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Superdiffusive transport in one-dimensional disordered Dirac model

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We have considered discrete one-dimensional Dirac equation in the framework of quantum dimer model with Bernoulli distribution. It has been shown, that superdiffusive transport regime is realized at several combinations of particle mass and amplitude of potential. For the case of non-zero mass it has been shown, that the only one point exists, where Lyapunov exponent is equal to zero. In the framework of dimer model with "random masses" it has been established, that localization is absent.

Keywords: low-dimensional systems, disordered systems, superdiffusive transport, Dirac equation.

1. Introduction

It is well known that in one-dimensional (1D) system any random potential $V(x)$ (x is coordinate) with continuous function of distribution $f(V)$ and absence of space correlations $\langle V(x)V(x')\rangle = \delta(x,x')$ leads to localization of the states [1]. In the case of discrete distribution of $V(x)$ and (or) space correlations between the values of $V(x)$ in different position the question about localization of the states is still open. In paper [2] the so called dimer model was considered. It was considered 1D lattice with host-lattice coordinates $x_n = a_0 n$, where $n = 1, 2, \ldots$ and a_0 is host-lattice spacing. It was assumed, that $f(V)$ is continuous function, but correlations between the values of potential $V(x_n)$ and the values of tunneling integrals τ_n on neighboring sites are present. It was shown numerically in [2] and analytically in [3,4], that for some combination of τ_n and $V(x_n)$ the dispersion D as the function of time t is described by superdiffusion dependency:

$$D(t) = \langle x(t)^2\rangle - \langle x(t)^2\rangle^2 \sim t^{3/2}, \quad t \sim 1. \tag{1}$$

It means, that random potential does not lead to the states localization. In papers [2,5,6] different realizations of dimer model were considered. The simplest case is dimer model with Bernoulli distribution. It is assumed, that $V(x_n) = \pm v_0$, where amplitude of the potential v_0 is constant. The probability $P(x)$ is

$$P(V(x_n) = v_0) = p,$$
$$P(V(x_n) = -v_0) = 1 - p, \quad 0 \le p \le 1. \tag{2}$$

Besides, dimer correlation conditions are imposed:

$$V(x_{2n}) = V(x_{2n+1}). \tag{3}$$

In [2,5,6] Schrödinger model was considered. At the same time it is known, that Dirac points (or extremely small gaps [7–9]) present in spectra of some particles (or quasiparticles) in solids. In this connection the question about localization of the states in the framework of Dirac model with random potential seems to be actual.

In this work we will consider 1D discrete Dirac equation in the framework of dimer model with Bernoulli distribution. In [10] it was shown that for massless case ($m = 0$) in some range of v_0 superdiffusive regime is realized. The question about localization for $m > 0$ has not been investigated yet.

In the present work we have investigated the dependencies of $D(t)$ just for this case. Besides, dimer model, where the masses are random, has been considered too.

2. Hamiltonian

In 1D case Dirac equation has the form

$$i\hbar\frac{\partial}{\partial t}\Phi(x,t) = \mathbf{H}\Phi(x,t) =$$
$$= \left(c\sigma_x p_x + mc^2\sigma_z + V(x)\mathbf{I}_2\right)\Phi(x,t). \tag{4}$$

Here σ_x, σ_z are Pauli matrices, $p_x = -i\hbar\frac{\partial}{\partial x}$, $\mathbf{I}_2$ is identity 2×2 matrix, m is the mass, c is the light velocity, and $\Phi(x,t)$ is spinor wave function:

$$\Phi(x,t) = \begin{cases}\varphi^{(1)}(x,t)\\ \varphi^{(2)}(x,t)\end{cases}.$$

In discrete case and in dimensionless units ($\hbar = c = 1$) one obtains

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$$\begin{cases} -i\left[\varphi^{(2)}(n,t)-\varphi^{(2)}(n-1,t)\right]+(V(n)+m)\varphi^{(1)}(n,t)=i\dfrac{\partial\varphi^{(1)}(n,t)}{\partial t} \\ -i\left[\varphi^{(1)}(n+1,t)-\varphi^{(1)}(n,t)\right]+(V(n)-m)\varphi^{(2)}(n,t)=i\dfrac{\partial\varphi^{(2)}(n,t)}{\partial t}. \end{cases} \quad (5)$$

Here $V(n)=V(x_n)$, $\varphi^{(1,2)}(n,t)=\varphi^{(1,2)}(x_n,t)$, $n = 0, 1, \ldots N-1$, where N is the total number of lattice nodes (system size). It should be noted, that first differences (in square brackets) in form (5) provide self-adjointness of $\mathbf{H}$.

