Low Temperature Physics/Fizika Nizkikh Temperatur, 2020, v. 46, No. 8, pp. 1014–1020

Dynamics of pair of coupled nonlinear systems. I. Magnetic systems

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Received April 23, 2020, published online June 22, 2020

In the framework of the Landau–Lifshitz equations for discrete systems, the dynamics of two classical magnetic moments modeling weakly coupled magnetic nanodots, layers of quasi-two-dimensional magnets and two-sublattice magnets are considered. Exact solutions of dynamic equations are found and investigated. Particular attention is paid to the study of essentially nonlinear inhomogeneous states with different levels of excitation for identical subsystems as a discrete analog for the magnetic solitons.

Keywords: nonlinear systems, Landau–Lifshitz equations, magnetic resonance, phase portrait, ferromagnetic and antiferromagnetic interactions.

Introduction

Although the nonlinear dynamics of dynamical systems is a traditional field of physics, the last half-century exhibits its essential progress, related to the active study of soliton excitations and their manifestations in the physics of condensed matter. Recently a particular interest is connected with the study of nonlinear discrete systems linking the areas of nonlinear oscillations and nonlinear waves. Under a weak localization of nonlinear excitations in discrete systems, the whole nonlinear dynamics is localized on several elements of the lattice. Recently, this problem has become more actual due to the active research and application in nanoobjects such as the coupled effective spin-torque oscillators [1], the cavities containing SQUIDs with Josephson junctions connections as the equipment for the quantum computer [2], high-gain weakly nonlinear flax-modulated Josephson parametric amplifier using a SQUID arrays [3,4], coupled micromechanical resonators [5], microelectromechanical (MEM) coupled cantilevers and nanoelectromechanical (NEM) systems [6], magnetic bilayers with F/N/F structures [7], bicomponent magnonic crystals [8], arrays of optical waveguides, optical switch and coupled modes in nonlinear optical waveguides [9]. As it was first demonstrated in [10,11], many elements of the solitary waves physics in systems with distributed parameters have their analogous in the systems with the final number of degrees of freedom, particularly in the systems with two elements [12]. Unfortunately, in the Hamiltonian systems with two degrees of freedom, in the absence of the additional integral of motion, there appears a chaotic component of the dynamics, which defaces such an important element of motion as the localization of excitations on one degree of freedom. In its evident form, this phenomenon manifests itself in the integrable systems with two degrees of freedom. Some examples of such systems are discussed in this article. As a first example, consider two bounded classical magnetic moments in Landau–Lifshitz equation approximation. In particular, it describes interacting magnetic nanodots, magnetic layers or two interacting magnetic sublattices.

1. The dynamics of two interacting magnetic moments

1.1. Ferromagnetic interaction

Let us study the dynamics of classical magnetic moments of two subsystems with the easy-axis single-ion magnetic anisotropy and the ferromagnetic type exchange interaction between the subsystems. The moments of the two elements $\mathbf{M}_1$ and $\mathbf{M}_2$ at ground state are oriented along the “easy axis” (Z axis). The total energy of such system reads [13]

$$E = -\varepsilon \mathbf{m}_1 \mathbf{m}_2 - (\omega_0 / 2)\left(m_1^2 + m_2^2\right) + \varepsilon + \omega_0,\ \varepsilon > 0, \quad (1)$$

where $\mathbf{m}_i = \mathbf{M}_i / M_0$ are the normalized magnetic moment, m_i are Z components of the moments, ω_0 is the constant of single-ion anisotropy (frequency of magnetic resonance),

ε is the constant of the exchange interaction, which is positive for the ferromagnetic case and negative for the antiferromagnetic case. (While choosing the energy in the form (1) the ground state of a "ferromagnetic" type has zero energy). In the classical approach, the dynamics of the magnetic system can be described in the framework of a discrete analog of the Landau–Lifshitz equation [14]:

$$d\mathbf{m}_j / dt = [\mathbf{m}_j \times \partial E / \partial \mathbf{m}_j].$$

In components $\psi_i = m_{ix} + i\, m_{iy}$ and $m_i = m_{iz}$ they have the form

$$i\, d\psi_i / dt = \omega_0 \psi_i m_i + \varepsilon\left(\psi_i m_j - \psi_j m_i\right). \quad (2)$$

