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Collective-mode dispersion of atomic Fermi gases in a honeycomb optical lattice: speed of sound of the attractive Kane–Mele–Hubbard model at half filling

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We examine the superfluid states that emerge in the Kane–Mele model as a result of the on-site short-range attractive interaction U. The collective-mode dispersion is defined by the solutions of the Bethe–Salpeter (BS) equation in the generalized random phase approximation. The slope of the low-energy (Goldstone) mode and the corresponding sound velocity at half filling, calculated within the BS formalism, has been compared with the corresponding results obtained previously by the T-matrix approximation. The difference between the two approaches is that the T-matrix approximation takes into account only the ladder diagrams, and neglects the bubble ones. For this reason, the sound velocity in the direction toward point M, calculated within the T-matrix approximation, is about 4% less than the result obtained by employing the BS equation.

Keywords: collective modes, sound velocity, superfluid state.

1. Introduction

We consider the so-called Kane–Mele–Hubbard (KMH) model, which can be used as a tight binding model of fermion atoms hopping in a honeycomb optical lattice (see Fig. 1). As in the case of charge carriers in graphene, at low energies the system admits an effective description in terms of massless Dirac fermions. Originally, Kane and Mele (KM) considered the Hamiltonian for the electrons in a graphene, which consists of two copies of the Haldane's model [1], one for spin-up electrons and one for spin-down electrons [2,3]. With Fourier transformation into momentum space, the KM's Hamiltonian can be written as a 4×4 matrix in the four-component spinor basis $(\psi_{A,k,\uparrow},\psi_{B,k,\uparrow},\psi_{A,k,\downarrow},\psi_{B,k,\downarrow})^T$, the Hamiltonian in the momentum space is given by the following 4×4 matrix [4]:

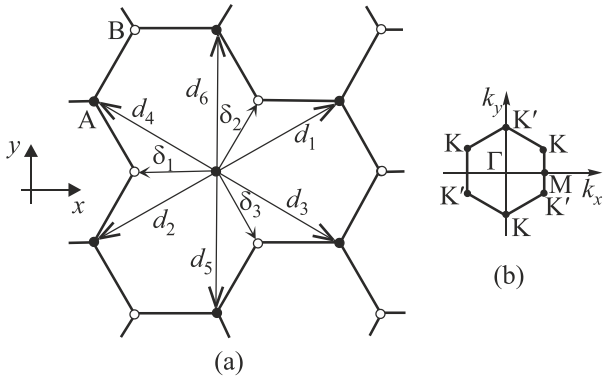

Fig. 1. Honeycomb lattice (a) and its Brillouin zone (b). $\delta_1 = (1/2, \sqrt{3}/2)$, $\delta_2 = (1/2, -\sqrt{3}/2)$, $\delta_3 = (-1, 0)$ are the nearest neighbor sites vectors, while $d_{1,2} = \pm(3/2, \sqrt{3}/2)$, $d_{3,4} = \pm(3/2, -\sqrt{3}/2)$, and $d_{5,6} = \pm(0, \sqrt{3})$ are the next nearest neighbor sites vectors. The lattice constant $a = 1$.

$$\widehat{H}_{KM}(\mathbf{k}) = \begin{pmatrix} H_{\uparrow}(\mathbf{k}) & 0 \\ 0 & H_{\downarrow}(\mathbf{k}) \end{pmatrix},$$

$$H_{\uparrow}(k) = \begin{pmatrix} Z(\mathbf{k}) - \Lambda & h(\mathbf{k}) \\ h^*(\mathbf{k}) & -Z(\mathbf{k}) - \Lambda \end{pmatrix}. \qquad (1)$$

Here $h(\mathbf{k}) = -t\left[x(\mathbf{k}) - iy(\mathbf{k})\right]$, $Z(\mathbf{k}) = -2\lambda z(\mathbf{k})$, t is the nearest-neighbor hopping amplitude, and λ is the strength of the intrinsic spin-orbit (ISO) interaction, and in the presence of time-reversal symmetry (TRS), $H_{\uparrow}(\mathbf{k}) = H_{\downarrow}^*(-\mathbf{k})$. The functions $x(\mathbf{k})$, $y(\mathbf{k})$, and $z(\mathbf{k})$ are defined as follows:

$$x(\mathbf{k}) = \cos(k_x) + 2\cos\left(\frac{k_x}{2}\right)\cos\left(\frac{\sqrt{3}k_y}{2}\right),$$

$$y(\mathbf{k}) = \sin(k_x) - 2\sin\left(\frac{k_x}{2}\right)\cos\left(\frac{\sqrt{3}k_y}{2}\right),$$

$$z(\mathbf{k}) = \sin(\sqrt{3}k_y) - 2\cos\left(\frac{3k_x}{2}\right)\sin\left(\frac{\sqrt{3}k_y}{2}\right).$$

Λ describes the possible energy offset between sites of A and B sublattices. Since the different site energies on sublattices A and B are controlled by the phases of the laser beams, we shall assume the case of honeycomb lattice with the same on-site energies ($\Lambda = 0$).

In the absence of ISO interaction ($\lambda = 0$), and if we take into account also the next nearest-neighbor hopping amplitude t', the eigenvalues of the Hamiltonian (1) are $\xi_{\pm}(\mathbf{k}) = -t' f(\mathbf{k}) \pm t\sqrt{3 + f(\mathbf{k})}$, where

$$f(\mathbf{k}) = 2\cos(\sqrt{3}k_y) + 4\cos\left(\frac{3k_x}{2}\right)\cos\left(\frac{\sqrt{3}k_y}{2}\right).$$

The band structure is as that of graphene where the Dirac cones at $K = \left(\frac{2\pi}{3a}, \frac{2\pi}{3\sqrt{3}a}\right)$ and $K' = \left(\frac{2\pi}{3a}, -\frac{2\pi}{3\sqrt{3}a}\right)$ are spin degenerate.

The KM model of fermions on the honeycomb lattice is a prototype for several phenomena in condensed matter physics, such as the quantum spin Hall effect, the Klein tunneling, the universal optical absorption, and a large amount of theoretical works predicted the existence of a number of exotic phases in the case of honeycomb lattice geometry. It is worth mentioning that most of the new phases in condensed matter physics are driven by the Coulomb interactions, while the KM model predicted that the ISO coupling alone would turn graphene into a completely new electronic state known as topological insulator. In other words, the ISO coupling makes graphene topologically different from graphene without ISO coupling.

In this paper, we examine single-particle and collective excitations in a honeycomb optical lattice by employing the *attractive* KMH model. In the presence of an attractive on-site interaction between the fermions, no matter how weak it is, the fermion atoms form bound (Cooper) pairs. As a result, the system becomes unstable against the formation of a s-wave spin-singlet superfluid ground state.