In the framework of "random masses" model we have considered 1D system with random distribution of positive (m^+) and negative (m^-) masses [11–13] so that $m^+ = |m^-|$. Imposing the conditions, similar to (2), (3) one can write, that $m = m(n) = \pm v_0$, $m(2n) = m(2n+1)$ and the equation (4) can be written in this case as

$$\begin{cases} -i\left[\varphi^{(2)}(n,t)-\varphi^{(2)}(n-1,t)\right]+V(n)\varphi^{(1)}(n,t)=i\dfrac{\partial\varphi^{(1)}(n,t)}{\partial t} \\ -i\left[\varphi^{(1)}(n+1,t)-\varphi^{(1)}(n,t)\right]-V(n)\varphi^{(2)}(n,t)=i\dfrac{\partial\varphi^{(2)}(n,t)}{\partial t}. \end{cases} \quad (6)$$

Here $V(n) = \pm v_0$ plays the role of random masses.

For the analysis of Lyapunov exponent and time dependence of dispersion D we will need also stationary versions of (5) and (6). Stationary 1D Dirac equation has the form

$$\mathbf{H}\Phi(x,\lambda)=\left(\sigma_x p_x+m\sigma_z+V(x)\mathbf{I}_2\right)\Phi(x,\lambda)=\lambda\Phi(x,\lambda). \quad (7)$$

Here λ are the eigenvalues and

$$\Phi(x,\lambda)=\begin{cases}\varphi^{(1)}(x,\lambda)\\ \varphi^{(2)}(x,\lambda)\end{cases}$$

are the eigenvectors of $\mathbf{H}$. In discrete case for the model with random potential one obtains

$$\begin{cases} -i\left[\varphi^{(2)}(n,\lambda)-\varphi^{(2)}(n-1,\lambda)\right]+(V(n)+m-\lambda)\varphi^{(1)}(n,\lambda)=0 \\ -i\left[\varphi^{(1)}(n+1,\lambda)-\varphi^{(1)}(n,\lambda)\right]+(V(n)-m-\lambda)\varphi^{(2)}(n,\lambda)=0, \end{cases} \quad (8)$$

and for "random masse" model:

$$\begin{cases} -i\left[\varphi^{(2)}(n,\lambda)-\varphi^{(2)}(n-1,\lambda)\right]+(V(n)-\lambda)\varphi^{(1)}(n,\lambda)=0, \\ -i\left[\varphi^{(1)}(n+1,\lambda)-\varphi^{(1)}(n,\lambda)\right]+(-V(n)-\lambda)\varphi^{(2)}(n,\lambda)=0. \end{cases} \quad (9)$$

3. The Lyapunov exponent

One of important characteristics of localization is Lyapunov exponent $\gamma(E)$, where E is the energy. Since localization length $r_0 \sim 1/\gamma(E)$, to investigate the delocalization conditions one needs to find the solution of the equation $\gamma(E) = 0$. Analysis of $\gamma(E)$ for the dimer model was carried out in [10,14]. It was shown, that for $m > 0$ Lyapunov exponent $\gamma(\tilde{E}) = 0$, if:

(I) $v_0 = \sqrt{2+m^2}$ and $\tilde{E} = 0$,

(II) $v_0 = 1/\sqrt{2}$ and $\tilde{E} = \pm 1/\sqrt{2} \pm \sqrt{2+m^2}$.

According to Furstenberg and Kesten Theorem [15], $\gamma(E)$ can be written as

$$\gamma(E)=\lim_{n\to\infty}\frac{1}{n}\log\left\|\prod_{k=n}^{1}\mathbf{T}_k(E)\right\|. \quad (10)$$

Here $\mathbf{T}_k(E)$ are the transfer matrices. It should be noted that in (10) each next matrix multiplied *on left* by previous ones, hence, the product in this expression is calculated from n to 1.

Transfer matrices for the equation (8) have the form

$$\mathbf{T}_k(E)=\begin{pmatrix}1+m^2-(E-V(k))^2 & i(E-V(k)+m)\\ i(E-V(k)-m) & 1\end{pmatrix}.$$

For these transfer matrices the following relation takes place:

$$\mathbf{T}_k(E)\begin{pmatrix}\varphi^{(1)}(k)\\ \varphi^{(2)}(k-1)\end{pmatrix}=\begin{pmatrix}\varphi^{(1)}(k+1)\\ \varphi^{(2)}(k)\end{pmatrix}.$$

Our numerical calculations of (10) show, that in the case (I) $\gamma(\tilde{E})$ is actually zero (Fig. 1, curve *3*), but in the case (II) $\gamma(\tilde{E}) > 0$ (Fig. 1, curve *2*). To find reason of such discrepancy we have performed additional analysis of γ.

Now, we will reproduce briefly the results of [10]. Behavior of $\gamma(E)$ is determined by module of transfer matrices trace. There are three important cases: 1) $|\mathrm{Tr}(T_k(E))| < 2$ (elliptic matrices), 2) $|\mathrm{Tr}(T_k(E))| = 2$ (parabolic matrices) and 3) $|\mathrm{Tr}(T_k(E))| > 2$ (hyperbolic matrices).