For the description of the magnetic moments, it is convenient to use the polar coordinate system with $\psi_i = \sin\vartheta_i \exp(i\varphi_i) = a_i \exp(i\varphi_i)$. Then the system of two complex equations (2) is reduced to the system of three first-order real equations for $\psi = \varphi_2 - \varphi_1$ and a_i (or $m_i = \sqrt{1-a_i^2}$):

$$da_1 dt = -\varepsilon\, m_1 a_2 \sin\psi, \quad (3)$$

$$da_2 dt = \varepsilon\, m_2 a_1 \sin\psi, \quad (4)$$

$$d\psi / dt = (m_1 - m_2)\left(\omega_0 - \varepsilon - \varepsilon \cos\psi\,(1 + m_1 m_2) / a_1 a_2\right). \quad (5)$$

In addition to the total energy (1), the system under consideration has the integral of motion — a complete Z projection of magnetization $M = \sum \cos\vartheta_i = m_1 + m_2$. This integral is connected with the total number of spin deviations $2 - M = N$ and plays the role of the number of elementary excitations in the quasi-classical quantization. The value of N is limited by number 2: $0 < N < 2$. (The value 2 corresponds to the configuration in which both the moments are perpendicular to the easy axis.) The presence of two integrals of motion leads to the complete integrability of the system under consideration and the possibility of obtaining its solution in the quadratures. The difficulty of the problem is connected with the choosing of the convenient variables which takes into account the presence of one of the integrals of motion (N). Let us introduce instead of two variables ϑ_i, one function P, such that $\cos\vartheta_i = m_i = M/2 \pm P$. Then the condition of the conserving for the total magnetization is fulfilled automatically. Finding from the expression for the energy

$$E = -\varepsilon\left(m_1 m_2 + a_1 a_2 \cos\psi\right) + \varepsilon - \omega_0\left(m_1^2 + m_2^2\right)/2 + \omega_0, \quad (6)$$

the connection $\psi = \psi(\vartheta_i, E)$ and substituting it in (3) in the form $dm_1 dt = \varepsilon\, a_1 a_2 \sin\psi$, we obtain the closed equation for the function $P(t)$:

$$\left(dP/dt\right)^2 = -A - BP^2 - CP^4, \quad (7)$$

where

$$A = \left(E - \omega_0\left(1 - M^2/4\right)\right)\left(E - (\omega_0 + 2\varepsilon)\left(1 - M^2/4\right)\right), \quad (8)$$

$$B = 2\left((\omega_0 - \varepsilon)E - \omega_0^2\left(1 - M^2/4\right) + 2\varepsilon^2\right), \quad (9)$$

$$C = \omega_0(\omega_0 - 2\varepsilon). \quad (10)$$

The solutions of the Eqs. (7)–(10) can be represented in terms of elliptical Jacoby functions, but at first, we research the obtained system qualitatively. It allows the single-frequency states (stationary states) which describe the synchronous pure rotations of two magnetization vectors: $\psi_i = a_i \exp(-i\omega t)$. These rotations permit the same amplitudes and phases (s), same amplitudes and the phases which differ in π (a) and different amplitudes with the same phases (nonuniform ones–(n)). Nonuniform states exist only when inequality $\varepsilon < \omega_0/2$ is valid. (In real magnets, the exchange interaction is essentially large then the energy of magnetic anisotropy ($\varepsilon >> \omega_0$), but in layered quasi-two-dimensional magnets and nanodots systems, the inequality we have used can be performed.) The amplitudes of the rotations of the moments are linked by the relation

$$(m_1 - m_2)\left((\omega_0 - \varepsilon)a_1 a_2 - \varepsilon(1 + m_1 m_2)\right) = 0. \quad (11)$$

In the in-phase and anti-phase states $m_1 = m_2$ and $a_1 = \pm a_2$, in the nonuniform stationary states $m_1 m_2 = -1 + M(1-\kappa)/\sqrt{1-2\kappa}$, where $\kappa = \varepsilon/\omega_0$. In stationary states, the frequencies of these excitations dependence on the number of spin deviations N have the forms

$$\omega_s = \omega_0(1 - N/2),\ a_2 = a_1, \quad (12)$$

$$\omega_a = \omega_0(1 + 2\kappa)(1 - N/2),\ a_2 = -a_1, \quad (13)$$

$$\omega_n = \omega_0\left(M - \sqrt{1-2\kappa}\right),\ a_2 \neq a_1. \quad (14)$$

These dependences are shown in Fig. 1a. At the critical level of the excitation $N_b = 2(1 - \sqrt{1-2\kappa})$ two dependences for the nonuniform rotations (n) split from the line of in-phase rotations.

