hat{W}\mathbf{S}_\perp - \mathbf{h}), \quad (7)$$

where we introduce the integral-differential operator

$$\hat{W} \equiv W_0 + \hat{A} + \hat{G} + \hat{C}.$$

Omitting in Eq. (7) both dissipation and external pumping, we get equations for SW and MSW eigenmodes and eigenfrequencies:

$$\mathbf{S}_\perp \equiv \mathbf{V}e^{-i\omega t}, \quad -i\omega\mathbf{V} = \hat{R}\hat{W}\mathbf{V}. \quad (8)$$

Herewith it is sufficient to consider positive frequencies only. Let the eigenmodes be enumerated by an index  $k$ . Since in a stable state the operator  $\hat{W}$  is positively defined, we can write

$$\omega_k \tilde{\mathbf{V}}_k = i\hat{W}^{1/2} \hat{R}\hat{W}^{1/2} \tilde{\mathbf{V}}_k, \quad \tilde{\mathbf{V}}_k \equiv \hat{W}^{1/2} \mathbf{V}_k. \quad (9)$$

The operator on right-hand side of the left of these two equations must be self-adjoint, hence, its eigenfunctions  $\tilde{\mathbf{V}}_k$  can be made mutually orthogonal. Which, for eigenmodes, results in the following orthogonality rule:

$$i \int \mathbf{S}_0 \cdot [\mathbf{V}_k \times \mathbf{V}_m^*] dr = \delta_{mk}, \quad (10)$$

with star denoting the complex conjugation. The same rule follows from the variational principle (6).

For a more general formulation of linear theory, let us turn from the “pre-made” dipole interaction operator  $\hat{G}$  directly to Maxwell's equations:

$$\frac{d\mathbf{S}_\perp}{dt} = -(\hat{R} - \gamma\hat{\Pi})\{(W_0 + \hat{A} + \hat{C})\mathbf{S}_\perp - \mathbf{h} - \mathbf{h}_S\}, \quad (11)$$

$$\nabla \cdot (\mathbf{h}_S + 4\pi\mathbf{S}_\perp) = 0, \quad \nabla \times \mathbf{h}_S = 0, \quad (12)$$

where we introduced a new vector field,  $\mathbf{h}_S$ , representing time-varying part of magnetic field, self-induced by time-

varying part of magnetization (that is, by  $\mathbf{s}$ ). As before, it is assumed that the sample is nonconducting.

Applying Fourier transform to the Eq. (11), in the frequency domain we have from (11) and (12)

$$\mathbf{S}_\perp = \hat{\chi}[\mathbf{h} + \mathbf{h}_S], \quad \nabla \cdot \hat{\mu}[\mathbf{h} + \mathbf{h}_S] = 0, \quad (13)$$

$$\hat{\chi} = \{i\omega + (\hat{R} - \gamma\hat{\Pi})(W_0 + \hat{A} + \hat{C})\}^{-1}(\hat{R} - \gamma\hat{\Pi}), \quad (14)$$

where  $\hat{\mu} \equiv 1 + 4\pi\hat{\chi}$ . Obviously, because of the presence of differential operator  $\hat{C}$  in denominator of the polarizability matrix  $\hat{\chi}$  in (14),  $\hat{\chi}$ , in fact, is an integral operator.

At this point let us apply the “magnetostatic approximation”, formulated and discussed in Sec. 1. Specifically, by neglecting the exchange operator  $\hat{C}$  from denominator of (14). Formally, this is equivalent to putting the exchange radius  $r_e$  equal to zero (of course, by this the exchange interaction is not neglected completely, since it remains responsible for the magnetization phenomenon itself). Strictly speaking, the static magnetization (i.e., the patterns  $\mathbf{S}_0$  and  $W_0$ ) must also be treated in this limit. But this does not matter in the case of uniform static magnetization, which is considered below.