Turning our attention to the collective-mode dispersion of the attractive KMH Hamiltonian, to the best of our knowledge, there exists only one paper where the collective-mode spectrum at zero temperature has been examined by employing the T-matrix approximation [5]. According to this approximation, the excitation spectrum of collective modes $\omega(\mathbf{Q})$ was derived by calculating the roots of the following secular 2×2 determinant:

$$\det \begin{vmatrix} U^{-1} - \Pi_{AA}(\omega, \mathbf{Q}) & -\Pi^{*}_{AB}(\omega, \mathbf{Q}) \\ -\Pi_{AB}(\omega, \mathbf{Q}) & U^{-1} - \Pi_{BB}(\omega, \mathbf{Q}) \end{vmatrix} = 0, \quad (2)$$

where

$$\Pi_{\lambda\mu} = \int \frac{d\Omega}{2\pi} \frac{d^2\mathbf{k}}{(2\pi)^2} G^{KM}_{\lambda\downarrow,\mu\downarrow}(\mathbf{k}+\mathbf{Q}, \Omega+\omega) G^{KM}_{\lambda\uparrow,\mu\uparrow}(\mathbf{k}, \Omega).$$

Here λ, μ are the sublattice indexes, and $G^{KM}_{\lambda\sigma,\mu\sigma}(\mathbf{k}, \omega)$ is the Fourier transforms of the KM single-particle Green's function $G^{KM}_{\lambda i\sigma,\mu j\sigma}(t-t') = -\left\langle \hat{T}_t\{\psi_{\lambda i\sigma}(t)\psi^{\dagger}_{\mu j\sigma'}(t')\}\right\rangle$. It is worth mentioning that the T-matrix approximation, also known as the ladder approximation to the Bethe–Salpeter (BS) equation [6] consists of the sum of ladder diagrams in the perturbation expansion in terms of U where the corresponding single-particle KM Green's functions are independent on the Hubbard interaction. The question that naturally arises here is about the contributions due to the bubble diagrams, neglected by the T-matrix approximation.

To answer the above question, we shall employ the Hubbard–Stratonovich transformation (HST). If no approximations were made in evaluating the corresponding functional integrals, it would not matter which of the possible HST is chosen. When approximations are taken, the final result depends on a particular form chosen. A possible approximation is to introduce the energy gap as an order parameter field, which allows us to integrate out the fermion fields and to arrive at an effective action. Next steps are to consider the state, which corresponds to the saddle point of the effective action, and to write the effective action as a series in powers of the fluctuations and their derivatives. The exact result can be obtained by explicitly calculating the terms up to second order in the fluctuations and their derivatives. This approximation, known as the Gaussian approximation, has been employed in the case of square geometry [7], but to the best of our knowledge, it has never been used in the case of honeycomb lattice.

Instead of introducing an order parameter field, we shall transform the quartic terms to quadratic forms by introducing a boson field which mediates the interaction of fermions. This assumption is similar to the situation in quantum electrodynamics, where the photons mediate the interaction of electric charges, and it allows us to derive the Schwinger–Dyson (SD) equation for the poles of the single-particle Green's function, as well as the BS equation in the generalized random phase approximation (GRPA) for the poles of the two-particle Green's function. In the GRPA, the single particle excitations are replaced with those obtained by diagonalizing the Hartree–Fock (HF) mean-field Hamiltonian, while the collective modes are obtained by solving the BS equation in which the single particle Green's functions are calculated in HF mean-field approximation, and the BS kernel is obtained by summing ladder and bubble diagrams

2. Mean-field approximation

When the attractive Hubbard interaction $H_U = -U\sum_{\mathbf{i}} \hat{n}_{\mathbf{i},\uparrow}\hat{n}_{\mathbf{i},\downarrow}$ is taken into account, the basis of the four-component wave function

$$\Psi^{\dagger} = (\psi^{\dagger}_{A,\mathbf{k},\uparrow}, \psi^{\dagger}_{A,\mathbf{k},\downarrow}, \psi^{\dagger}_{B,\mathbf{k},\uparrow}, \psi^{\dagger}_{B,\mathbf{k},\downarrow})$$

becomes a basis of the eight-component wave function $\Psi^\dagger = (\psi^\dagger_{A,\mathbf{k},\uparrow}, \psi^\dagger_{A,\mathbf{k},\downarrow}, \psi^\dagger_{B,\mathbf{k},\uparrow}, \psi^\dagger_{B,\mathbf{k},\downarrow}, \psi_{A,-\mathbf{k},\uparrow}, \psi_{A,-\mathbf{k},\downarrow}$ $\psi_{B,-\mathbf{k},\uparrow}, \psi_{B,-\mathbf{k},\downarrow})$. We further assume that the BCS mean-field order parameters are real constants, i.e., $\Delta_{A(B)}(\mathbf{k}) = U\left\langle \psi_{A(B),-\mathbf{k},\downarrow}\psi_{A(B),\mathbf{k},\uparrow}\right\rangle = \Delta$. The order parameter Δ and the corresponding chemical potentials $\mu_A = \mu_B = \mu$ are defined by the solutions of the number and the gap equations. Thus, the mean-field Hamiltonian in the momentum space on the basis of the eight-component wave function is represented by the following 8×8 matrix:

$$\widehat{H}_{MF}(\mathbf{k}) = \begin{pmatrix} \widehat{H}^{AA}(\mathbf{k}) & \widehat{H}^{AB}(\mathbf{k}) \\ \widehat{H}^{AB\dagger}(\mathbf{k}) & \widehat{H}^{BB}(\mathbf{k}) \end{pmatrix}, \tag{3}$$

where the corresponding 4×4 blocks are defined by the following block-matrices:

$$\widehat{H}^{AB}(\mathbf{k}) = \begin{pmatrix} h(\mathbf{k}) & 0 & 0 & 0 \\ 0 & h(\mathbf{k}) & 0 & 0 \\ 0 & 0 & -h(\mathbf{k}) & 0 \\ 0 & 0 & 0 & -h(\mathbf{k}) \end{pmatrix},$$

$$\widehat{H}^{AA}(\mathbf{k}) = \begin{pmatrix} -t'f(\mathbf{k}) + Z(\mathbf{k}) - \mu & 0, & 0 & \Delta \\ 0 & -t'f(\mathbf{k}) - Z(\mathbf{k}) - \mu & -\Delta & 0 \\ 0 & -\Delta & t'f(\mathbf{k}) + Z(\mathbf{k}) + \mu & 0 \\ \Delta & 0 & 0 & t'f(\mathbf{k}) - Z(\mathbf{k}) + \mu \end{pmatrix},$$

$$\widehat{H}^{BB}(\mathbf{k}) = \begin{pmatrix} -t'f(\mathbf{k}) - Z(\mathbf{k}) - \mu & 0, & 0 & \Delta \\ 0 & -t'f(\mathbf{k}) + Z(\mathbf{k}) - \mu & -\Delta & 0 \\ 0 & -\Delta & t'f(\mathbf{k}) - Z(\mathbf{k}) + \mu & 0 \\ \Delta & 0 & 0 & t'f(\mathbf{k}) + Z(\mathbf{k}) + \mu \end{pmatrix}.$$