Corresponding dependences of energy on the norm $N = 2 - M$ for the three types of stationary rotations of magnetic moments have the forms

$$E_s = \omega_0 N - \omega_0 N^2/4, \quad (15)$$

$$E_a = (\omega_0 + 2\varepsilon)\left(N - N^2/4\right), \quad (16)$$

$$E_n = (\omega_0 + 2\varepsilon)/2 - \omega_0\left(M - \sqrt{1-2\kappa}\right)^2/2, \quad (17)$$

$$E_q = \left(\omega_0^2\left(N - N^2/4\right) - 2\varepsilon^2\right)/(\omega_0 - \varepsilon). \quad (18)$$

These dependences are represented in Fig. 1b. (On the line $E_q = E_q(N)$ ((18) and (q) in Fig. 1b) the value B in (9) changes the signs and this state is not stationary.) First of all, we see that for the single-frequency states the well-known mechanical relation $dE/dN = \omega$ holds. In addition, dependences (15), (16) follow from formula (8)

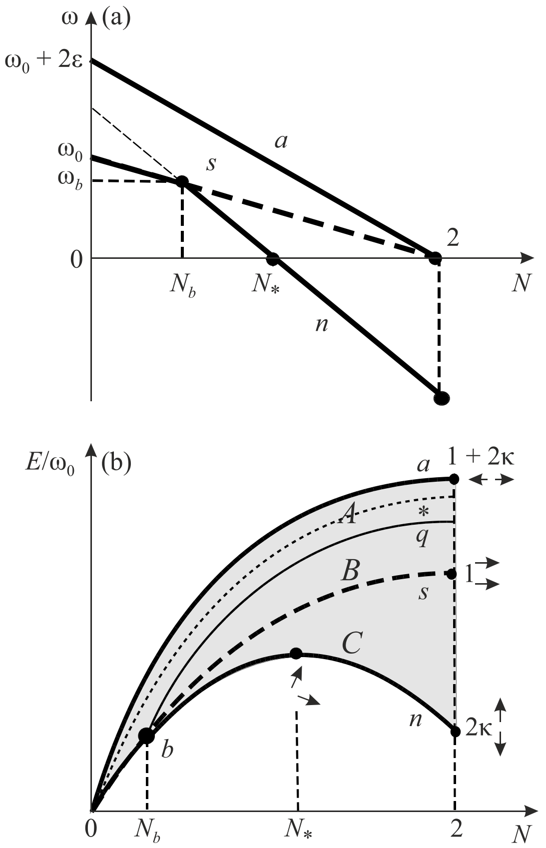

Fig. 1. (a) The dependences of stationary state frequencies on the number of spin deviations; (b) the domains of the existence for the solutions with different dynamics on the plane of the integrals of motion (E, N).

for $A = 0$, since we have $dP/dt = P = 0$ for the states with $m_1 = m_2$. The domain $N > 2$ of the plane (E, N) corresponds to the similar excitations above another ground state with $M = -2$. In the limit $N \to 2$ $(M \to 0)$, the frequency of in-phase and anti-phase rotations of magnetization tends to zero and it appears the static configurations with collinear and anti-collinear configurations of magnetization vectors in the "heavy" plane. The nonuniform state in this limit represents the anti-collinear configuration with moments along the easy axis with precession frequency $\omega = -\omega_0\sqrt{1-2\kappa}$ (they rotate in opposite direction). The frequency of inhomogeneous excitations turns to zero at $N_* = 2 - \sqrt{1-2\kappa}$. In this case, the vectors of two moments are orthogonal as it is shown in Fig. 1b: $m_1 = a_2 = $ $= \sqrt{1+\sqrt{1-4\kappa^2}}/\sqrt{2}$ and $m_2 = -a_1 = -\sqrt{1-\sqrt{1-4\kappa^2}}/\sqrt{2}$.

