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Effective many-body interactions in one-dimensional dilute Bose gases

A.K. Kolezhuk

Institute of High Technologies, T. Shevchenko National University, Kyiv 03022, Ukraine

Institute of Magnetism NAS of Ukraine and MES of Ukraine, Kyiv 03142, Ukraine
E-mail: kolezhuk@gmail.com

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We propose a new method of constructing the effective theory for a multicomponent low-dimensional dilute Bose gas. The method is based on obtaining and solving renormalization group equations for all many-body interactions of density-density type in the one-loop approximation. In contrast to the standard approach based on two-body interactions, our method does not rely on the introduction of density-dependent infrared cutoff, and is able to reproduce exactly the leading term in the energy density of the one-dimensional dilute Bose gas, and to obtain the next term with the error of less than 2%.

Keywords: ultracold atoms, spinor bosons, low-dimensional Bose gas.

1. Introduction

Multicomponent dilute ultracold Bose gases continue to attract the attention of researchers, not only in the context of spinor Bose condensates [1,2] and heteronuclear Bose mixtures [3,4], but also in the context of frustrated magnets, where strong magnetic fields may lead to condensation of magnons at several wave vectors [5–10]. Multicomponent systems can exhibit a rich phase behavior, which is usually studied using the macroscopic description in terms of the multicomponent-field Gross–Pitaevsky energy functional containing only quadratic density-density interaction terms. In the case of low-dimensional systems (provided by highly anisotropic trap geometries in the case of atomic Bose mixtures, or by magnetic materials with strongly anisotropic exchange coupling in the case of magnon condensates), effective two-body couplings of the Gross–Pitaevsky-type theory are strongly renormalized from their bare microscopic values [11].

For the one-component dilute Bose gas, there is a well-known renormalization group (RG) approach [12–16] based on deriving the RG equations at the critical point of the zero chemical potential (vanishing particle density), and terminating the RG flow at a certain density-dependent infrared cutoff scale. For one-dimensional Bose gas, this approach correctly reproduces the energy in the dilute limit to be proportional to the third power of density. A slightly different but essentially equivalent version of this approach [17,18], instead of the infrared momentum cutoff in the momentum space, utilizes the so-called off-shell regularization where the effective couplings are obtained by taking energy-dependent two-body scattering amplitudes at a finite negative energy determined by the chemical potential. This sort of approach has been generalized to the multicomponent case [9,19] and successfully applied to magnon condensation in frustrated spin chains and ladders [9,10]. Among other things, it has been shown that for spinor bosons the renormalization tends to enhance the interaction symmetry, effectively diminishing the spin-dependent part of the coupling [19,20]. However, the presence of several different densities causes an ambiguity in the procedure of the RG flow termination in the multicomponent case.

The aim of this study is to explore a different route to the analysis of dilute multicomponent Bose systems. Instead of adopting the density-dependent wave vector or energy cutoff, we assume the presence of many-body couplings from the outset, and study their renormalization in the one-loop approximation, without assuming any infrared cutoffs. We show that such approach leads to a system of the renormalization group equations with a remarkable structure that an equation for n-body coupling involves only k-body couplings with $k \le n$, so this system can be solved sequentially. In this approach, we are able to reproduce exactly the leading term in the energy density of the one-dimensional dilute Bose gas, and the next term is obtained within 2% error margin.

2. One-component Bose gas

We start by illustrating and testing our approach on the well-known case of the 1D dilute Bose gas for which a number of exact benchmark results are known [21]. The system is described by the euclidean action of a single Bose field ψ:

$$A[\psi]=\int d\tau\int dx\{\psi^*(\partial_\tau-\mu)\psi+\frac{|\nabla\psi|^2}{2m}+U(\rho)\},$$

$$U(\rho)=\sum_{n=2}^{\infty}\frac{\Gamma_n}{n!}\rho^n,\quad \rho=|\psi|^2, \tag{1}$$

where m is the particle mass, the interaction energy $U(\rho)$ is assumed to contain many-body couplings $\Gamma_{n\geq 3}$ beside the familiar two-body coupling Γ_2, and we have set the Planck constant to unity. We decompose field $\psi=\Phi+\phi$ into the "fast" component ϕ with wave vector k in the interval $\Lambda<|k|<\Lambda+d\Lambda$ and the "slow" component Φ with $|k|<\Lambda$, where Λ is the running cutoff.