Let us consider firstly case 1 (transfer matrices are elliptic) and denote $\alpha = E - v_0$, $\beta = E + v_0$,

$$\mathbf{T}(E,\alpha)=\begin{pmatrix}1+m^2-\alpha^2 & i(\alpha+m)\\ i(\alpha-m) & 1\end{pmatrix},$$

$$\mathbf{T}(E,\beta)=\begin{pmatrix}1+m^2-\beta^2 & i(\beta+m)\\ i(\beta-m) & 1\end{pmatrix}.$$

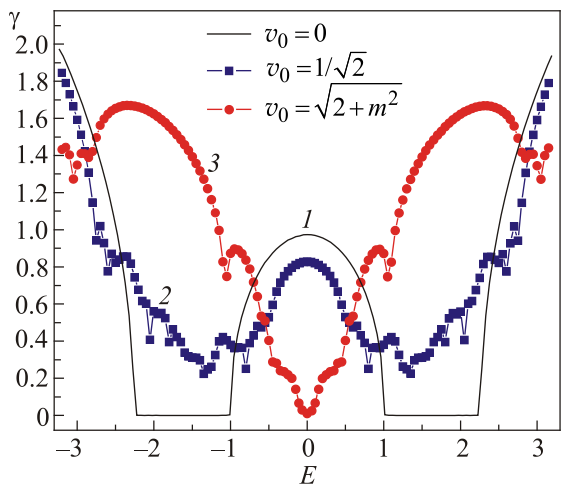

Fig. 1. The dependence of Lyapunov exponent (10) on E for different values of v_0. The mass $m = 1$.

From the condition $\left|\mathrm{Tr}(\mathbf{T}(E,\alpha))\right| < 2$ (analogously for $\mathbf{T}(E,\beta)$) it follows that $\mathbf{T}(E,\alpha)$ and $\mathbf{T}(E,\beta)$ are both elliptic, if $|\alpha|, |\beta| \in (m, \sqrt{4+m^2})$ and, therefore, in this range $\gamma(E) > 0$. The exception is one point only. Due to (3) $V(n)$ acquires the same values on the nodes $2n$ and $2n+1$. Hence, the expression (10) includes the squares of $\mathbf{T}(E,\alpha)$ and $\mathbf{T}(E,\beta)$. Then

$$\left|\mathrm{Tr}(\mathbf{T}^2(E,\alpha))\right| = |2+\alpha^4+4m^2+m^4-2\alpha^2(2+m^2)|.$$

Analogously,

$$\left|\mathrm{Tr}(\mathbf{T}^2(E,\beta))\right| = |2+\beta^4+4m^2+m^4-2\beta^2(2+m^2)|.$$

From the conditions $\left|\mathrm{Tr}(\mathbf{T}^2(E,\alpha))\right| = \left|\mathrm{Tr}(\mathbf{T}^2(E,\beta))\right| = 2$ (the squares of both transfer matrices are parabolic ones) it follows that $\alpha,\beta = \pm\sqrt{2+m^2}$. It means that $\gamma(\tilde{E}) = 0$, if $v_0 = \sqrt{2+m^2}$ and $\tilde{E} = 0$.

For parabolic transfer matrices (case 2) $\gamma(E) > 0$. Now, let us consider case of hyperbolic transfer matrices (case 3). This situation realizes, if $|\alpha| < m$ or $|\alpha| > \sqrt{4+m^2}$. Let us consider the orbit of $T(E,\alpha)$ eigenvector:

$$e^{\varepsilon} = \begin{pmatrix} \alpha^2-m^2+\varepsilon\sqrt{(\alpha^2-m^2)(\alpha^2-m^2-4)} \\ 2(\alpha-m) \end{pmatrix}, \quad \varepsilon = \pm 1.$$

If $\mathbf{T}(E,\beta)$ is hyperbolic or parabolic, then $\gamma(E) > 0$. In the case of elliptic $\mathbf{T}(E,\beta)$ Lyapunov exponent $\gamma(E) > 0$, if $\mathbf{T}(E,\beta)e^{\varepsilon} \neq e^{-\varepsilon}$. Now, let us consider the case of $\mathbf{T}(E,\beta)e^{\varepsilon} = e^{-\varepsilon}$. Then,

$$(1+m^2-\beta^2)(\alpha^2-m^2+\varepsilon u)-4(m^2-\beta\alpha)+(\alpha^2-m^2-\varepsilon u),$$

where $u = \sqrt{(\alpha^2-m^2)(\alpha^2-m^2-4)} \neq 0$. Hence, $\beta^2 = 2+m^2$, $\alpha = \beta \pm \sqrt{2}$ and, thus, $v_0 = 1/\sqrt{2}$ and $\tilde{E} = \pm\sqrt{2+m^2} - 1/\sqrt{2}$. Analogous consideration for $\mathbf{T}(E,\beta)$ eigenvector gives $\tilde{E} = \pm\sqrt{2+m^2} + 1/\sqrt{2}$. It means, that in Dirac model with random potential (2) but without dimer correlations (3) $\gamma(\tilde{E}) = 0$, if $v_0 = 1/\sqrt{2}$ and $\tilde{E} = \pm 1/\sqrt{2} \pm \sqrt{2+m^2}$ (see Fig. 2).