The single-particle excitations in the mean-field approximation manifest themselves as poles of the Matsubara single-particle Green's function, defined as $\widehat{G}(\mathbf{k}, i\omega_m) = (i\omega_m \hat{1} - \widehat{H}_{MF}(\mathbf{k}))^{-1}$. Here, the Matsubara fermion energies are $\omega_m = (2\pi/\beta)(m + 1/2)$, $m = 0, 1, 2...$, $\beta = 1/(k_B T)$, and k_B is the Boltzmann constant (throughout this paper we have assumed $\hbar = k_B = 1$). The corresponding zero-temperature Green's function $\widehat{G}(\mathbf{k}, \omega)$ is an 8×8 matrix:

$$\widehat{G}(\mathbf{k},\omega) = \begin{pmatrix} G_{11}(\mathbf{k},\omega) & 0 & 0 & G_{14}(\mathbf{k},\omega) & G_{15}(\mathbf{k},\omega) & 0 & 0 & G_{18}(\mathbf{k},\omega) \\ 0 & G_{22}(\mathbf{k},\omega) & G_{23}(\mathbf{k},\omega) & 0 & 0 & G_{26}(\mathbf{k},\omega) & G_{27}(\mathbf{k},\omega) & 0 \\ 0 & G_{23}(\mathbf{k},\omega) & G_{33}(\mathbf{k},\omega) & 0 & 0 & G_{36}(\mathbf{k},\omega) & G_{37}(\mathbf{k},\omega) & 0 \\ G_{41}(\mathbf{k},\omega) & 0 & 0 & G_{44}(\mathbf{k},\omega) & G_{45}(\mathbf{k},\omega) & 0 & 0 & G_{48}(\mathbf{k},\omega) \\ G_{51}(\mathbf{k},\omega) & 0 & 0 & G_{54} & G_{55}(\mathbf{k},\omega) & 0 & 0 & G_{58}(\mathbf{k},\omega) \\ 0 & G_{62}(\mathbf{k},\omega) & G_{63}(\mathbf{k},\omega) & 0 & 0 & G_{66}(\mathbf{k},\omega) & G_{67}(\mathbf{k},\omega) & 0 \\ 0 & G_{73}(\mathbf{k},\omega) & G_{73}(\mathbf{k},\omega) & 0 & 0 & G_{76}(\mathbf{k},\omega) & G_{77}(\mathbf{k},\omega) & 0 \\ G_{81}(\mathbf{k},\omega) & 0 & 0 & G_{84}(\mathbf{k},\omega) & G_{85}(\mathbf{k},\omega) & 0 & 0 & G_{88}(\mathbf{k},\omega) \end{pmatrix}, \tag{4}$$

where the elements of the above matrix can be written in the following form:

$$G_{n_1,n_2}(\mathbf{k},\omega) = \frac{A_{n_1,n_2}(\mathbf{k})}{\omega - \omega_1(\mathbf{k}) + i0^+} + \frac{B_{n_1,n_2}(\mathbf{k})}{\omega + \omega_1(\mathbf{k}) - i0^+} + \frac{C_{n_1,n_2}(\mathbf{k})}{\omega - \omega_2(\mathbf{k}) + i0^+} + \frac{D_{n_1,n_2}(\mathbf{k})}{\omega + \omega_2(\mathbf{k}) - i0^+}, \quad \{n_1, n_2\} = 1,\ 2, \ldots 8,$$

$$\omega_{1,2}(\mathbf{k}) = \sqrt{\Delta^2 + \Omega_1^2(\mathbf{k}) \pm \Omega_2^2(\mathbf{k})}, \qquad \Omega_1^2(\mathbf{k}) = [t'f(\mathbf{k}) + \mu]^2 + t^2[3 + f(\mathbf{k})]^2 + \lambda^2 z^2(\mathbf{k}), \tag{5}$$

$$\Omega_2^2(\mathbf{k}) = 2[t'f(\mathbf{k}) + \mu]\sqrt{t^2[3 + f(\mathbf{k})]^2 + \lambda^2 z^2(\mathbf{k})}.$$

The poles $\pm\omega_{1,2}(\mathbf{k})$ of the single-particle Green's function describe the dispersion of the single-particle excitations in the mean-field approximation. The functions $A_{n_1,n_2}(\mathbf{k})$, $B_{n_1,n_2}(\mathbf{k})$, $C_{n_1,n_2}(\mathbf{k})$ and $D_{n_1,n_2}(\mathbf{k})$ can be numerically calculated by inverting the matrix $(i\omega_m \hat{1} - \widehat{H}_{MF}(\mathbf{k}))$.

The momentum distribution for the spin components $n_{\uparrow(\downarrow)}(\mathbf{k})$ can be evaluated using the corresponding elements of the 8×8 Green's function matrix:

$$
\begin{aligned}
n_{\uparrow}(\mathbf{k}) &= \left\langle \psi^{\dagger}_{A,\mathbf{k},\uparrow}\psi_{A,\mathbf{k},\uparrow}\right\rangle + \left\langle \psi^{\dagger}_{B,\mathbf{k},\uparrow}\psi_{B,\mathbf{k},\uparrow}\right\rangle = \beta^{-1}\sum_{i\omega_m}\left(G_{11}(\mathbf{k},i\omega_m)+G_{55}(\mathbf{k},i\omega_m)\right) = \\
&= f_{F-D}(\omega_1(\mathbf{k}))(A_{11}(\mathbf{k})+A_{55}(\mathbf{k}))+\left(1-f_{F-D}(\omega_1(\mathbf{k}))\right)(B_{11}(\mathbf{k})+B_{55}(\mathbf{k})) = \\
&= f_{F-D}(\omega_2(\mathbf{k}))(C_{11}(\mathbf{k})+C_{55}(\mathbf{k}))+\left(1-f_{F-D}(\omega_2(\mathbf{k}))\right)(D_{11}(\mathbf{k})+D_{55}(\mathbf{k})), \\
n_{\downarrow}(\mathbf{k}) &= \left\langle \psi^{\dagger}_{A,\mathbf{k},\downarrow}\psi_{A,\mathbf{k},\downarrow}\right\rangle + \left\langle \psi^{\dagger}_{B,\mathbf{k},\downarrow}\psi_{B,\mathbf{k},\downarrow}\right\rangle = \beta^{-1}\sum_{i\omega_m}\left(G_{22}(\mathbf{k},i\omega_m)+G_{66}(\mathbf{k},i\omega_m)\right) = \\
&= f_{F-D}(\omega_1(\mathbf{k}))(A_{22}(\mathbf{k})+A_{66}(\mathbf{k}))+\left(1-f_{F-D}(\omega_1(\mathbf{k}))\right)(B_{22}(\mathbf{k})+B_{66}(\mathbf{k})) = \\
&= f_{F-D}(\omega_2(\mathbf{k}))(C_{22}(\mathbf{k})+C_{66}(\mathbf{k}))+\left(1-f_{F-D}(\omega_2(\mathbf{k}))\right)(D_{22}(\mathbf{k})+D_{66}(\mathbf{k})),
\end{aligned}
\tag{6}
$$

where $f_{F-D}(x) = (\mathrm{e}^{\beta x}+1)^{-1}$ is the Fermi–Dirac distribution function. The symbol $\sum_{\omega_m}$ is used to denote $\beta^{-1}\sum_m$. Assuming the zero-temperature case, and by using the relationship $A_{ij}(\mathbf{k})+B_{ij}(\mathbf{k})+C_{ij}(\mathbf{k})+D_{ij}(\mathbf{k}) = \delta_{i,j}$, we find:

$$
\begin{aligned}
n_{\uparrow\{\downarrow\}}(\mathbf{k}) &= 1-\frac{1}{2}[A_{11\{22\}}(\mathbf{k})-B_{11\{22\}}(\mathbf{k})+ \\
&+C_{11\{22\}}(\mathbf{k})-D_{11\{22\}}(\mathbf{k})]-\frac{1}{2}[A_{55\{66\}}(\mathbf{k})- \\
&-B_{55\{66\}}(\mathbf{k})+C_{55\{66\}}(\mathbf{k})-D_{55\{66\}}(\mathbf{k})].
\end{aligned}
\tag{7}
$$

By summing over all vectors from the first Brillouin zone, we derive the number equation for the filing factor:

$$
f = \sum_{k\in BZ}\left(n_{\uparrow}(\mathbf{k})+n_{\downarrow}(\mathbf{k})\right). \tag{8}
$$

Very similarly, one can derive the gap equation

$$
\Delta = U\left\langle \psi_{A,-\mathbf{k},\downarrow}\psi_{A,\mathbf{k},\uparrow}\right\rangle = U\beta^{-1}\sum_{i\omega_m}G_{23}(\mathbf{k},i\omega_m),
$$

which at a zero temperature assumes the form:

$$
\Delta = U\sum_{k\in BZ}\left(B_{23}(\mathbf{k})+D_{23}(\mathbf{k})\right). \tag{9}
$$

At fixed filling factor f, one can obtain the chemical potential and the gap by solving Eqs. (8) and (9).