Following this assumption,  $\hat{\chi}$  turns into an algebraic expression, literally, becoming a matrix, and the problem reduces to the solution of purely differential equations for the field  $\mathbf{h}_S$ . Direct analytical calculation yields a very simple expression for the polarizability:

$$\hat{\chi} = \frac{(\bar{W}_0 + A_1 + A_2)\hat{\Pi} - \hat{A}_\perp + i\tilde{\omega}\hat{R}}{(\bar{W}_0 + A_1)(\bar{W}_0 + A_2) - \tilde{\omega}^2}, \quad (15)$$

$$\hat{A}_\perp \equiv \hat{\Pi}\hat{A}\hat{\Pi}, \quad \bar{W}_0 \equiv W_0 - i\gamma\tilde{\omega}, \quad \tilde{\omega} \equiv \frac{\omega}{1 + \gamma^2}.$$

Here  $A_1$  and  $A_2$  are those two eigenvalues of matrix  $\hat{A}_\perp$ , which correspond to the pair of its eigenvectors perpendicular to  $\mathbf{S}_0$  (and to one another):  $A_\perp \mathbf{a}_{1,2} = A_{1,2} \mathbf{a}_{1,2}$ . We enumerate them so that  $\mathbf{S}_0 \cdot [\mathbf{a}_1 \times \mathbf{a}_2] > 0$ . In practically interesting ferrite samples  $\gamma \lesssim 10^{-3}$  and, therefore,  $\gamma^2$  plays no role.

### 4. Propagator of magnetostatic waves in films

Let the time-varying field  $\mathbf{h}$  be produced by some conductors, placed outside the ferromagnetic sample and carrying ac currents  $I_n$ , distributed with densities  $I_n \mathbf{J}_n$  ( $n = 1, 2, \dots$ ). Then

$$\mathbf{h} = \sum_n \mathbf{h}_n I_n, \quad \nabla \cdot \mathbf{h}_n = 0, \quad \nabla \times \mathbf{h}_n = \frac{4\pi}{c} \mathbf{J}_n. \quad (16)$$

Here  $\mathbf{h}_n$  is the magnetic field, created by a unit-value current in  $n$ th conductor. The same quantity determines voltage (emf),  $\varepsilon_n$ , induced in the  $n$ th conductor by the time-varying magnetization in the sample:

$$\varepsilon_n = \int \left( \mathbf{h}_n \cdot \frac{d\mathbf{s}}{dt} \right) dr. \quad (17)$$

The fields  $\mathbf{h}_n$  as well as the self-induced field  $\mathbf{h}_S$  can be represented in the potential form.

In the linear regime, when  $\mathbf{s} \simeq \mathbf{S}_\perp$ , the response of the sample divides into a sum of partial responses:

$$\mathbf{h}_n = -\nabla U_n, \quad \mathbf{h}_S = \sum_n \hat{\mathbf{h}}_n^S I_n, \quad \hat{\mathbf{h}}_n^S = -\nabla \hat{U}_n^S,$$

where  $\hat{U}_n^S$  is the potential of the field, produced by the sample in response to influence of the  $n$ th conductor. After obtaining  $\mathbf{h}_S$  together with  $\mathbf{S}_\perp$  from (17), we will determine mutual impedances of the conductors,  $\hat{Z}_{nm}$ , due to their interaction via the ferromagnet:  $\varepsilon_n = \sum_m \hat{Z}_{nm} I_m$  (the hat marks convolution operators).

Now concretize the sample as a plate (film, formally an infinite plate), whose in-plane dimensions are much larger, compared to its thickness. At sufficiently large bias field,  $\mathbf{H}_0$ , the state of uniform magnetization becomes stable. In real finite-size films the stable state may contain closure domains at film's edges, whose width is of the order of a few  $D$  ( $D$  is film thickness). This conditions applicability of the theory for an infinite uniformly magnetized film to the MSW in real plates.