Let us take into account conditions (3). It is easy to see, that for $\beta^2 = 2+m^2$, $\left|\mathrm{Tr}\left(\mathbf{T}(E,\beta)\right)\right| = 0$ (elliptic case), but $\left|\mathrm{Tr}(\mathbf{T}^2(E,\beta))\right| = 2$, so $\mathbf{T}^2(E,\beta)$ is parabolic. According to [10], it means, that $\gamma(\tilde{E}) > 0$. Hence, if correlation conditions (3) are imposed, then the zeros of $\gamma(\tilde{E})$ disappear (Fig. 1, curve *2*). This situation is similar to discrete 1D Schrödinger model with random Kronig–Penney potential. The zeros of Lyapunov exponent disappear if one adds into the model any periodic potential [16].

Finalizing the consideration of the model with $m > 0$ it should be noted the following:

— Our numerical calculations indicate, that without dimer correlations the states are always localized, despite the fact that in case (II) $\gamma(\tilde{E}) = 0$ (if $v_0 = 1/\sqrt{2}$ and $\tilde{E} = \pm\sqrt{2+m^2} \pm 1/\sqrt{2}$).

— In the framework of dimer model with $m > 0$ only one point exists, where Lyapunov exponent $\gamma(\tilde{E}) = 0$ and states can be delocalized: $v_0 = \sqrt{2+m^2}$ and $\tilde{E} = 0$.

The dependencies of $\gamma(E)$ for the model with $m = 0$ are presented on Fig. 3. In agreement with [10,14] $\gamma(|E| = v_0) = 0$, if $v_0 \leq 1$ (curves *1* and *2*). Besides, $\gamma(E = 0) = 0$, if $v_0 = \sqrt{2}$ (curve *3*).

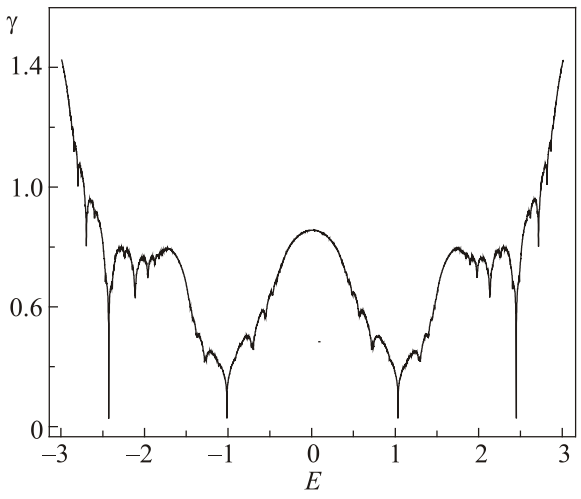

Fig. 2. The dependence of Lyapunov exponent (10) on E for $m = 1$ and $v_0 = 1/\sqrt{2}$ without dimer correlation (3).

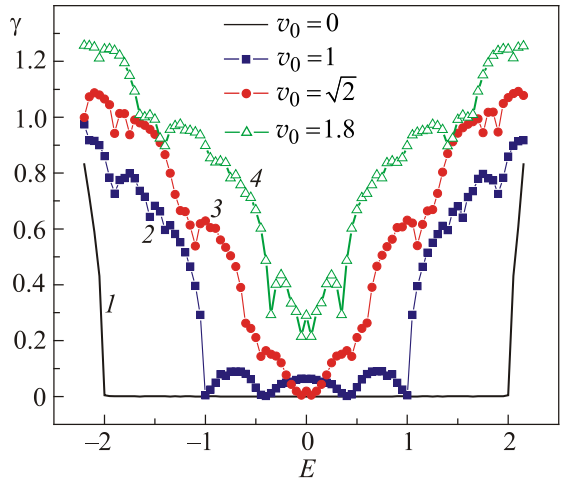

Fig. 3. The dependence of Lyapunov exponent (10) on E for different values of v_0. The mass $m = 0$.