To compare our results with the T-matrix approximation, presented in Ref. 5, we have solved the gap equation at half filling ($\mu = 0$ and $f = 1$) setting $U/t = 2.69$, $\lambda/t = 0.1$ and $t' = 0$. The gap equation provides $\Delta/t = 0.151$.

3. GRPA for the collective modes

The Green's functions in the functional-integral approach are defined by means of the so-called generating functional with sources for the boson and fermion fields. In our problem, the corresponding functional integrals cannot be evaluated exactly because the interaction part of the KMH Hamiltonian is quartic in the Grassmann fermion fields. We transform the quartic terms to a quadratic forms by introducing a boson field which mediates the interaction of fermions. The boson fields in square and triangular lattices are four-component fields [8,9], but in the honeycomb lattice we have to work with a model system which consists of a eight-component boson field $A_{\alpha}(z)$ ($\alpha = 1, 2, ..., 8$) interacting with eight-component fermion spinor fields $\hat{\hat{\Psi}}(y)$ and $\hat{\Psi}(x) = \hat{\hat{\Psi}}^{\dagger}(x)$:

$$
\begin{aligned}
\hat{\hat{\Psi}}(y) &= (\psi^{\dagger}_{A,\uparrow}(y)\psi^{\dagger}_{A,\downarrow}(y)\psi_{A,\uparrow}(y)\psi_{A,\downarrow}(y), \\
&\psi^{\dagger}_{B,\uparrow}(y)\psi^{\dagger}_{B,\downarrow}(y)\psi_{B,\uparrow}(y)\psi_{B,\downarrow}(y)).
\end{aligned}
$$

Here, we have introduced composite variables, $z = (\mathbf{r}_j, v) = (j, v)$, $y = \{\mathbf{r}_i, u\} = \{i, u\}$ and $x = \{\mathbf{r}_{i'}, u'\} = \{i', u'\}$, where $\mathbf{r}_i$, $\mathbf{r}_{i'}$ are the lattice site vectors, and according to imaginary-time (Matsubara) formalism the variable u, u' and v range from 0 to β.

The action of this model system is assumed to be of the following form $S = S_0^{(F)} + S_0^{(B)} + S^{(F-B)}$, where:

$$
S_0^{(F)} = \hat{\hat{\psi}}(y)\hat{G}^{(0)-1}(y;x)\hat{\psi}(x),
$$

$$
S_0^{(B)} = \frac{1}{2}A_{\alpha}(z)D^{(0)-1}_{\alpha\beta}(z,z')A_{\beta}(z'),
$$

$$
S^{(F-B)} = \hat{\hat{\psi}}(y)\hat{\Gamma}^{(0)}_{\alpha}(y,x\,|\,z)\hat{\psi}(x)A_{\alpha}(z).
$$

Here we use the summation-integration convention: that repeated variables are summed up or integrated over. The action $S_0^{(F)}$ describes the fermion part of the system. The inverse Green's function of free fermions $\hat{G}^{(0)-1}(y;x)$ is given by the following matrix:

$$
\begin{aligned}
&\hat{G}^{(0)-1}(y;x) = \\
&= \sum_{\mathbf{k},\omega_m}\exp\left[i\mathbf{k}(\mathbf{r}_i-\mathbf{r}_{i'})-\omega_m(u-u')\right]G^{(0)-1}_{n_1n_2}(\mathbf{k},i\omega_m),
\end{aligned}
$$

where the non-interacting Green's function is defined as $\hat{G}^{(0)-1}(\mathbf{k},i\omega_m) = (i\omega_m\hat{1}-\hat{H}_0(\mathbf{k}))$. The non-interacting Hamilton $\hat{H}_0(\mathbf{k})$ is obtained from $\hat{H}_{MF}(\mathbf{k})$, but with $\Delta = 0$.

The action $S_0^{(B)}$ describes the boson field which mediates the fermion–fermion on-site interaction in the Hubbard Hamiltonian. The Fourier transform of the bare boson propagator in $S_0^{(B)}$ is defined as:

$$\hat{D}^{(0)}(z,z') = \frac{1}{N}\sum_{\mathbf{k}}\sum_{\omega_p} e^{\{i[\mathbf{k}(\mathbf{k}_j-\mathbf{k}_{j'})-\omega_p(v-v')]\}}\hat{D}^{(0)}(\mathbf{k}),$$

$$\hat{D}^{(0)}(\mathbf{k}) = \begin{pmatrix} 0 & U & 0 & 0 & 0 & 0 & 0 & 0 \\ U & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & U & 0 & 0 \\ 0 & 0 & 0 & 0 & U & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (10)$$

Here, the Matsubara boson energies are $\omega_p = 2m\pi/\beta$; $p = 0, 1, 2....$

The interaction between the fermion and the boson fields is described by the action $S^{(F-B)}$. The bare vertex

$$\hat{\Gamma}^{(0)}_{\alpha}(y_1;x_2\,|\,z) = \hat{\Gamma}^{(0)}_{\alpha}(i_1,u_1;i_2,u_2\,|\,j,v) =$$

$$= \delta(u_1-u_2)\delta(u_1-v)\delta_{i_1i_2}\delta_{i_1j}\hat{\Gamma}^{(0)}_{\alpha}$$

is a 8×8 matrix $\hat{\Gamma}^{(0)}_{\alpha} = \begin{pmatrix} \hat{\Gamma}^{(A)}_{\alpha} & \hat{0} \\ \hat{0} & \hat{\Gamma}^{(B)}_{\alpha} \end{pmatrix}$, where the 4×4 blocks $\hat{\Gamma}^{(A/B)}_{\alpha}$ are defined in terms of the Dirac matrix $\hat{\gamma}_0$ and the matrices $\hat{\alpha}_i$ ($\hat{\alpha}_i$ matrices also appear in superconductivity [10]):

$$\begin{aligned} \hat{\Gamma}^{(A)}_{\alpha} &= \frac{1}{2}(\hat{\gamma}_0+\hat{\alpha}_z)\delta_{\alpha1}+\frac{1}{2}(\hat{\gamma}_0-\hat{\alpha}_z)\delta_{\alpha2}+ \\ &+\frac{1}{2}(\hat{\alpha}_x+i\hat{\alpha}_y)\delta_{\alpha3}+\frac{1}{2}(\hat{\alpha}_x-i\hat{\alpha}_y)\delta_{\alpha4}, \\ \hat{\Gamma}^{(B)}_{\alpha} &= \frac{1}{2}(\hat{\gamma}_0+\hat{\alpha}_z)\delta_{\alpha5}+\frac{1}{2}(\hat{\gamma}_0-\hat{\alpha}_z)\delta_{\alpha6}+ \\ &+\frac{1}{2}(\hat{\alpha}_x+i\hat{\alpha}_y)\delta_{\alpha7}+\frac{1}{2}(\hat{\alpha}_x-i\hat{\alpha}_y)\delta_{\alpha8}, \end{aligned} \quad (11)$$

$$\hat{\gamma}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \hat{\alpha}_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_y\sigma_i\sigma_y \end{pmatrix}, \; i = x, y, z.$$