The expression (8) can be represented in the following form: $A = 4(E - E_s)(E - E_a)$. Therefore the value A is negative in the area of the parameters between the lines *a* and *s* in Fig. 1b and is positive in the area between lines *s* and *n*. Constant C is positive in the whole domain of the acceptable parameters of the solutions. At last, $B > 0$ for $E_q < E < E_a$ and $B < 0$ for $E_n < E < E_q$. So the phase portrait of the system in the "phase plane" $(P, \dot{P})$ has different structure for $N > N_b$ and $N < N_b$, and for $E_q < E < E_a$ and $E_n < E < E_q$ in the domain $N > N_b$ (see Fig. 2). We are able to represent all the trajectories only in two figures for the oscillations close to in-phase and to anti-phase one. These two types of orbits are separated by phase trajectory with the largest size which corresponds to the line $E_*(N)$ in Fig. 1b with

$$E_* = 2(\omega_0 + \varepsilon)\left(N - N^2/4\right). \tag{19}$$

The phase portrait of the system for $N < N_b$ is demonstrated in Fig. 2a. It is similar to the portrait for linear system. The two maximal orbits in two parts of the figure are the same and correspond to the energy $E = E_*$.

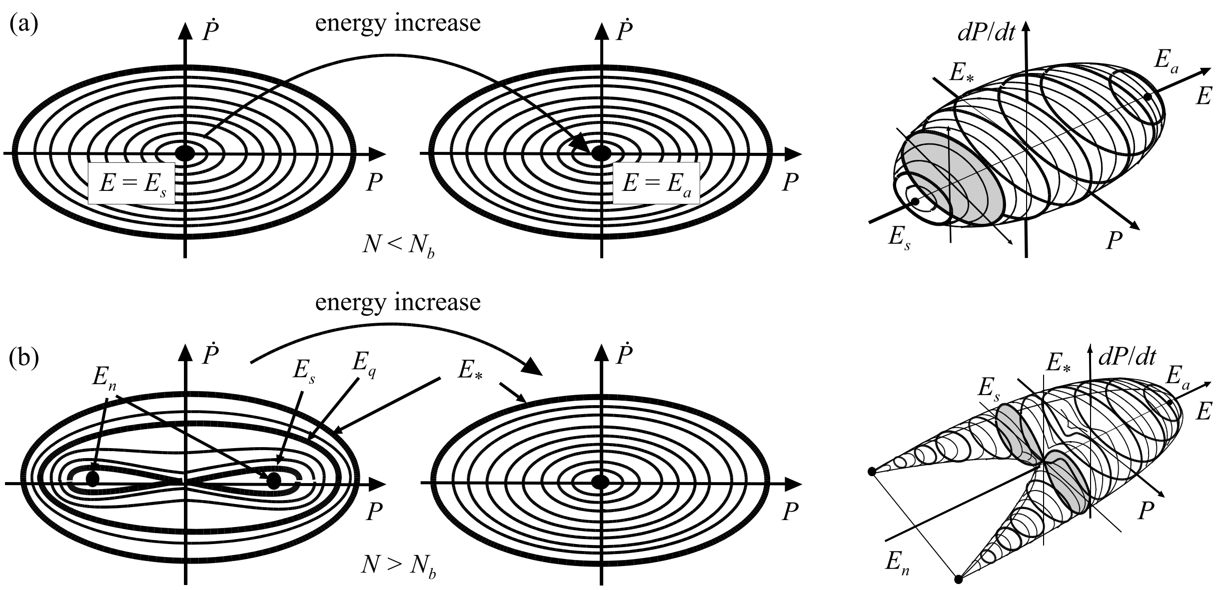

Fig. 2. Phase portraits of two coupled magnetic moments for $N < N_b$ (a) and $N > N_b$ (b).

In the region $N > N_b$, the phase portrait is more complicated. The separatrix E_s "begins" and "ends" in unstable saddle point for in-phase oscillations ($P = 0, \dot{P} = 0$) and separates the domain with the nonuniform rotations of two moments. Two stable "center"-type critical points with $E = E_n$ correspond to the states in which only one from two moments oscillates.