Analyzing $\gamma(E)$ we have obtained the dependencies of Lyapunov exponent minimal value as the function of v_0:

$$\gamma_{\min} = \min_E \gamma(E).$$

These dependencies are also obtained numerically. In upper parts of Fig. 4, and Fig. 5 the behavior of $\gamma_{\min}$ are presented. Figure 4 correponds to the case of $m = 1$. There are two regions, where $\gamma_{\min} << 1$: in the vicinity of $v_0 = 0$ (it is evident) and in the vicinity of $v_0 = \sqrt{2+m^2}$. There are no other regions, where $\gamma(E) \to 0$. Figure 5 corresponds to $m = 0$ case. As seen $\gamma_{\min} << 1$ for $v_0 \lesssim 1.75$.

Let us consider now "random masses" model. The analysis of $\gamma(E)$ is similar to one carried out above. By analogy we will introduce transfer matrices:

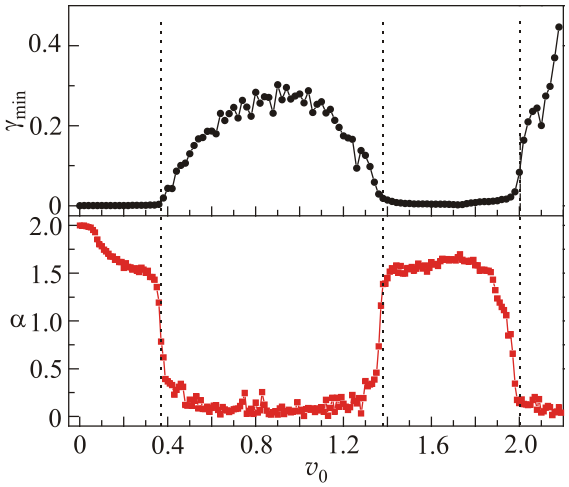

Fig. 4. Upper part: the dependence of Lyapunov exponent minimal value $\gamma_{\min}$ on v_0. Lower part: the dependence of parameter α (14) on v_0. The mass $m = 1$.

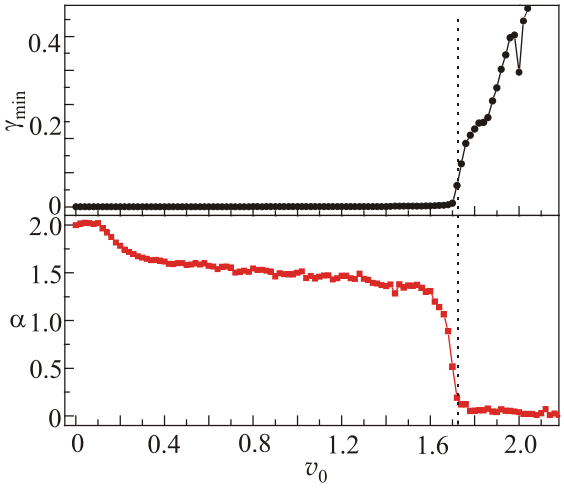

Fig. 5. Upper part: the dependence of Lyapunov exponent minimal value $\gamma_{\min}$ on v_0. Lower part: the dependence of parameter α (14) on v_0. The mass $m = 0$.

$$\mathbf{T}(E,\alpha) = \begin{pmatrix} 1+v_0^2-E^2 & i(E+v_0) \\ i(E-v_0) & 1 \end{pmatrix},$$

$$\mathbf{T}(E,\beta) = \begin{pmatrix} 1+v_0^2-E^2 & i(E-v_0) \\ i(E+v_0) & 1 \end{pmatrix}.$$

Again, the most important is the case, when both matrices are elliptic: $\left|\mathrm{Tr}(\mathbf{T}(\tilde{E},\alpha)0\right| < 2$, $\left|\mathrm{Tr}(\mathbf{T}(\tilde{E},\beta))\right| < 2$. These inequalities are fulfilled, if $|E| \in (v_0, \sqrt{4+v_0^2})$. Imposing correlation conditions (3) one obtains, that $\gamma(\tilde{E}) = 0$, if $\left|\mathrm{Tr}(\mathbf{T}^2(\tilde{E},\alpha))\right| = \left|\mathrm{Tr}(\mathbf{T}^2(\tilde{E},\beta))\right| = 2$. Corresponding equation has the form

$$\left|2+\tilde{E}^4+4v_0^2+v_0^4-2\tilde{E}^2(2+v_0^2)\right| = 2.$$

Solution of the equation gives: $\tilde{E} = \pm v_0$, $\tilde{E} = \pm\sqrt{4+v_0^2}$ and $\tilde{E} = \pm\sqrt{2+v_0^2}$. First two roots are out of $(v_0, \sqrt{4+v_0^2})$ region. Hence, $\gamma(\tilde{E}) = 0$, if $\tilde{E} = \pm\sqrt{2+v_0^2}$. It means, that for "random masses" model localization is absent, because for any v_0 value there exists $\tilde{E} = \pm\sqrt{2+v_0^2}$, so, that $\gamma(\tilde{E}) = 0$. The dependence of $\gamma(E)$ for "random masses" model is presented on Fig. 6. As seen from upper part of Fig. 7 $\gamma_{\min} = 0$ with the accuracy of numerical calculations.