The basic assumption in our BS formalism is that the bound states of two Fermi atoms in an optical lattice at zero temperature are described by the BS wave functions (BS amplitudes). The BS amplitude determines the probability amplitude to find the first atom at the site **i** at the moment t_1 and the second atom at the site **j** at the moment t_2. The BS amplitude depends on the relative internal time t_1-t_2 and on the "center-of-mass" time $(t_1+t_2)/2$ [11]. Since the boson propagator $\hat{D}^{(0)}(\mathbf{k})$ is frequency independent, the spectrum of the collective modes will be obtained by solving the following BS equation for the equal-time BS amplitude $\Psi^{\mathbf{Q}}_{n_2,n_1}, \{n_1,n_2\} = 1, 2, ..., 8$:

$$\Psi^{\mathbf{Q}}_{n_2n_1} = K^{(0)}\begin{pmatrix} n_1 & n_3 \\ n_2 & n_4 \end{pmatrix}\Big|\omega(\mathbf{Q})\Big)\times$$

$$\times\left[I_d\begin{pmatrix} n_3 & n_5 \\ n_4 & n_6 \end{pmatrix}+I_{\text{exc}}\begin{pmatrix} n_3 & n_5 \\ n_4 & n_6 \end{pmatrix}\right]\Psi^{\mathbf{Q}}_{n_6,n_5}. \quad (12)$$

In the GRPA the two-particle propagator $K^{(0)}$ is written in terms of the mean-field single-particle Green's functions:

$$K^{(0)}\begin{pmatrix} n_1 & n_3 \\ n_2 & n_4 \end{pmatrix}\Big|\omega(\mathbf{Q})\Big) \equiv K^{(0)}_{n_1n_3n_4n_2} =$$

$$= \int\frac{d\Omega}{2\pi}\int\frac{d^2\mathbf{k}}{(2\pi)^2}G_{n_1n_3}(\mathbf{k}+\mathbf{Q},\Omega+\omega(\mathbf{Q}))G_{n_4n_2}(\mathbf{k},\Omega). \quad (13)$$

The kernel of the BS equation is a sum of the direct $I_d = \delta\Sigma^F/\delta G$ and exchange $I_{\text{exc}} = \delta\Sigma^H/\delta G$ interactions, written as derivatives of the Fock Σ^F and the Hartree Σ^H parts of the self-energy. This means that the BS equation and the corresponding the SD equation for the self-energy have to be solved self-consistently. In the Appendix A, we have presented an approximation which allows us to decouple the BS and SD equations, and to obtain the following expressions for the BS kernel:

$$\begin{aligned} I_d\begin{pmatrix} n_1 & n_3 \\ n_2 & n_4 \end{pmatrix} &= -\Gamma^{(0)}_{\alpha}(n_1,n_3)D^{(0)}_{\alpha\beta}\Gamma^{(0)}_{\beta}(n_4,n_2), \\ I_{\text{exc}}\begin{pmatrix} n_1 & n_3 \\ n_2 & n_4 \end{pmatrix} &= \frac{1}{2}\Gamma^{(0)}_{\alpha}(n_1,n_2)D^{(0)}_{\alpha\beta}\Gamma^{(0)}_{\beta}(n_4,n_3). \end{aligned} \quad (14)$$

Here $\Gamma^{(0)}_{\alpha}(n_1,n_2)$ is the corresponding matrix element of $\hat{\Gamma}^{(0)}_{\alpha}$. The BS equation, written in the matrix form, is $(\hat{I}+U\hat{Z})\hat{\Psi} = 0$, where $\hat{I}$ is the unit matrix, and the condition for the existence of non-trivial solution requires the 64×64 determinant $\det\left|U^{-1}\hat{I}+\hat{Z}\right| = 0$. By applying simple matrix algebra, the 64×64 determinant can be simplified to a 20×20 one of the following form

$$\det\begin{vmatrix} A_{4\times4}(\omega,\mathbf{Q}) & 0 & 0 \\ 0 & B_{4\times4}(\omega,\mathbf{Q}) & 0 \\ 0 & 0 & C_{12\times12}(\omega,\mathbf{Q}) \end{vmatrix} = 0. \quad (15)$$

The elements of the above blocks are given in the Appendix B. The elements of the $A_{4\times4}$ and $B_{4\times4}$ blocks are different, but $\det|A_{4\times4}(\omega,\mathbf{Q})| = \det|B_{4\times4}(\omega,\mathbf{Q})|$. The above 20×20 determinant vanishing if $\det|A_{4\times4}(\omega,\mathbf{Q})| = 0$, but our numerical calculations at half filling show that $\det|A_{4\times4}(\omega=0,\mathbf{Q}=0)| \neq 0$, which means that the determinant $\det|A_{4\times4}$ cannot provide the Goldstone mode dispersion. Therefore, the BS formalism provides only one secular determinant $C_{12\times12}(\omega,\mathbf{Q})$. We have used the mean-field system parameters at half filling, introduced in Sec. 2, to

obtain the sound velocity $u = \alpha at/\hbar$ in the direction of point Γ toward point M. The slope of the linear part of the collective-mode dispersion has been calculated numerically by using four points with $Qy = 0$ and $aQx = 0, 0.0025\pi$ and 0.0050π. The corresponding slope is $\alpha = 1.45$, and therefore, the sound velocity becomes $u/v_F = 0.97$, where we have introduced the Fermi velocity $v_F = (3/2)ta/\hbar$ in a honeycomb lattice. For the similar system parameters, the slope, obtained from Fig. 5(b) in Ref. 5, is $\alpha = 1.34$, that is about 4% difference.

4. Conclusion

To summarize, we have examined the superfluid states of the attractive Kane–Mele–Hubbard model. We have obtained the spectrum of the single-particle excitations within the mean-field approximation. We have also derived the Bethe–Salpeter equation for the attractive Kane–Mele–Hubbard Hamiltonian based on the generalized random phase approximation. The collective modes are defined by the roots of the corresponding secular determinant. The slope of the Goldstone sound mode has been numerically calculated at half filling. Our numerical result shows that the T-matrix approximation, which does not take into account the exchange interaction, underestimates the sound velocity by about 4%.

It is worth mentioning that the Gaussian approximation also neglects the exchange interaction, but in square or triangular lattices it overestimates the slop of the Goldstone mode compared with the BS formalism.This is because the Gaussian and the T-matrix approximation are using different single-particle Green's functions. The Gaussian approximation uses mean-field single particle Green's functions, while in the T-matrix approximation the KM single-particle Green's functions are employed.

Although we restricted our numerical analysis to half filling, we plan in a subsequent work to use our approach in the case when the system is away from half filling. Our preliminary results show that away from half filling, the sound speed decreases. For example, for the following system parameters, $U/t = 3$, $f = 0.93$, $\lambda/t = 0.2$, and $t'/t = -0.15$, we found $\mu/t = -0.75$, $\Delta/t = 0.65$, and $u/v_F = 0.52$.