Let our film occupy the region  $-D/2 < z < D/2$ . The Fourier transforms of various functions with respect to time and in-plane coordinates,  $x$  and  $y$  will be marked by tilde. Let us also denote  $\mathbf{k} = \{k_x, k_y\}$ ,  $\nabla = \{ik_x, ik_y, \nabla_z\}$ . In film's interior  $\nabla^2 \tilde{U}_n = 0$ , and, therefore, the potentials of conductors have the form

$$\tilde{U}_n(\mathbf{k}, z) = \Phi_n(\mathbf{k}) \exp\{|\mathbf{k}|(\sigma_n z - D/2)\}, \quad (18)$$

where  $\sigma_n = 1$  ( $-1$ ), if  $n$ th conductor is placed above (below) film, and form-factor  $\Phi_n(\mathbf{k})$  describes distribution of  $n$ th current. In combination with (12) and (17) the latter formula implies relation between impedances, on one hand, and values of the potentials, taken at film's surfaces, on the other hand:

$$Z_{nm} = \frac{i\omega}{2\pi} \int |\mathbf{k}| \tilde{U}_m^S \left( \omega, \mathbf{k}, \sigma_n \frac{D}{2} \right) \tilde{U}_n \left( -\mathbf{k}, \sigma_n \frac{D}{2} \right) d\mathbf{k}, \quad (19)$$

where  $d\mathbf{k} \equiv dk_x dk_y / (2\pi)^2$ . The potentials are taken at the surface, closest to the receiving antenna ( $n$ th conductor). In contrast to  $\tilde{U}_n$ , potentials  $\tilde{U}_n^S$  are frequency-dependent.

The Eq. (13), or equivalently,

$$(\nabla \cdot \hat{\mu} \nabla)(\tilde{U}_n^S + \tilde{U}_n) = 0, \quad (20)$$

should be solved under standard boundary conditions [4,15]. To write the answer, introduce the unit-length vectors

$$\mathbf{v} \equiv \{k_x/|\mathbf{k}|, k_y/|\mathbf{k}|, 0\}, \quad \bar{\mathbf{z}} \equiv \{0, 0, 1\},$$

and the matrix

$$M = \begin{bmatrix} \mu_{vv} & \mu_{vz} \\ \mu_{zv} & \mu_{zz} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{v} \cdot \hat{\mu} \mathbf{v} & \mathbf{v} \cdot \hat{\mu} \bar{\mathbf{z}} \\ \bar{\mathbf{z}} \cdot \hat{\mu} \mathbf{v} & \bar{\mathbf{z}} \cdot \hat{\mu} \bar{\mathbf{z}} \end{bmatrix}. \quad (21)$$

As usual, the solution is composed of two exponents:

$$\tilde{U}_n^S + \tilde{U}_n = \sum_{\pm} u_{n\pm} \exp(q_{\pm} z), \quad q_{\pm} = \lambda_{\pm} |\mathbf{k}|, \quad (22)$$

$$\lambda_{\pm} \equiv \lambda_0 \pm \Lambda, \quad \lambda_0 = \frac{\mu_{vz} + \mu_{zv}}{2i\mu_{zz}}, \quad (23)$$

$$\Lambda = \sqrt{\frac{\mu_{vv}}{\mu_{zz}} - \left( \frac{\mu_{vz} + \mu_{zv}}{2\mu_{zz}} \right)^2}. \quad (24)$$

Let us emphasize, however, that in general case (at arbitrary orientation of the vector  $\mathbf{S}_0$ ) the exponents  $q_{\pm}$  are neither poorly imaginary nor poorly real but complex (that is, MSW are not standing in  $Z$  direction).

If we'd take into account the exchange radius  $r_e > 0$  and deal with the operator-valued polarizability matrix (14), then in place of (22) one would get a sum of at least six terms, where transverse wave numbers of the order of  $|\mathbf{k}|$  (as  $q_{\pm}$  in (22)) are more or less hybridized with real or imaginary wave numbers of the order of  $\pi/r_e$ . From the point of view of our aims, such complication would be too big price to pay for a small increase in accuracy. But it may be necessary when considering short SW or small-scale details of long MSW on background of a nonuniform domain structure. A more natural approach to these harder problems would be the direct analysis of the system of Eqs. (11) and (12).