4. Calculations of $D(t)$

According to (1), one needs to calculate the 1st and 2nd moments of x. The expression for kth moment has the form:

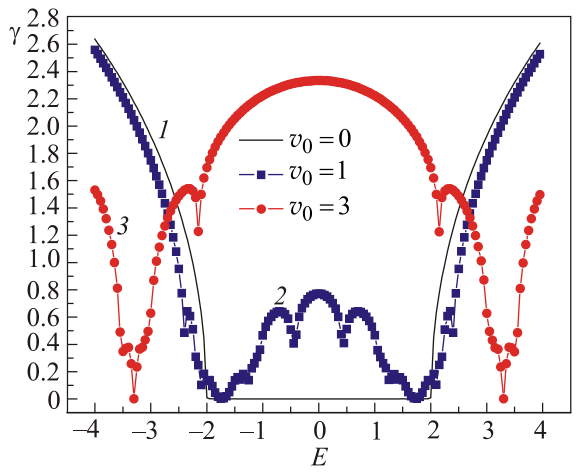

Fig. 6. "Random masses" model. The dependence of Lyapunov exponent (10) on E for different values of v_0.

$$< x_{\pm}^{k}(t) > \equiv x_{\pm}(k,t) = \sum_{n=0}^{N-1} n^k \,|\, \Phi_{\pm}(n,t) \,|^2 =$$

$$= \sum_{n=0}^{N-1} n^k \left(|\, \varphi_{\pm}^{(1)}(n,t) \,|^2 + |\, \varphi_{\pm}^{(2)}(n,t) \,|^2 \right).$$

Index "+" denotes the solution of Cauchy problem for (5) or (6) with initial values $\varphi^{(1)}(n,0) = \delta(n,n_0)$, $\varphi^{(2)}(n,0) = 0$, i.e. the particle with spin "up" at initial moment $t = 0$ is localized on node n_0, and "–" corresponds to the case, when $\varphi^{(1)}(n,0) = 0$, $\varphi^{(2)}(n,0) = \delta(n,n_0)$, i.e. the particle with spin "down" is localized on node n_0 at $t = 0$. Then

$$D(t) = < \sum_{\sigma=\pm} \left[x_{\sigma}(2,t) - (x_{\sigma}(1,t))^2 \right] >_{\{\xi\}} .$$

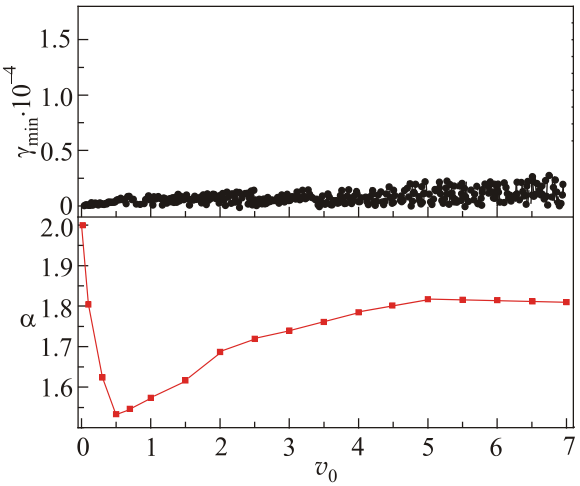

Fig. 7. "Random masses" model. Upper part: the dependence of Lyapunov exponent minimal value $\gamma_{\min}$ on v_0. Lower part: the dependence of parameter α (14) on v_0.

Here $< \ldots >_{\{\xi\}}$ means averaging over the realizations of potential V.

The dependencies of $D(t)$ can be also obtained using stationary versions of Dirac equations (that is, using (8) and (9)). In this case the expression of $x_{\pm}(k,t)$ has the form

$$x_{\pm}(k,t) = (\Phi_{\pm}^{0}, e^{-i\mathbf{H}t} x^k e^{i\mathbf{H}t} \Phi_{\pm}^{0}). \qquad (11)$$

Here $\Phi_{\pm}^{0}$ are the wavefunctions with spin "up" and "down", localized on node n_0:

$$\Phi_{+}^{0} = \begin{cases} \delta(n,n_0) \\ 0 \end{cases}, \quad \Phi_{-}^{0} = \begin{cases} 0 \\ \delta(n,n_0). \end{cases}$$

Then expression (11) can be written as

$$x_{\pm}(k,t) = \sum_{n=0}^{N-1} n^k \left(\sum_{\mu=0}^{1} \left| \sum_{\lambda} \Phi^{*}(n_0,\lambda) \beta^{\mu} \Phi_{\pm}(n,\lambda) e^{i\lambda t} \right|^2 \right). \qquad (12)$$