Appendix A

There is one-to-one correspondence between the KMH model and our model system, which is based on the following Hubbard–Stratonovich transformation for the fermion operators:

$$\int \mu[A] \exp\left[\hat{\bar{\psi}}(y)\hat{\Gamma}^{(0)}_{\alpha}(y;x\,|\,z)\hat{\psi}(x)A_{\alpha}(z)\right] = \exp\left[-\frac{1}{2}\hat{\bar{\psi}}(y)\hat{\Gamma}^{(0)}_{\alpha}(y;x\,|\,z)\hat{\psi}(x)D^{(0)}_{\alpha,\beta}(z,z')\hat{\bar{\psi}}(y')\hat{\Gamma}^{(0)}_{\beta}(y';x'\,|\,z')\hat{\psi}(x')\right].$$

The functional measure $D\mu[A]$ is chosen to be:

$$\mu[A] = DA e^{-\frac{1}{2}A_{\alpha}(z)D^{(0)-1}_{\alpha,\beta}(z,z')A_{\beta}(z')}, \quad \int \mu[A] = 1.$$

According to the field-theoretical approach, the expectation value of a general operator $\hat{O}(u)$ can be expressed as a functional integral over the boson field A and the Grassmann fermion fields $\hat{\bar{\psi}}$ and $\hat{\psi}$:

$$\left\langle \hat{T}_u(\hat{O}(u)) \right\rangle = \frac{1}{Z[J,M]} \times$$

$$\times \int D\mu[\hat{\bar{\psi}},\hat{\psi},A]\hat{O}(u)\exp\left[J_{\alpha}(z)A_{\alpha}(z) - \hat{\bar{\psi}}\hat{M}\hat{\psi}\right]|_{J=M=0},$$

where the symbol $<...>$ means that the thermodynamic average is made. The functional $Z[J,M]$ is defined by

$$Z[J,M] = \int D\mu[\hat{\bar{\psi}},\hat{\psi},A]\exp\left[J_{\alpha}(z)A_{\alpha}(z) - \hat{\bar{\psi}}\hat{M}\hat{\psi}\right],$$

where the functional measure

$$D\mu[\hat{\bar{\psi}},\hat{\psi},A] = DAD\hat{\bar{\psi}}D\hat{\psi}\exp(S)$$

satisfies the condition $\int D\mu[\hat{\bar{\psi}},\hat{\psi},A] = 1$. The quantity $J_{\alpha}(z)$ is the source of the boson field. The sources $M_{ij}(y;x)$ of the fermion fields are included in the $\hat{\bar{\Psi}}(y)\hat{M}(y,x)\hat{\Psi}(x)$ term, where $\hat{M}(y,x)$ is an 8×8 matrix:

$$\hat{M}(y,x) = \begin{pmatrix} M_{11}(y;x) & ... & M_{18}(y;x) \\ M_{21}(y;x) & ... & M_{28}(y;x) \\ . & ... & . \\ M_{81}(y;x) & ... & M_{88}(y;x) \end{pmatrix}.$$

In what follows, we introduce complex indexes $1 = \{n_1, y_1\}$, and $2 = \{n_2, x_2\}$, so in short notations we have $M_{n_1 n_2}(y_1;x_2) = M(1;2)$.

By means of the definition of the thermodynamic average, one can express all Green's functions in terms of the functional derivatives with respect to the corresponding sources of the generating functional of the connected Green's functions $W[J,M] = \ln Z[J,M]$.

The boson Green's function is $D_{\alpha\beta}(z,z')$ is a 8×8 matrix defined as

$$D_{\alpha\beta}(z,z') = -\frac{\delta^2 W}{\delta J_{\alpha}(z)\delta J_{\beta}(z')}.$$

The single-fermion Green's function $\widehat{G}(x,y) = -\left\langle \hat{T}_u \widehat{\Psi}(x) \otimes \widehat{\overline{\Psi}}(y) \right\rangle$ includes all possible thermodynamic averages. Its matrix elements are $G_{n_1 n_2}(x_1; y_2) = -\delta W / \delta M_{n_2 n_1}(y_2; x_1)$. The Fourier transform of the single-particle Green's function is given by

$$\widehat{G}(1;2) = \frac{1}{N}\sum_{k}\sum_{\omega_m} \exp\left\{ i\left[\mathbf{k}(\mathbf{r}_{i_1} - \mathbf{r}_{i_2}) - \omega_m (u_1 - u_2) \right] \right\} \widehat{G}(\mathbf{k}, i\omega_m).$$

The two-particle Green's function $K\begin{pmatrix} n_1, x_1 & n_3, y_3 \\ n_2, y_2 & n_4, x_4 \end{pmatrix}$ is defined as

$$K\begin{pmatrix} n_1, x_1 & n_3, y_3 \\ n_2, y_2 & n_4, x_4 \end{pmatrix} = K\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \frac{\delta^2 W}{\delta M_{n_2 n_1}(y_2; x_1) \delta M_{n_3 n_4}(y_3; x_4)} = -\frac{\delta G_{n_1 n_2}(x_1; y_2)}{\delta M_{n_3 n_4}(y_3; x_4)}.$$

The vertex function $\hat{\Gamma}_\alpha(2;1|z)$ for a given α is a 8×8 matrix whose elements are:

$$\hat{\Gamma}_\alpha(i_2, u_2; i_1, u_1 | v, j)_{n_2 n_1} = -\frac{\delta G^{-1}_{n_2 n_1}(i_2, u_2; i_1, u_1)}{\delta J_\beta(z')} D^{-1}_{\beta\alpha}(z', z).$$

Since the single-particle and the two-particle (collective) excitations manifest themselves as poles of the corresponding Green's functions, our next step is to obtain equations of the boson and fermion Green's functions. First, we shall obtain the SD equations, and they will be used to define the fermion self-energy (fermion mass operator) $\hat{\Sigma}(1;2)$. The simplest way to derive the SD equations is to use the fact that the measure $D\mu[\overline{\psi}, \psi, A]$ is invariant under the translations $\overline{\psi} \to \overline{\psi} + \delta\overline{\psi}$ and $A \to A + \delta A$:

$$D^{(0)-1}_{\alpha\beta}(z, z') R_\beta(z') + \frac{1}{2}\mathrm{Tr}\left(\widehat{G}(1;2) \hat{\Gamma}^{(0)}_\alpha(2;1|z) \right) + J_\alpha(z) = 0,$$

$$\widehat{G}^{-1}(1;2) - \widehat{G}^{(0)-1}(1;2) + \hat{\Sigma}(1;2) + \widehat{M}(1;2) = 0,$$

where $R_\alpha(z) = \delta W / \delta J_\alpha(z)$ is the average boson field. The fermion self-energy $\hat{\Sigma}$, is a 8×8 matrix which can be written as a sum of Hartree $\hat{\Sigma}^H$ and Fock $\hat{\Sigma}^F$ parts. The Hartree part is a diagonal matrix whose elements are:

$$\Sigma^H(i_1, u_1; i_2, u_2)_{n_1 n_2} = \frac{1}{2}\hat{\Gamma}^{(0)}_\alpha(i_1, u_1; i_2, u_2 | j, v)_{n_1 n_2} \times D^{(0)}_{\alpha\beta}(j, v; j', v') \hat{\Gamma}^{(0)}_\beta(i_3, u_3; i_4, u_4 | j', v')_{n_3 n_4} \times G_{n_4 n_3}(i_4, u_4; i_3, u_3).$$

The Fock part of the fermion self-energy is given by:

$$\Sigma^F(i_1, u_1; i_2, u_2)_{n_1 n_2} = -\hat{\Gamma}^{(0)}_\alpha(i_1, u_1; i_6, u_6 | j, v)_{n_1 n_6} \times D^{(0)}_{\alpha\beta}(j, v; j', v') \hat{\Gamma}^{(0)}_\beta(i_4, u_4; i_5, u_5 | j', v')_{n_4 n_5} \times K\begin{pmatrix} n_5, i_5, u_5 & n_3, i_3, u_3 \\ n_4, i_4, u_4 & n_6, i_6, u_6 \end{pmatrix} G^{-1}_{n_3 n_2}(i_3, u_3; i_2, u_2).$$

The Fock part of the fermion self-energy depends on the two-particle Green's function K; therefore the SD equations and the BS equation for K have to be solved self-consistently.