Consider the susceptibility matrix  $M$ . Introducing spherical coordinate system, such that its azimuthal angle  $\theta$  is the angle between  $Z$  axis and  $\mathbf{S}_0$ . In the  $XY$  plane (film's plane), we introduce quantities  $v_{\parallel}$  and  $v_{\perp}$  as cosine and sine, respectively, of the angle (counted clockwise) between projection of  $\mathbf{S}_0$  onto this plane and the above defined unit vector  $\mathbf{v}$ , lying in it. Additionally, in the plane  $\mathbf{a}_1 \mathbf{a}_2$  (perpendicular to  $\mathbf{S}_0$ ) let us define the polar angle  $\psi$  as the angle between the plane  $Z\mathbf{S}_0$  and the vector  $\mathbf{a}_1$  (definition of vectors  $\mathbf{a}_1$  and  $\mathbf{a}_2$  is given at the end of Sec. 3). Finally, introducing the quantities  $A_{\pm} \equiv (A_2 \pm A_1)/2$ , and letting  $\Omega$  be the numerator of the polarizability matrix (15),

$$\Omega = (\bar{W}_0 + A_1 + A_2) \hat{\Pi} - \hat{A}_{\perp} + i\bar{\omega} \hat{R}$$

we get for its components

$$\Omega_{zz} = (\bar{W}_0 + A_+ + A_- \cos 2\psi) \sin^2 \theta, \quad (25)$$

$$\Omega_{vv} = (\bar{W}_0 + A_+)(v_{\perp}^2 + v_{\parallel}^2 \cos^2 \theta) + A_- \{ (v_{\parallel}^2 \cos^2 \theta - v_{\perp}^2) \cos 2\psi - 2v_{\parallel} v_{\perp} \sin 2\psi \cos \theta \}, \quad (26)$$

$$\Omega_{zv, vz} = \Omega_x \mp i\omega v_{\perp} \sin \theta, \quad (27)$$

$$\begin{aligned} \Omega_x \equiv & \sin \theta \{ A_{-} v_{\perp} \sin 2\psi - \\ & - v_{\parallel} (\overline{W}_0 + A_{+} + A_{-} \cos 2\psi) \cos \theta \}. \end{aligned} \quad (28)$$

These formulas make it evident that effects of anisotropy are determined by  $A_{-}$ , while  $A_{+}$  merely redefines the magnitude of static internal field,  $W_0$ .

Then, for given in-plane orientation of the wave,  $\mathbf{v}$ , introduce the following characteristic frequencies:

$$\omega_0^2 \equiv (\overline{W}_0 + A_{+})^2 - A_{-}^2, \quad (29)$$

$$\omega_u^2 \equiv \omega_0^2 + 4\pi\Omega_{zz}, \quad (30)$$

$$\begin{aligned} \omega_{1,2}^2 \equiv & \omega_0^2 + 2\pi(\Omega_{zz} + \Omega_{vv}) \mp \\ & \mp 2\pi\sqrt{(\Omega_{zz} + \Omega_{vv})^2 - (2v_{\perp} \sin \theta)^2 \omega_0^2}, \end{aligned} \quad (31)$$

$$\omega_3^2 \equiv \{\omega_1^2 + \omega_2^2 + (4\pi v_{\perp} \sin \theta)^2\} / 2. \quad (32)$$

The frequency  $\omega_u$ , which is independent on the in-plane wave vector  $\mathbf{k}$ , is the uniform precession frequency. In terms of these frequencies,

$$\mu_{zz} = \frac{\omega_u^2 - \omega^2}{\omega_0^2 - \omega^2}, \quad \mu_{zv} - \mu_{vz} = -\frac{8\pi i \omega v_{\perp} \sin \theta}{\omega_0^2 - \omega^2}, \quad (33)$$

$$\lambda_0 = \frac{4\pi i \Omega_x}{\omega^2 - \omega_u^2}, \quad \Lambda = \frac{\sqrt{(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}}{\omega^2 - \omega_u^2}. \quad (34)$$

Besides, below we will need the determinant

$$\Delta \equiv \det M = \frac{2\omega_3^2 - \omega_0^2 - \omega^2}{\omega_0^2 - \omega^2}. \quad (35)$$

It appears that despite  $\Delta$  being a quadratic function of the matrix elements of  $M$  and  $\hat{\chi}$ , it always has only a simple pole.