In all the area outside the separatrix E_s (upper in energy from the line s in Fig. 1b), the exact solution of Eq. (7) is expressed in term of the elliptic Jacobi function:

$$P = a\,\mathrm{cn}\left(\sqrt{a^2 + b^2}\sqrt{C}\,t, k\right),\ k = a/\sqrt{a^2 + b^2} \quad (20)$$

with C from (10). For $N < N_b$ with $E_s < E < E_a$ and for $N > N_b$ in the domain $E_q < E < E_a$ parameters a and b read

$$a^2 = (B/2C)\left(\sqrt{4|A|C/B^2 + 1} - 1\right), \quad (21)$$

$$b^2 = (B/2C)\left(\sqrt{4|A|C/B^2 + 1} + 1\right). \quad (22)$$

For the small levels of excitations with $N < N_b$ at the borders of area for the solutions $E = E_a, E_s$ parameter A tends to zero and $a \to 0$ with $k \to 0$. So the function (20) becomes trigonometric. The solution (20)–(22) describes the relative oscillations of two magnetic moments with the frequency $\Omega = \pi a\sqrt{C}/2kK(k)$. These oscillations are accompanied by the common rotation of all the systems with the frequency close to the resonant frequency ω_0 for small exchange interaction.

At $N > N_b$ the solution (20) preserves its form for energies $E_s < E < E_q$ but with another definition of the parameters:

$$a^2 = (|B|/2C)\left(\sqrt{4|A|C/B^2 + 1} + 1\right), \quad (23)$$

$$b^2 = (|B|/2C)\left(\sqrt{4|A|C/B^2 + 1} - 1\right). \quad (24)$$

Now at the line of in-phase oscillations E_s (s in Fig. 1b) parameter $b \to 0$ and the modulus k tends to unity: the solution on the separatrix E_s becomes aperiodic and one spin deviates from another passing through the easy axis.

At last in the domain of the parameters $E_n < E < E_s$, as in the previous case, nonuniform distribution of energy between the two moments takes place and the appropriate solution of Eq. (7) has the form

$$P = a\,\mathrm{dn}\left(at\sqrt{C}, k\right), \quad k = \sqrt{1 - b^2/a^2}, \quad (25)$$

with

$$a^2 = (|B|/2C)\left(1 + \sqrt{1 - 4AC/B^2}\right), \quad (26)$$

$$b^2 = (|B|/2C)\left(1 - \sqrt{1 - 4AC/B^2}\right). \quad (27)$$

In the limit $E \to E_n$, as it follows from (7), $B^2 = 4AC$ and modulus k is equal to zero. From this it follows that the parameter $P = a(N)$ does not depend on time and so $\vartheta_i = \mathrm{const}$: magnetic moments rotate around the direction of the easy axis as a common object.

We note that formulae (20) and (25) describe the oscillations of the polar angles $\vartheta_i(t)$ of the magnetization vectors $\mathbf{m}_1$ with frequencies, depending on $a(N,E)$ and $b(N,E)$ in (21), (22), (26), (27). After substituting the time dependence of $P(t)$ into formulas (2), (5), one can find additional azimuthal oscillations of the angles $\varphi_i(t)$ and $\psi(t)$ with their additional frequencies. The general motion of magnetization is two-frequency with incommensurable frequencies of azimuthal and polar motions. In Fig. 1a only the frequencies of the general magnetization rotation in particular cases of single-frequency motions are given.

1.2. Antiferromagnetic interaction

We proceed to consider a system of two magnetic moments with antiferromagnetic exchange interaction and single-axis anisotropy of the "easy axis" type. This problem is interesting in the following aspect. In contrast to the one-dimensional ferromagnetic chain studied in the framework of the Landau–Lifshitz equation, the one-dimensional antiferromagnetic chain studied in the long-wavelength approximation is described by evolutionary equations that are not fully integrable [15–19]. In the case of the antiferromagnetic interaction in the expression for energy (1), it is necessary to replace the sign of the exchange interaction constant. In terms of the variables m_i and ψ_i introduced above, and energy will take the form

$$E = \varepsilon\left(m_1 m_2 + \left(\psi_1\bar{\psi}_2 + \bar{\psi}_1\psi_2\right)/2\right) -$$
$$-\omega_0\left(m_1^2 + m_2^2\right)/2 + \varepsilon + \omega_0, \quad \varepsilon > 0, \quad (28)$$

where the constants are chosen so that in the ground state the energy turns to zero. Dynamic equations (2) are transformed as follows:

$$i\,d\psi_i/dt = \omega_0\psi_i m_i - \varepsilon\left(\psi_i m_j - \psi_j m_i\right). \quad (29)$$

We start with a qualitative consideration of the problem by analyzing single-frequency stationary states. Representing the magnetization components in the form $\psi_i = a_i\exp(-i\omega t)$ and substituting them in this form in (29), we obtain the relation similar to (11):

$$(m_1 - m_2)\left((\omega_0 + \varepsilon)a_1 a_2 + \varepsilon(1 + m_1 m_2)\right) = 0. \quad (30)$$

As before, it admits three types of solutions: (s) in-phase solution with $m_1 = m_2$ and $a_1 = a_2$, (a) anti-phase solution with a phase shift of rotation by π and with $m_1 = m_2$ and $a_1 = -a_2$, and (n) an inhomogeneous state with $m_1 \neq m_2$ and with opposite signs of a_1 and a_2 in contrast to the ferromagnetic case.