Here $\beta^0 = I_2$, $\beta^1 = \sigma_x$, and

$$\Phi_{+}(n_0,\lambda) = \begin{cases} \varphi^{(1)}(n_0,\lambda) \\ 0 \end{cases}, \quad \Phi_{-}(n_0,\lambda) = \begin{cases} 0 \\ \varphi^{(2)}(n_0,\lambda). \end{cases}$$

It should be noted, that expression $\Phi^{*}(n_0,\lambda)\beta^{\mu}\Phi_{\pm}(n,\lambda)$ ($\mu = 0,1$) is nothing else that two first components of charge-current density matrix $j^{\mu}(\lambda,x,y)$. Hence, one can write $x_{\pm}(k,t)$ as

$$x_{\pm}(k,t) = \sum_{n=0}^{N-1} n^k \sum_{\mu=0}^{1} \left| \mathbf{j}^{\mu}(n_0,n,t) \right|^2,$$

where

$$j^{\mu}(n_0,n,t) = \sum_{\lambda} \mathbf{j}^{\mu}(n_0,n,\lambda) e^{i\lambda t}.$$

The systems of differential equations (5) and (6) have been solved using Runge–Kutta method of the 8th order. To improve accuracy the norm of $\Phi(n,t)$ has been restored on each step of calculations. To avoid boundary effects the probability of a particle discovering on the last node has been checked:

$$|\, \Phi(N-1,t) \,| \lesssim \varepsilon = 10^{-10}. \qquad (13)$$

Besides, the calculations have been carried out for $n_0 = 0$ and $n_0 = N/2$, i.e., in the first case it has been assumed, that at $t = 0$ the particle is localized on the system edge, and in the second case it has been assumed, that at $t = 0$ the particle is localized in the middle of the system. Both results are in good agreement.

Additionally, to check the obtained results the dependencies, $D(t)$ has been calculated also, using (12). For this it is convenient to introduce scalar wavefunction $\tilde{\varphi}(n)$ so, that:

$$\tilde{\varphi}(2n) = \varphi^{(1)}(n), \quad \tilde{\varphi}(2n+1) = \varphi^{(2)}(n), \quad (n = 0,1,\ldots,N-1).$$

Note, that in (8) and (9) the matrices **H** are complex tridiagonal. For numerical calculations of $x_{\pm}(\alpha,t)$ it is convenient to turn to the presentation, where **H** is real symmetric and, therefore, $\tilde{\varphi}$ is real too:

$$\mathbf{H} = \mathbf{U}^{-1}\mathbf{H}\mathbf{U},$$

where $\mathbf{U}_{n,m} = i\delta(n,m)(-1)^n$. Then even rows of $\tilde{\mathbf{H}}$ for (8) have the form $\{-1, m+V([n/2]), -1\}$, and odd rows are $\{1, -m+V([n/2]), 1\}$, where $n = 0, 1, \ldots, 2N-1$.

For (9) matrix $\tilde{\mathbf{H}}$ has the form: $\{-1, V([n/2]), -1\}$ and $\{1, -V([n/2]), 1\}$. Symbol […] denotes integer part of a number.

All the numerical calculations have been carried out for system sizes $N = 5000$ and $N = 10000$. If $n_0 = 0$, then the condition (13) is fulfilled at least for $t \lesssim N/2$, and if $n_0 = N/2$, then this condition is true at least for $t \lesssim N/4$. The probability p in (2) is equal to $1/2$.

5. Results and discussion

The examples of $D(t)$ dependencies, obtained numerically are presented on Fig. 8. Curve *1* corresponds to the case, where localization of the states is absent, and curve *2* corresponds to the case of localized states. For clarity both these curves are presented on the same figure, but at different scales.

In the region of $t >> 1$ these dependencies have been approximated by the function:

$$D(t) = c_0 t^{\alpha}. \qquad (14)$$

There are several different regimes of $D(t)$ behavior: If $\alpha = 0$, then the states are localized. If $0 < \alpha < 1$, then it is subdiffisive regime. $\alpha = 1$ corresponds to diffusive regime. If $1 < \alpha < 2$, then it is superdiffisive regime and if $\alpha = 2$, then it is ballistic regime, corresponding to free motion.

The dependence of parameter α on amplitude v_0 for $m = 1$ is presented in lower part of Fig. 4. As seen, there are two regions of delocalization: $v_0 \lesssim 0.4$, where $1.7 \lesssim \alpha < 2$ and $1.4 \lesssim v_0 \lesssim 2$, where $\alpha \approx 1.7$.

Similar dependence for $m = 0$ is presented on Fig. 5. Parameter $\alpha \gtrsim 3/2$ in the range $0 \le v_0 \lesssim 1.7$. For $v_0 > 1.7$ localization of the states takes place.