We omit trivial but tremendous evaluation of the surface potentials appearing in Eq. (19). The result, for the surface, closest to a given antenna, is

$$\tilde{U}_n^S(\omega, \mathbf{k}, \sigma_n D / 2) = \Phi_n(\mathbf{k}) P(\omega, \mathbf{k}), \quad (36)$$

$$\begin{aligned} P(\omega, \mathbf{k}) \equiv & \frac{1 - \Delta - i(\mu_{zv} - \mu_{vz})}{1 + \Delta + 2\mu_{zz} \Lambda \coth(\Lambda |\mathbf{k}| D)} = \\ & = \frac{\omega_0^2 - \omega_3^2 - 4\pi\omega v_{\perp} \sin \theta}{G(\omega, \mathbf{k})}, \end{aligned} \quad (37)$$

where denominator is given by

$$G(\omega, \mathbf{k}) \equiv \omega_3^2 - \omega^2 + (\omega_u^2 - \omega^2) \Lambda \coth(\Lambda |\mathbf{k}| D). \quad (38)$$

For brevity, we do not mark dependencies of the factors  $\Delta$  and  $\Lambda$  on  $\omega$  and  $\mathbf{k}$  as well as dependencies of  $\omega_{1,2,3}$  on

$\mathbf{k}$  (or, to be precise, on direction of the in-plane wave vector  $\mathbf{k}$ ). Combining these formulas and Eq. (19), we obtain the mutual impedance between two antennae, located on the same side of film:

$$Z_{nm} = \frac{i\omega}{2\pi} \int |k| \Phi_n(-k) \Phi_m(k) P(\omega, k) dk. \quad (39)$$

The latter formulas are the main results of the paper and, as far as we know, can be a useful addition to results of Damon and Eshbach [2] and other authors (see Sec. 1). Function  $P(\omega, k)$  is the sought propagator of linear (weak) magnetostatic excitations from one antenna to another. At the same time,  $P(\omega, k)$  contains complete information about the spectrum of MSW. The condition that its denominator turns into zero,  $G(\omega, k) = 0$  (in the absence of dissipation, at  $\gamma = 0$ ) yields a set of dispersion laws for all possible types of MSW. Their study will be the subject of a separate work.

## 5. Mutual impedance of wire antennae

To give a specific example of application of these formulas, consider relatively simple but practically interesting case of straight cylindrical wire antennae, parallel to the film's surface. Besides, let all of them be oriented along the  $Y$  axis and located at  $X$ -positions  $x_n$ , on the same side of the film at distances  $\rho_n$  from its closest face. In this case

$$\Phi_n(k) = \frac{4\pi^2}{ick_x} \exp(-|k_x| \rho_n - ik_x x_n) \delta(k_y)$$

(here  $c$  is the speed of light), and Eq. (39) becomes

$$\begin{aligned} \frac{Z_{nm}[\text{Ohm}]}{w[\text{cm}]f[\text{GHz}]} = & 4\pi i \int_0^{\infty} \exp\{-q(\rho_n + \rho_m)\} \times \\ & \times \frac{(1 - \Delta) \cos(qx) + (\mu_{zv} - \mu_{vz}) \sin(qx)}{1 + \Delta + 2\mu_{zz} \Lambda \coth(\Lambda q D)} \frac{dq}{q}. \end{aligned} \quad (40)$$

Here on the left  $w$  is the film's width (formally infinite) measured in centimeters along antennae (i.e., in  $Y$  direction),  $f$  is the frequency expressed in GHz, while on the right-hand side  $x \equiv x_n - x_m$ , and the integral is taken over  $q \equiv k_x$  at  $k_y \rightarrow 0$ . The latter means that matrix elements of the magnetic susceptibility matrix  $M$  and the functions  $\Lambda$  and  $\Delta$  are calculated at  $\mathbf{v} = \{1, 0, 0\}$ .

Note that the dimensionless circular frequency  $\omega$ , entering all these functions, is connected with the actual frequency  $f$ , expressed in GHz, by the relation (see Sec. 2)

$$\omega = f / f_0, \quad f_0 \equiv (2\pi\tau_0)^{-1} = gM_s.$$

The impedance (40) possesses evident asymmetry with respect to sign of  $x$ , if  $\mu_{zv} - \mu_{vz} \neq 0$ . This is another example of the nonreciprocity, inherent to wave propagation in presence of static magnetization.