For an inhomogeneous configuration, from (30) it is easy to obtain the relations $m_1 m_2 = -1 + M(1+\kappa)/\sqrt{1+2\kappa}$ and $a_1 a_2 = -M\kappa/\sqrt{1+2\kappa}$. Using them, from Eqs. (29) for stationary states, we find the dependences of the frequencies of single-frequency solutions on the excitation amplitude. It is natural to characterize it by the number of spin deviations N, which can be associated with the full z component of the magnetization M: $N = M$ in the case of an antiferromagnet. These dependences for three types of stationary states are:

$$\omega_s = \omega_0 N/2,\ a_2 = a_1, \quad (31)$$

$$\omega_a = \omega_0 (1-2\kappa) N/2,\ a_2 = -a_1, \quad (32)$$

$$\omega_n = \omega_0 \left(N - \sqrt{1+2\kappa}\right),\ a_2 \neq a_1. \quad (33)$$

Accordingly, the dependence of energy on the number of excitations takes for these states the form

$$E_s = \omega_0 (1+2\kappa) - \omega_0 N^2/4, \quad (34)$$

$$E_a = \omega_0 - \omega_0 (1-2\kappa) N^2/4, \quad (35)$$

$$E_n = \omega_0 \sqrt{1+2\kappa} N - \omega_0 N^2/2, \quad (36)$$

$$E_q = \omega_0 \left(1+2\kappa - N^2/4\right)/(1+\kappa). \quad (37)$$

These dependences are shown in Fig. 3b as the lines *s*, *a*, *n*. The line *q* corresponds to the nonstationary special state (see below).

In contrast to a ferromagnetically coupled pair of spins, in the case of an antiferromagnetic coupling, the inhomogeneous state is split off from the line of antisymmetric oscillations (rotation of moments with phase difference π). The frequency and number of spin deviations at the bifurcation point are equal to $\omega_b = \omega_0(1-2\kappa)/\sqrt{1+2\kappa}$ and $N_b = 2/\sqrt{1+2\kappa}$. On the plane of the integrals of motion (N, E) in Fig. 3b, the region of the existence of nonlinear solutions of a general form is hatched. These are two-frequency solutions, which are easy to find in the same way as solutions for ferromagnetic coupling, using the representation of moments in cylindrical coordinates (ϑ_i, φ_i) or in variables (m_i, ψ).

In order to take into account the conservation of complete magnetization $M = N$, we introduce a variable P such that $m_{1,2} = M/2 \pm P$. The parameter P is associated with the antiferromagnetism vector $\mathbf{l} = \mathbf{m}_1 - \mathbf{m}_2$, namely $P = l_z/2 = \cos\vartheta$, where ϑ is the polar angle of the antiferromagnetism vector $\mathbf{l}$. The equation for the function P coincides with Eq. (7) in the case of ferromagnetic interaction, but the parameters of the equation have a different form:

$$A = (E - E_s)(E - E_a), \quad (38)$$

$$B = 2\left((\omega_0 + \varepsilon)E + \omega_0^2 N^2/4 - \omega_0(\omega_0 + 2\varepsilon)\right), \quad (39)$$

$$C = \omega_0(\omega_0 + 2\varepsilon). \quad (40)$$

The shaded region for the existence of solutions in Fig. 3b is divided into three subdomains with $A > 0$, $B < 0$, $C > 0$ (I), $A < 0$, $B < 0$, $C > 0$ (II), and $A < 0$, $B > 0$, $C > 0$ (III). In region I, the solution has the form (25)–(27), in region II — the form (20), (23), (24) and in region III — (20)–(22). Knowledge of the solution for $P(N, E, t)$ allows us to find the time dependence for all characteristics of the dynamics of magnetization:

$$m_i(t) = N/2 \pm P(N, E, t)$$

and

$$\frac{1}{\omega_0}\frac{d\varphi_i}{dt} = m_i - \kappa m_j + \frac{m_i}{1-m_i^2}\left(\frac{E}{\omega_0} + \frac{N^2}{2} - (1+\kappa)(1+m_1 m_2)\right). \quad (41)$$