Then, $\mathcal{A}[\psi]=\mathcal{A}[\Phi]+\mathcal{A}_{\rm int}[\Phi,\phi]$, and the quadratic in ϕ part of the action is

$$\mathcal{A}_{\rm int}[\Phi,\phi]=\int d\tau\int dx\{\phi^*(\partial_\tau-\mu)\phi+\frac{|\nabla\phi|^2}{2m}+ \tag{2}$$

$$+\Sigma_{11}[\Phi]|\phi|^2+\frac{1}{2}[\Sigma_{12}[\Phi]\phi^2+\Sigma_{12}^*[\Phi](\phi^*)^2]\},$$

where

$$\Sigma_{11}[\Phi]=\sum_{n=2}\frac{n\Gamma_n}{(n-1)!}|\Phi|^{2(n-1)},$$

$$\Sigma_{12}[\Phi]=\sum_{n=2}\frac{\Gamma_n}{(n-2)!}|\Phi|^{2(n-2)}(\Phi^*)^2. \tag{3}$$

We assume that at the equilibrium $\mu=\partial U(\rho)/\partial\rho$ with $\rho=|\Phi|^2$, which translates into the condition

$$\Sigma_{11}[\Phi]-\mu=|\Sigma_{12}[\Phi]| \tag{4}$$

that may be viewed as the consequence of the Hugenholtz–Pines theorem [22].

Integrating over the "fast" field

$$\int D\phi\int D\phi^* e^{-\mathcal{A}_{\rm int}[\Phi,\phi]}=e^{-\delta\mathcal{A}[\Phi]}$$

yields the action renormalization

$$\delta\mathcal{A}[\Phi]=\frac{1}{2}\int d\tau\int dx\sum_{\Lambda<|k|<\Lambda+d\Lambda}(E_k-T_k), \tag{5}$$

$$T_k=\frac{k^2}{2m}+|\Sigma_{12}[\Phi]|,\quad E_k=\{T_k^2-|\Sigma_{12}[\Phi]|^2\}^{1/2}. \tag{6}$$

Expanding $\delta\mathcal{A}[\Phi]$ in powers of $|\Phi|^2$, we obtain the following one-loop RG equations for couplings Γ_n:

$$\frac{d\Gamma_2}{d\Lambda}=-\frac{m}{\pi\Lambda^2}\Gamma_2^2,$$

$$\frac{d\Gamma_3}{d\Lambda}=\frac{6}{\pi}\left\{\left(\frac{m}{\Lambda^2}\right)^2\Gamma_2^3-\frac{m}{\Lambda^2}\Gamma_2\Gamma_3\right\}, \tag{7}$$

$$\frac{d\Gamma_4}{d\Lambda}=\frac{12}{\pi}\left\{6\left(\frac{m}{\Lambda^2}\right)^2\Gamma_2^3\Gamma_3-5\left(\frac{m}{\Lambda^2}\right)^3\Gamma_2^4-\frac{m}{\Lambda^2}(\Gamma_3^2+\Gamma_2\Gamma_4)\right\},$$

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The RG equations have a remarkable structure: an equation for Γ_n involves only Γ_k with $k\leq n$, so this system can be solved sequentially. Setting the bare values of the couplings at the initial microscopic scale Λ_0 as $\Gamma_2(\Lambda_0)=g$, $\Gamma_k(\Lambda_0)=0$, $k\geq 3$, one obtains the following behavior at $\Lambda\to 0$:

$$\Gamma_2\to\frac{\pi\Lambda}{m}-\left(1-\frac{mg}{\pi\Lambda_0}\right)\left(\frac{\pi\Lambda}{m}\right)^2\frac{1}{g}+O(\Lambda^3),$$

$$\Gamma_3\to\frac{\pi^2}{m}-\left(1-\frac{mg}{\pi\Lambda_0}\right)\frac{12\pi^3\Lambda}{5gm^2}+O(\Lambda^2), \tag{8}$$

$$\Gamma_4\to-\frac{8\pi^4}{5gm^2}\left(1-\frac{mg}{\pi\Lambda_0}\right)+O(\Lambda).$$