One can also see that the impedance is a function of the dimensionless ratios  $D/\rho_n$  and  $x/D$  only. This is the consequence of the scale invariance of the dipole interaction, which is sensitive to the shape of the sample, but not to its size.

In obvious way, one can generalize Eq. (40) for multi-component antennae, consisting of several (mutually parallel) wires, each with alternated signs of current in them. It should be also noted that all the formulas (25)–(35) are useful for analytical calculations, but, when using computer, it is sufficient to numerically calculate the matrix (21) and then (24) and (35) only (for this reason we have expressed the impedance directly in terms of  $\Lambda$  and  $\Delta$ ). In general, of course, first of all, one must find the static distribution of the magnetization vector  $\mathbf{S}_0$ , but this is also not a hard task for a computer.

For the case of two parallel wires, situated on the opposite sides of the film, evaluation of the corresponding boundary potentials yields (in the same units):

$$\frac{Z_{nm}}{wf} = 2\pi i \int_{-\infty}^{\infty} e^{-|q|(\rho_n + \rho_m) + iqx} \times \left\{ e^{-|q|D} - \frac{2\mu_{zz}\Lambda \exp(\lambda_0 |q| D)}{(1 + \Delta) \sinh(\Lambda |q| D) + 2\mu_{zz}\Lambda \cosh(\Lambda |q| D)} \right\} \frac{dq}{|q|} \quad (41)$$

Of course, here in the integrand  $\mathbf{v} = \{\text{sign}(q), 0, 0\}$ , and  $\lambda_0$  is defined in (34).

For the simplest example, let us evaluate the self-impedance,  $Z_{11}$ , of a straight wire antenna (to be precise, the part of full self-impedance due to the presence of the film) in the special case, when the bias magnetic field,  $\mathbf{H}_0$  lies in the film's plane. For concreteness, let it be oriented along  $Y$  axis. Additionally, we assume that the anisotropy field is small compared to  $|\mathbf{H}_0| + 4\pi M_s$  and neglect it. Finally, we orient the antenna parallel to  $\mathbf{H}_0$ , so that the impedance is primarily caused by the so-called surface MSW, discovered by Damon and Eshbach [2]. Under the above formulated conditions, these waves occupy the sector  $|k_y/k_x| < \sqrt{4\pi M_s / |\mathbf{H}_0|}$  in the  $\mathbf{k}$  plane. But sufficiently (infinitely) long antenna excites mainly the waves with  $|k_y/k_x| \rightarrow 0$ , running perpendicularly to the field. Apparently, the latter case is the only case when the dispersion law of MSW (Damon–Eshbach waves with  $\mathbf{k} \perp \mathbf{H}_0$ , or DE-waves) can be written in the evident analytical form [2]:

$$\omega_{DE}(\mathbf{k}) = \sqrt{|\mathbf{H}_0| (|\mathbf{H}_0| + 4\pi) + 4\pi^2 [1 - \exp(-2D|\mathbf{k}|)]}. \quad (42)$$

Here the field and frequency are expressed in the dimensionless units, introduced in Sec. 2.

In this case there is a good analytical approximation for the integral (40), which yields

$$\frac{R_{11}[\text{Ohm}]}{w[\text{cm}]f[\text{GHz}]} \approx \frac{4\pi\omega_t X}{1 - X^2} \left( \frac{1 - X}{1 + X} \right)^{\rho/2D} \left[ \ln \frac{1 + X}{1 - X} \right]^{-1}, \quad (43)$$

$$X \equiv \frac{\omega^2 - \omega_u^2}{4\pi^2}, \quad \omega = \frac{f}{f_0}, \quad \omega_u = \sqrt{|\mathbf{H}_0| (|\mathbf{H}_0| + 4\pi)},$$

$$\omega_t = |\mathbf{H}_0| + 2\pi.$$

In this formula,  $R_{11} = \text{Re } Z_{11}$  is the film-induced contribution to the resistance of the antenna ( $Z = R - 2\pi ifL$ ); magnetic field is dimensionless, expressed in units of  $M_s$ ; frequency belongs to the interval  $f_0\omega_u < f < f_0\omega_t$ ;  $\omega_u$  is dimensionless frequency of uniform precession and, at the same time, the lower bound of Damon–Eshbach spectrum, while  $\omega_t$  is its upper bound (and of the MSW spectrum in general). Outside this frequency range, there are no waves, perpendicular to the antenna, and hence  $R_{11}$  turns into zero (or, to be more precise, becomes comparatively small).