Let us analyze the general dynamics of magnetization in the simplest case $N = 0$, which, nevertheless, describes the main general features of this nonlinear dynamics. In this particular case at low energies $E < \omega_0$, the solution for magnetic moments has the form

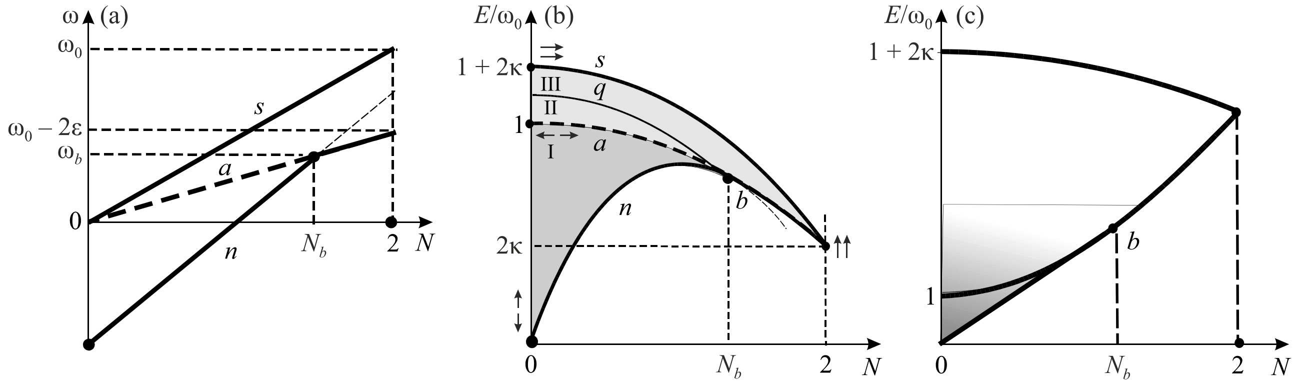

Fig. 3. (a) The dependences of stationary state frequencies on the number of spin deviations; (b) the domains of the existence for the solutions with different dynamics on the plane of the integrals of motion (E, N); (c) the same for $\kappa >> 1$.

$$m_{1,2} = \pm\sqrt{1-\frac{E}{\omega_0+2\varepsilon}}\,\mathrm{dn}\left(\sqrt{1+2\kappa-E/\omega_0}\,\omega_0 t, k\right),$$
$$k=\sqrt{\frac{2\kappa E}{\omega_0+2\varepsilon-E}}. \quad (42)$$

In the limit $E \to 0$ $(k \to 0)$, we obtain the linear polarized oscillations of the magnetization

$$m_{1,2} \approx \pm\left(1-E\frac{1-2\kappa+2\kappa\cos 2\omega_0 t}{2(\omega_0+2\varepsilon)}\right) \quad (43)$$

with the frequency $2\omega_0$. This is an additional oscillation to the azimuthal rotation with the frequency $\omega = \omega_0\sqrt{1+2\kappa}$ from (33): the general motion is two-frequency one.

In the opposite limit $E \to \omega_0$ $(k \to 1)$, the magnetization trajectory becomes aperiodic, and the static state noted in Fig. 3b as 1 becomes unstable:

$$m_{1,2} \to \pm\sqrt{\frac{2\kappa}{1+2\kappa}\frac{1}{\mathrm{cn}\left(\sqrt{2\kappa}\omega_0 t\right)}}. \quad (44)$$

This situation is similar to that considered in the previous section and corresponds to the separatrix E_s in Fig. 2b.

Finally, in the region III with $N = 0$ in Fig. 3b, the solution takes the form

$$m_{1,2} = \pm\sqrt{1-E/(\omega_0+2\varepsilon)}\,\mathrm{cn}\left(\sqrt{2\varepsilon E}t, k\right), \quad a_1 = a_2,$$
$$k=\sqrt{(\omega_0+2\varepsilon-E)/2\kappa E}. \quad (45)$$

In the limit $E \to \omega_0 + 2\varepsilon$, the linear small-amplitude oscillations

$$m_{1,2} = \pm\mu\cos\Omega t, \ \psi = \mu 2\sqrt{1+1/2\kappa}\sin\Omega t,$$

represent the synchronous rotation of the vectors $\mathbf{m}_i$ around a certain direction in XY plane with the frequency $\Omega = \omega_0\sqrt{2\kappa(1+2\kappa)}$. In this case, the frequency of the general rotation around the easy axis becomes zero.