The microscopic scale Λ_0 is roughly the inverse of the atom size (for the gas in the continuum) or of the lattice constant (for a lattice system), so we will assume that $\Lambda_0\gg mg$. Using the effective couplings at $\Lambda\to 0$, one obtains the following expansion of the energy density $e(\rho)$ of the Bose gas in the dilute limit $\rho\to 0$:

$$e(\rho)\approx\frac{\pi^2}{6m}\rho^3-\frac{\pi^4}{15gm^2}\rho^4+\ldots. \tag{9}$$

Comparing with the known exact results [21], one can see that the coefficient at the cubic term in density is exact, while the coefficient at the quartic term is less than 1.5% off (its exact value is $2\pi^2/3$). It should be emphasized that we have not used any heuristic RG flow cutoffs.

3. Two-component Bose gas

Now we extend the approach presented in the previous section to the model of a two-component dilute Bose gas described by the following action:

$$\mathcal{A}=\int d\tau\int dx\{\psi_a^*(\partial_\tau-\mu_a)\psi_a+\frac{|\nabla\psi_a|^2}{2m}|+U(\rho_1,\rho_2)\}, \tag{10}$$

where a = 1, 2 and the interaction energy

$$U(\rho_1,\rho_2)=\frac{1}{2!}\Gamma_2^{ab}\rho_a\rho_b+\frac{1}{3!}\Gamma_3^{abc}\rho_a\rho_b\rho_c+$$

$$+\frac{1}{4!}\Gamma_4^{abcd}\rho_a\rho_b\rho_c\rho_d\ldots \tag{11}$$

includes many-body terms depending only on densities $\rho_a=|\psi_a|^2$. Obviously, many-body interactions $\Gamma_n^{a_1a_2\cdots a_n}$ are symmetric with respect to permutations of upper indices.

Decomposing the fields into "slow" and "fast" components $\psi_a = \Phi_a + \phi_a$, expanding the action up to quadratic terms in "fast" fields ϕ_a, imposing the conditions $\mu_a = \partial U(\rho_1, \rho_2)/\partial \rho_a$, and integrating the "fast" fields out, one obtains the following one-loop RG equations:

$$-\frac{d\Gamma_2^{aa}}{d\Lambda} = \frac{m_a}{\pi\Lambda^2}(\Gamma_2^{aa})^2, \quad -\frac{d\Gamma_2^{12}}{d\Lambda} = \frac{2m_1 m_2}{(m_1+m_2)\pi\Lambda^2}(\Gamma_2^{12})^2, \quad -\frac{d\Gamma_3^{aaa}}{d\Lambda} = \frac{6m_a}{\pi\Lambda^2}\Gamma_2^{aa}\Gamma_3^{aaa} - \frac{6m_a^2}{\pi\Lambda^4}(\Gamma_2^{aa})^3, \tag{12}$$

$$-\frac{d\Gamma_3^{aab}}{d\Lambda} = -\frac{8(m_a+2m_b)m_a^2 m_b}{(m_a+m_b)^2\pi\Lambda^4}\Gamma_2^{aa}(\Gamma_2^{ab})^2 + \frac{2m_a}{\pi\Lambda^2}\Gamma_2^{aa}\Gamma_3^{aab} - \frac{8m_a m_b}{(m_a+m_b)\pi\Lambda^2}\Gamma_2^{ab}\Gamma_3^{aab},$$

$$-\frac{d\Gamma_4^{aaaa}}{d\Lambda} = \frac{12m_a}{\pi\Lambda^2}(\Gamma_2^{aa}\Gamma_4^{aaaa} + (\Gamma_3^{aaa})^2) - \frac{72m_a^2}{\pi\Lambda^4}(\Gamma_2^{aa})^2\Gamma_3^{aaa} + \frac{60m_a^3}{\pi\Lambda^6}(\Gamma_2^{aa})^4,$$