Our approach to the Hubbard model allows us to obtain exact equations of the Green's functions by using the field-theoretical technique, in particular, the Legendre transforms. We can go over from the functional $W[J, M]$ to a new functional $V[R, G] = W[J[R, G], M[R, G]] - J_\alpha[R, G] R_\alpha + \mathrm{Tr}(M[R, G] G)$, such that the conjugate equations hold:

$$\frac{\delta V}{\delta R_\alpha}(z) = -J_\alpha(z); \quad \frac{\delta V}{\delta G_{n_1 n_2}(1;\, 2)} = M_{n_1 n_2}(1;\, 2).$$

By means of the SD equations and the identity

$$\delta(1-3)\delta(2-4)\delta_{n_1, n_3}\delta_{n_2, n_4} = \frac{\delta M_{n_1 n_2}(1;\, 2)}{\delta M_{n_3 n_4}(3;\, 4)} = \frac{\delta M_{n_1 n_2}(1;\, 2)}{\delta R_\alpha(z)} \frac{\delta R_\alpha(z)}{\delta M_{n_3 n_4}(3;\, 4)} + \frac{\delta M_{n_1 n_2}(1;\, 2)}{\delta G_{n_5 n_6}(5;\, 6)} \frac{\delta G_{n_5 n_6}(5;\, 6)}{\delta M_{n_3 n_4}(3;\, 4)},$$

one sees that two-particle Green's function satisfies the BS equation

$$K^{-1}\begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix} = K^{(0)-1}\begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix} - I\begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix}.$$

Here,

$$K^{(0)}\begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix} = G_{n_2 n_3}(i_2, u_3; i_2, u_2) G_{n_4 n_1}(i_4, u_4; i_1, u_1)$$

is the two-particle free propagator constructed from a pair of fully dressed generalized single-particle Green's functions. The kernel $I = \delta\Sigma / \delta G$ of the BS equation can be expressed as a functional derivative of the fermion self-energy $\hat{\Sigma}$. Since $\hat{\Sigma} = \hat{\Sigma}^H + \hat{\Sigma}^F$, the BS kernel $I = I_{\mathrm{exc}} + I_d$ is a sum of functional derivatives of the Hartree Σ^H and Fock Σ^F contributions to the self-energy:

$$I_{\rm exc}\begin{pmatrix} n_2,i_2,u_2 & n_3,i_3,u_3 \\ n_1,i_1,u_1 & n_4,i_4,u_4 \end{pmatrix} = \frac{\delta\Sigma^H(i_2,u_2;i_1,u_1)_{n_2n_1}}{\delta G_{n_3n_4}(i_3,u_3;i_4,u_4)},$$

$$I_d\begin{pmatrix} n_2,i_2,u_2 & n_3,i_3,u_3 \\ n_1,i_1,u_1 & n_4,i_4,u_4 \end{pmatrix} = \frac{\delta\Sigma^F(i_2,u_2;i_1,u_1)_{n_2n_1}}{\delta G_{n_3n_4}(i_3,u_3;i_4,u_4)}.$$

The BS equation and the SD equations have to be solved self-consistently. In order to decouple them, we note that the identity

$$K^{(0)}\begin{pmatrix} n_2,i_2,u_2 & n_3,i_3,u_3 \\ n_1,i_1,u_1 & n_4,i_4,u_4 \end{pmatrix}\times$$
$$\times\hat{\Gamma}_\beta(i_4,u_4;i_3,u_3\mid z')_{n_4n_3}D_{\beta\alpha}(z',z) =$$
$$= K\begin{pmatrix} n_2,i_2,u_2 & n_3,i_3,u_3 \\ n_1,i_1,u_1 & n_4,i_4,u_4 \end{pmatrix}\times$$
$$\times\hat{\Gamma}^{(0)}_\beta(i_4,u_4;i_3,u_3\mid z')_{n_4n_3}D^{(0)}_{\beta\alpha}(z',z)$$

allows us to rewrite the Fock term as

$$\Sigma^F(i_1,u_1;i_2,u_2)_{n_1n_2} = -\hat{\Gamma}^{(0)}_\alpha(i_1,u_1;i_3,u_3\mid j,v)_{n_1n_3}\times$$
$$\times D_{\alpha\beta}(j,v;j',v')G_{n_3n_4}(i_3,u_3;i_4,u_4)\times$$
$$\times\hat{\Gamma}_\beta(i_4,u_4;i_2,u_2\mid j',v')_{n_4n_2}.$$

To decouple SD and BS equations, we replace D and $\hat{\Gamma}$ by the free boson propagator $D^{(0)}$ and by the bare vertex $\hat{\Gamma}^{(0)}$, respectively. In this approximation the Fock term assumes the form:

$$\Sigma^F_0(i_1,u_1;i_2,u_2)_{n_1n_2} = -\hat{\Gamma}^{(0)}_\alpha(i_1,u_1;i_3,u_3\mid j,v)_{n_1n_3}\times$$
$$\times D^{(0)}_{\alpha\beta}(j,v;j',v')\hat{\Gamma}^{(0)}_\beta(i_4,u_4;i_2,u_2\mid j',v'))_{n_4n_2}\times$$
$$\times G_{n_3n_4}(i_3,u_3;i_4,u_4).$$

The total self-energy is $\hat{\Sigma}(i_1,u_1;i_2,u_2)=\hat{\Sigma}^H(i_1,u_1;i_2,u_2)+$ $+\hat{\Sigma}^F(i_1,u_1;i_2,u_2)$. The Hartree part of the fermion self-energy is a diagonal 8×8 matrix, but in the mean-field approximation, the elements on the major diagonal of $\hat{\Sigma}^H(i_1,u_1;i_2,u_2)$ will be included into the chemical potential. To obtain an analytical expression for the single-particle Green's function in the mean-field approximation, we neglect the frequency dependence of the Fourier transform of the Fock part of the fermion self-energy. In this approximation, the Fock term is an 8×8 matrix with non-zero elements $\Sigma^F_0(i_1,u_1;i_2,u_2)_{n_1n_2}=\delta(u_1-u_2)\delta(r_{i_1}-r_{i_2})\Delta_{A/B}$ for $\{n_1n_2\}=\{14\}$, $\{23\}$,$\{32\}$, $\{41\}$ for Δ_A and $\{n_1n_2\}=\{58\}$, $\{67\}$, $\{76\}$, $\{85\}$ for Δ_B.