Since usually in magnetic systems the exchange interaction substantially exceeds single-ion anisotropy, we consider the problem in the limit of large values of the parameter κ (see Fig. 3c). The parameter P is associated with the antiferromagnetism vector $\mathbf{l} = \mathbf{m}_1 - \mathbf{m}_2$, namely $l_z = 2P = 2\cos\vartheta$, where ϑ is the polar angle of the antiferromagnetic vector. In terms of the magnetization vector $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$ and the antiferromagnetic vector $\mathbf{l}$, introduced above, the Eqs. (29) take the form

$$d\mathbf{l}/dt = \varepsilon[\mathbf{l},\mathbf{m}] - (\omega_0/2)\left([\mathbf{m},\mathbf{n}](\mathbf{l},\mathbf{n}) + [\mathbf{l},\mathbf{n}](\mathbf{m},\mathbf{n})\right), \quad (46)$$

$$d\mathbf{m}/dt = -(\omega_0/2)\left([\mathbf{m},\mathbf{n}](\mathbf{m},\mathbf{n}) + [\mathbf{l},\mathbf{n}](\mathbf{l},\mathbf{n})\right), \quad (47)$$

where $\mathbf{n}$ is the unit vector along the easy axis.

In the commonly used approximation [15–17] the first equation defines the relationship

$$\mathbf{m} \approx -\left[\mathbf{l},\dot{\mathbf{l}}\right]/4\varepsilon. \quad (48)$$

In the spherical coordinates for $\mathbf{l} = 2(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, from (48) under the assumption $\mathbf{m}_1 \approx -\mathbf{m}_2$, it follows the expression for the integral of motion $M = N$:

$$M = m_z = m_1 + m_2 = -\frac{1}{\varepsilon}\sin^2\vartheta\frac{d\varphi}{dt}. \quad (49)$$

When substituting the value of $\mathbf{m}$ from (48) into (47), we obtain in the main approximation the equation

$$\frac{d^2\vartheta}{dt^2} + \left(2\omega_0^2\kappa - \left(\frac{d\varphi}{dt}\right)^2\right)\sin\vartheta\cos\vartheta = 0. \quad (50)$$

Using the expression for the angle φ from (49), after integration for the variable P we obtain the final equation

$$(dP/dt)^2 = 2\varepsilon\left(\omega_0 + \varepsilon M^2/2 - E\right) + 2\varepsilon(E - 2\omega_0)P^2 + 2\varepsilon\omega_0 P^4, \quad (51)$$

which coincides with Eq. (7) in the limit $\varepsilon >> \omega_0$. Therefore, the general solution of Eq. (50) follows from the solutions with parameters (38)–(40) in this limit. From the foregoing, it follows that the solutions of the approximate equations (50) are valid only in the shaded area in Fig. 3c with small energy of the system.

Conclusions

We investigated the nonlinear dynamics for the integrable systems of two identical coupled magnetic moments and paid attention to some interesting features of this dynamics. The most interesting facts consist in the appearance of the additional states with the average mismatching distribution of the energy between the degrees of freedom. This mismatching nonlinear mode appears in the bifurcation way at the critical value of the total energy. These states can be treated as the analogies of solitons in the system with the finite numbers of the degrees of freedom.

This work was supported by the scientific project of the National Academy of Sciences of Ukraine No. 4.17-N and the scientific program 1.4.10.26/F-26-4.

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Динаміка пари зв'язаних нелінійних систем. I. Магнітні системи

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В межах рівнянь Ландау–Ліфшиця для дискретних систем розглянуто динаміку двох класичних магнітних моментів, що моделюють слабкозв'язані магнітні нанодоти, шари квазідвовимірних магнетиків та двопідграткові магнетики. Знайдено й досліджено точні розв'язки динамічних рівнянь. Особливу увагу приділено дослідженню суттєво нелінійних неоднорідних станів з різним рівнем збудження ідентичних підсистем.

Ключові слова: нелінійні системи, рівняння Ландау–Ліфшица, магнітний резонанс, фазовий портрет, феромагнітні та антиферромагнітні взаємодії.