$$-\frac{d\Gamma_4^{1122}}{d\Lambda} = \frac{2}{\pi\Lambda^2}(m_1(\Gamma_3^{112})^2 + m_2(\Gamma_3^{122})^2) + \frac{2}{\pi\Lambda^2}\Gamma_4^{1122}((m_1\Gamma_2^{11} + m_2\Gamma_2^{22}) + \frac{8m_1 m_2}{m_1+m_2}\Gamma_2^{12}) -$$
$$-\frac{16m_1^2 m_2(m_1+2m_2)}{\pi\Lambda^4(m_1+m_2)^2}\Gamma_3^{112}\Gamma_2^{12}(\Gamma_2^{12} + 2\Gamma_2^{22}) - \frac{16m_2^2 m_1(m_2+2m_1)}{\pi\Lambda^4(m_1+m_2)^2}\Gamma_3^{122}\Gamma_2^{12}(\Gamma_2^{12} + 2\Gamma_2^{11}) +$$
$$+\frac{32m_1^2 m_2^2(m_1^2 + 3m_1 m_2 + m_2^2)}{(m_1+m_2)^3\pi\Lambda^6}(\Gamma_2^{12})^2((\Gamma_2^{12})^2 + 2\Gamma_2^{11}\Gamma_2^{22}),$$

$$-\frac{d\Gamma_4^{aaab}}{d\Lambda} = \frac{6m_a}{\pi\Lambda^2}(\Gamma_3^{aab}\Gamma_3^{aaa} + \Gamma_2^{aa}\Gamma_4^{aaab}) + \frac{12m_a m_b}{(m_a+m_b)\pi\Lambda^2}(\Gamma_2^{ab}\Gamma_4^{aaab} + (\Gamma_3^{aab})^2) - \frac{18m_a^2}{\pi\Lambda^4}(\Gamma_2^{aa})^2\Gamma_3^{aab} -$$
$$-\frac{24m_a^2 m_b(m_a+2m_b)}{(m_a+m_b)^2\pi\Lambda^4}((\Gamma_2^{ab})^2\Gamma_3^{aaa} + 2\Gamma_2^{aa}\Gamma_2^{ab}\Gamma_3^{aab}) + \frac{24(3m_a^2 + 9m_a m_b + 8m_b^2)m_a^3 m_b}{(m_a+m_b)^3\pi\Lambda^6}(\Gamma_2^{aa})^2(\Gamma_2^{ab})^2.$$

Let us consider the simplest case of equivalent components, with equal masses $m_1 = m_2 = m$ and interactions symmetric with respect to the permutation $1 \leftrightarrow 2$. If the bare values of couplings defined at the microscopic scale Λ_0 are $\Gamma_2^{11} = \Gamma_2^{22} = g_{11}$, $\Gamma_2^{12} = \Gamma_2^{21} = g_{12}$, then at $\Lambda \to 0$ one obtains the following behavior of the renormalized couplings:

$$\Gamma_2^{ab} \to 0, \quad \Gamma_3^{abc} \to \frac{\pi^2}{m}, \quad \Gamma_4^{1111} \to -\frac{8\pi^4}{5m^2 g_{11}}, \tag{13}$$

$$\Gamma_4^{1112} \to -\frac{4\pi^4}{5m^2}\left(\frac{1}{g_{11}} + \frac{1}{g_{12}}\right),$$

$$\Gamma_4^{1122} \to -\frac{8\pi^4}{15m^2}\left(\frac{1}{g_{11}} + \frac{2}{g_{12}}\right).$$

Thus, the energy density of the dilute two-component Bose gas, in the one-loop approximation, becomes

$$e(\rho_1, \rho_2) = \frac{\pi^2}{6m}(\rho_1 + \rho_2)^3 - \tag{14}$$
$$-\frac{\pi^4}{15m^2}\left\{\frac{(\rho_1+\rho_2)^4}{g_{11}} + 2\left(\frac{1}{g_{12}} - \frac{1}{g_{11}}\right)\rho_1\rho_2(\rho_1+\rho_2)^2\right\}.$$

The minimum of this energy is achieved in a phase-separated state if $g_{12} > g_{11}$ and in a mixed state otherwise.