Appendix B

The blocks in Eq. (15) are given by the following matrices:

$$A_{4\times4}(\omega,\mathbf{Q}) = \begin{pmatrix} U^{-1}-K^{(0)}_{2211} & -K^{(0)}_{6215} & -K^{(0)}_{3214} & -K^{(0)}_{7218} \\ -K^{(0)*}_{6215} & U^{-1}-K^{(0)}_{6655} & -K^{(0)}_{6345} & -K^{(0)}_{7658} \\ -K^{(0)*}_{3214} & -K^{(0)*}_{6345} & U^{-1}-K^{(0)}_{3344} & -K^{(0)}_{7348} \\ -K^{(0)*}_{7218} & -K^{(0)*}_{7658} & -K^{(0)*}_{7348} & U^{-1}-K^{(0)}_{7788} \end{pmatrix},$$

$$B_{4\times4}(\omega,\mathbf{Q}) = \begin{pmatrix} U^{-1}-K^{(0)}_{4433} & -K^{(0)}_{8437} & -K^{(0)*}_{5436} & -K^{(0)}_{4123} \\ -K^{(0)*}_{8437} & U^{-1}-K^{(0)}_{8877} & -K^{(0)}_{8567} & -K^{(0)}_{8127} \\ -K^{(0)}_{5436} & -K^{(0)*}_{8567} & U^{-1}-K^{(0)}_{5566} & -K^{(0)}_{5126} \\ -K^{(0)*}_{4123} & -K^{(0)*}_{8127} & -K^{(0)*}_{5126} & U^{-1}-K^{(0)}_{1122} \end{pmatrix}.$$

The elements of $C_{12\times12}(\omega,\mathbf{Q}) = \begin{pmatrix} C_{11} & C^\dagger_{21} \\ C_{21} & C_{22} \end{pmatrix}$ will be given by four 6×6 blocks:

$$C_{11}(\omega,\mathbf{Q}) = \begin{pmatrix} U^{-1}+K^{(0)}_{1144} & K^{(0)*}_{1414} & K^{(0)*}_{8145} & K^{(0)*}_{5148} & 0 & 0 \\ K^{(0)}_{1414} & U^{-1}+K^{(0)}_{4411} & K^{(0)*}_{8415} & K^{(0)*}_{5418} & 0 & 0 \\ K^{(0)}_{8145} & K^{(0)}_{8415} & U^{-1}+K^{(0)}_{8855} & K^{(0)*}_{5858} & 0 & 0 \\ K^{(0)}_{5148} & K^{(0)}_{5418} & K^{(0)}_{5858} & U^{-1}+K^{(0)}_{5588} & 0 & 0 \\ 0 & 0 & 0 & 0 & U^{-1}+K^{(0)}_{2233} & K^{(0)*}_{2323} \\ 0 & 0 & 0 & 0 & K^{(0)}_{2323} & U^{-1}+K^{(0)}_{3322} \end{pmatrix},$$

$$C_{21}(\omega,\mathbf{Q})=\begin{pmatrix} 0 & 0 & 0 & 0 & K^{(0)}_{7236} & K^{(0)}_{7326} \\ 0 & 0 & 0 & 0 & K^{(0)}_{6237} & K^{(0)}_{6327} \\ K^{(0)}_{4144} & K^{(0)}_{4414} & K^{(0)}_{8445} & K^{(0)}_{5448} & -K^{(0)}_{2232} & -K^{(0)}_{2322} \\ K^{(0)}_{8148} & K^{(0)}_{8418} & K^{(0)}_{8858} & K^{(0)}_{8588} & -K^{(0)}_{6236} & -K^{(0)}_{6326} \\ -K^{(0)}_{1114} & -K^{(0)}_{1411} & -K^{(0)}_{8115} & -K^{(0)}_{5118} & K^{(0)}_{3233} & K^{(0)}_{3323} \\ -K^{(0)}_{5145} & -K^{(0)}_{5415} & -K^{(0)}_{5855} & -K^{(0)}_{5558} & K^{(0)}_{7237} & K^{(0)}_{7327} \end{pmatrix}.$$

$$C_{22}(\omega,\mathbf{Q})=\begin{pmatrix} U^{-1}+K^{(0)}_{7766} & & & & & \\ K^{(0)}_{6767} & U^{-1}+K^{(0)}_{6677} & & \text{H.c.} & & \\ -K^{(0)}_{7226} & -K^{(0)}_{6227} & K^{(0)}_{2222}+K^{(0)}_{4444} & & & \\ -K^{(0)}_{6766} & -K^{(0)}_{6676} & K^{(0)}_{6226}+K^{(0)}_{8448} & K^{(0)}_{6666}+K^{(0)}_{8888} & & \\ K^{(0)}_{7336} & K^{(0)}_{6337} & 2U^{-1}-K^{(0)}_{2332}-K^{(0)}_{4114} & 0-K^{(0)}_{6336}-K^{(0)}_{8118} & K^{(0)}_{1111}+K^{(0)}_{3333} & \\ K^{(0)}_{7767} & K^{(0)}_{76777} & -K^{(0)}_{5445}-K^{(0)}_{7227} & 2U^{-1}-K^{(0)}_{6776}-K^{(0)}_{8558} & K^{(0)}_{5115}+K^{(0)}_{7337} & K^{(0)}_{5555}+K^{(0)}_{7777} \end{pmatrix}.$$

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Дисперсія колективних мод в атомних фермі-газах в оптичній гратці стільникового типу: швидкість звуку в моделі Кейна–Меле–Хаббарда з притяганням при половинному заповненні

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Досліджено надплинний стан, що виникає у моделі Кейна–Меле внаслідок локальної притягальної взаємодії *U*. Дисперсія колективної моди визначається рівнянням Бете–Солпітера (БС) в узагальненому наближенні випадкових фаз. Нахил низькоенергетичної моди (Голдстоуна) та відповідна йому швидкість звуку при половинному заповненні, які розраховано у рамках формалізму БС, порівнюються з відповідними результатами, що отримані раніше в наближенні Т-матриці. Різниця між цими двома підходами полягає в тому, що наближення Т-матриці враховує тільки сходові діаграми та нехтує пухирцевими. З цієї причини швидкість звуку в напрямку до точки *M*, яка розрахована в наближенні Т-матриці, приблизно на 4% менше, ніж результат, який отримано за допомогою рівняння БС.

Ключові слова: колективні моди, швидкість звуку, надплинний стан.

Дисперсия коллективных мод в атомных ферми-газах в оптической решетке сотового типа: скорость звука в модели Кейна–Меле–Хаббарда с притяжением при половинном заполнении

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Исследовано сверхтекучее состояние, возникающее в модели Кейна–Меле в результате локального притягивающего взаимодействия *U*. Дисперсия коллективной моды определяется уравнением Бете–Солпитера (БС) в обобщенном приближении случайных фаз. Наклон низкоэнергетической моды (Голдстоуна) и соответствующая ему скорость звука при половинном заполнении, рассчитанные в рамках формализма БС, сравниваются с соответствующими результатами, полученными ранее в приближении Т-матрицы. Разница между этими двумя подходами состоит в том, что приближение Т-матрицы учитывает только лестничные диаграммы и пренебрегает пузырьковыми. По этой причине скорость звука по направлению к точке *M*, рассчитанная в приближении Т-матрицы, примерно на 4% меньше, чем результат, полученный с помощью уравнения БС.

Ключевые слова: коллективные моды, скорость звука, сверхтекучее состояние.