As seen from upper curves on Fig. 4, Fig. 5 the dependencies α on v_0 correlate well with the behavior of $\gamma_{\min}$. Indeed, the delocalization regions correspond to the intervals, where $\gamma_{\min} \to 0$.

Of particular interest is the "random masses" model. As it was mentioned above, in this case localization is absent

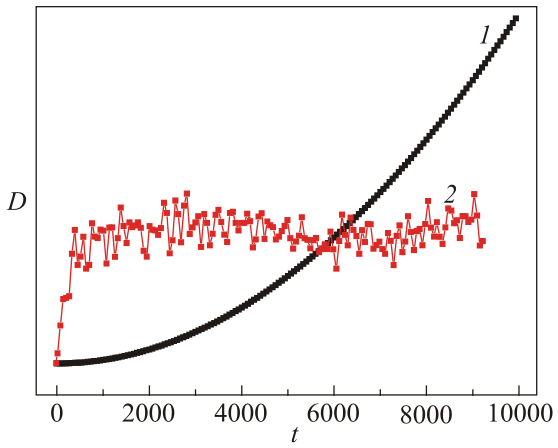

Fig. 8. The examples of $D(t)$ dependencies. Curve *1* corresponds to the case, when localization of the states is absent, and curve *2* corresponds to case of localized states. For clarity both these curves are presented on the same figure, but at different scales.

for arbitrary v_0 value and power index for this model corresponds to superdiffusive regime in whole v_0 range (see lower part of Fig. 7). It is important to note, that parameter c_0 (14) for the model with random potential depends of v_0 slightly. At the same time, for "random masses" model c_0 decays exponentially with v_0 (see. Fig. 9). Physically c_0 means the square of inverse mass. Hence, one can say, that for "random masses" model with increase of v_0 effective mass increases exponentially, slowing the particles motion.

The maximal values of power index α for random potential model ($\alpha \approx 1.7$) and for "random masses" model ($\alpha \approx 1.8$) are close to ballistic regime ($\alpha = 2$). It means, that the random potential (or random masses) affects weakly on transport characteristics of the system under consideration. There are strong reasons to suggest that

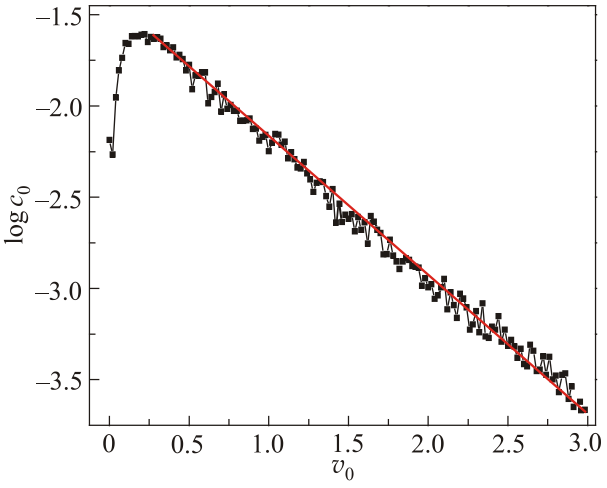

Fig. 9. The dependence of parameter c_0 (see (14)) on v_0 in "random masses" model, presented in logarithmic scale. Solid line corresponds to monoexponential decay.

such unusual transport behavior can manifests itself in other dynamic properties of the model (for example, in the conductive properties). We plan to investigate this problem in near future.

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Супердифузійний транспорт в одновимірній невпорядкованій моделі Дірака

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Розглянуто одновимірне рівняння Дірака в рамках квантової димерної моделі з розподілом Бернуллі. Показано, що при певних співвідношеннях маси частинки і амплітуди потенціалу реалізується супердифузійний режим руху. У разі ненульовий маси частки показано, що існує єдина точка, в якій показник Ляпунова обертається в нуль. У рамках димерної моделі «випадкових мас» показано, що локалізація станів відсутня.

Ключові слова: низьковимірні системи, невпорядковані системи, супердифузійний транспорт, рівняння Дірака.

Супердиффузионный транспорт в одномерной неупорядоченной модели Дирака.

В. Славин, Ю. Савин

Рассмотрено одномерное уравнение Дирака в рамках квантовой димерной модели с распределением Бернулли. Показано, что при определенных соотношениях массы частицы и амплитуды потенциала реализуется супердиффузионный режим движения. В случае ненулевой массы частицы показано, что существует единственная точка, в которой показатель Ляпунова обращается в нуль. В рамках димерной модели «случайных масс» показано, что локализация состояний отсутствует.

Ключевые слова: низкоразмерные системы, неупорядоченные системы, супердиффузионный транспорт, уравнение Дирака.