4. Summary

In summary, we have proposed a novel approach to the analysis of multicomponent low-dimensional dilute Bose gases. In contrast to the standard method that is based on the Gross–Pitaevsky theory limited to two-body interactions and relies on the introduction of an infrared cutoff fine-tuned to reproduce known exact results, we explicitly construct the effective action with many-body couplings of the density-density type, by obtaining and solving renormalization group equations for the entire set of many-body couplings in the one-loop approximation. It is shown that the proposed approach is able to reproduce exactly the leading term in the energy density of the one-dimensional dilute Bose gas, and to obtain the next term with the error of less than 2%.

1. Yuki Kawaguchi and Masahito Ueda, *Phys. Rep.* **520**, 253 (2012).
2. M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen(De), and U. Sen, *Adv. Phys.* **56**, 243 (2007).

3. G. Modugno, M. Modugno, F. Riboli, G. Roati, and M. Inguscio, *Phys. Rev. Lett.* **89**, 190404 (2002).
4. J. Catani, L. De Sarlo, G. Barontini, F. Minardi, and M. Inguscio, *Phys. Rev. A* **77**, 011603(R) (2008).
5. T. Nikuni and H. Shiba, *J. Phys. Soc. Jpn.* **64**, 3471 (1995).
6. G. Jackeli and M.E. Zhitomirsky, *Phys. Rev. Lett.* **93**, 017201 (2004).
7. A. Kolezhuk and T. Vekua, *Phys. Rev. B* **72**, 094424 (2005).
8. M. Arlego, F. Heidrich-Meisner, A. Honecker, G. Rossini, and T. Vekua, *Phys. Rev. B* **84**, 224409 (2011).
9. A.K. Kolezhuk, F. Heidrich-Meisner, S. Greschner, and T. Vekua, *Phys. Rev. B* **85**, 064420 (2012).
10. I.T. Shyiko, I.P. McCulloch, J.V. Gumenjuk-Sichevska, and A.K. Kolezhuk, *Phys. Rev. B* **88**, 014403 (2013).
11. see, e.g., Chapter 17 of L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation*, Oxford University Press (2003).
12. D.S. Fisher and P.C. Hohenberg, *Phys. Rev. B* **37**, 4936 (1988).
13. D.R. Nelson and H.S. Seung, *Phys. Rev. B* **39**, 9153 (1989).
14. E.B. Kolomeisky and J.P. Straley, *Phys. Rev. B* **46**, 11749 (1992).
15. S. Sachdev, *Quantum Phase Transitions*, Cambridge University Press (1999).
16. D. Uzunov, *Phys. Lett. A* **87**, 11 (1981).
17. A.K. Kolezhuk, *Phys. Rev. A* **81**, 013601 (2010).
18. M.D. Lee, S.A. Morgan, M.J. Davis, and K. Burnett, *Phys. Rev. A* **65**, 043617 (2002); see also arXiv:cond-mat/0305416 (unpublished).
19. S.A. Morgan, M.D. Lee, and K. Burnett, *Phys. Rev. A* **65**, 022706 (2002).
20. A.K. Kolezhuk, *Fiz. Nizk. Temp.* **36**, 947 (2010) [*Low Temp. Phys.* **36**, 752 (2010)].
21. H.T.C. Stoof, K.B. Gubbels, and D.B.M. Dickerscheid, *Ultracold Quantum Fields*, Springer, Dordrecht (2009), p. 485.
22. E.H. Lieb and W. Liniger, *Phys. Rev.* **130**, 1605 (1963).

Ефективні багаточастинкові взаємодії в одновимірних розріджених Бозе-газах

О.К. Колежук

Запропоновано новий метод побудови ефективної теорії для багатокомпонентного низьковимірного розрідженого Бозе-газу. Метод базується на розв'язанні в однопетлевому наближенні системи рівнянь ренормгрупи для усіх багаточастинкових взаємодій типу густина–густина. На відміну від стандартного підходу, що оснований лише на двочастинкових взаємодіях, запропонований метод не спирається на використання залежного від густини інфрачервоного відсікання. Основний доданок для густини енергії одновимірного Бозе-газу відтворюється цим методом точно, наступний доданок отримується з похибкою менше 2%.

Ключові слова: надхолодні атоми, спінорні бозони, низьковимірний Бозе-газ.