Let us notice that  $X \rightarrow 1$  when  $f \rightarrow f_0\omega_t$ . Therefore, under condition  $\rho/2D < 1$ , the resistance (43) tends to infinity at the upper edge of the DE spectrum. This is because the group velocity of DE-waves,  $v_g$ , goes to zero and the density of states (DE-wave modes) tends to infinity.

Under the opposite condition,  $\rho/2D > 1$ , this effect is canceled by sufficient weakness of excitation of short DE-waves. The presence of the exponent in (43), which depends on dimensionless geometric parameters of the system, is, eventually, the consequence of the scale invariance of dipole interaction.

It is interesting, that in the lower part of DE spectrum the resistance  $R_{11}$  is almost independent on the film's thickness,  $D$ , although seemingly the emf and thus the resistance must be proportional to the amount of magnetic moments (spins) under excitation and thus to  $D$ . The matter is that the energy outflow from the antenna,  $p$ , is proportional to the group velocity,  $v_g$ , of the excited DE-waves:  $p \propto Dv_g |\mathbf{S}_\perp|^2$  (where  $\mathbf{S}_\perp$  represents magnitude of spin precession). On the other hand, we can write  $p \propto \varepsilon_1^2/R_{11}$ , while  $\varepsilon_1 \propto D|\mathbf{S}_\perp|$ . These three relations result in  $R_{11} \propto D/v_g$ . But, as it follows from (42), group velocity of long DE-waves is proportional to the film thickness,  $v_g \propto D$ . This is the reason for the insensitivity of  $R_{11}(\omega \rightarrow \omega_u)$  to variation of  $D$ .

A simple analytical estimate for the inductance,  $L_{11} = -\text{Im } Z_{11} / (2\pi f)$ , can be deduced at  $\rho/D \sim 1$  only, and then

$$\frac{L_{11}[\text{nH}]}{w[\text{cm}]} \approx -(2 + |\mathbf{H}_0|/\pi) \exp(-X) \text{Ei}(X) \quad (44)$$

(Ei is the integral exponent function). Clearly,  $L_{11}$  can be both positive and negative.

## 6. Conclusions

In brief, we have expressed (i) the propagator (37) of magnetostatic waves (MSW) running in infinite ferromagnetic film from one antenna to another and (ii) the linear (small-amplitude) mutual impedance of the antennae, for

the case of arbitrary orientation of uniform static magnetization of the ferromagnetic film with an arbitrary magnetic anisotropy.

The equation  $G(\omega, \mathbf{k}) = 0$ , with  $G$  being the denominator (38) in (37), determines the dispersion laws for various linear eigenmodes of MSW, thus, allowing generalizations of classical results of Ref. 2.

Finally, let us discuss the applicability of formulas, obtained for continuous spectrum of MSW in infinite film, to real finite-size samples, where the MSW spectrum is discrete [16]. When in-plane dimensions of a film decrease, the characteristic frequency separation between neighboring MSW modes increases, but, at the same time, selection of the modes by any simple (for instance, straight-line wire) antenna becomes progressively worse. The formulas, derived for infinite film, can give a good estimates for impedances of antennae, interacting with real films as confirmed by comparison between the analytical estimates and measurements of impedances, induced by millimeter-size ferrite films, as well as by results of our numerical simulations. Moreover, numerical simulations of the torque equation (1) with dipole–dipole interactions between spins show that spatiotemporal patterns of spin precession even in rather small films (with length to thickness ratio  $\sim 30$ – $100$ ) and even in essentially nonlinear regimes possess clear imprints of qualitative and quantitative characteristics inherent to linear MSW modes in infinite system.

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