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Critical current of dc SQUID on Josephson junctions with unconventional current-phase relation

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Here we present calculations of the critical current of dc SQUID based on Josephson junction with unconventional current-phase relation. We analyzed two cases of current-phase relation of junction: with anharmonic and Majorana term. It is shown that the changing of the critical current in the case of the small geometrical inductance of dc SQUID based on Josephson junctions with unconventional current-phase relation is determined by the amplitude of the second term in current-phase relation, the geometrical inductance of dc SQUID and external magnetic field. In the case of the high inductance of dc SQUID unconventional terms can be ignored.

Keywords: dc SQUID, current-phase relation, anharmonic and Majorana terms.

Introduction

It is well known that dc SQUID consists of two Josephson junctions in parallel, including to superconducting loop (Fig. 1). Suppose that a magnetic flux Φ_e passes through the interior of the loop. Detail description of dc SQUID on conventional superconductor based Josephson junctions were presented in Refs. 1, 2. In the investigations of the dynamics of dc SQUID, the conventional current-phase relation of Josephson junctions [1, 2]

$$I = I_{c0} \sin\varphi \tag{1}$$

were used. The relationship $I = I_{c0} \sin\varphi$ is fulfilled with high accuracy for Josephson junctions on low-temperature superconductors [3]. In the case of Josephson junctions on high-temperature superconductors, the current-phase relation becomes anharmonic [4, 5]:

$$I = I_{c0} f_\alpha(\varphi) = I_{c0}(\sin\varphi + \alpha \sin 2\varphi) \ , \tag{2}$$

where anharmonicity parameter α depends on the junction preparation technology. In general, anharmonicity in the current-phase relation for high temperature and Fe-based superconductors based junctions are associated with the *d*-wave behavior of the order parameter and many band character of superconducting state in new compounds [6]. The ratio between the second and the first harmonics α ranges from 0 at temperature 900 mK to a saturated value 0.77 below 100 mK in a $YBa_2Cu_3O_7$ grain boundary biepitaxial Josephson junctions [4]. The anharmonic current-phase relation of junctions based on Fe-based superconductors widely discussed in Refs. 7–12. The origin of anharmonicity in such junctions related to many-band character of superconducting state and sign-reversal symmetry of order parameter [7–12]. Dynamical properties of single Josephson junctions with an anharmonic current-phase relation (2) were previously studied in Refs. 13–15.

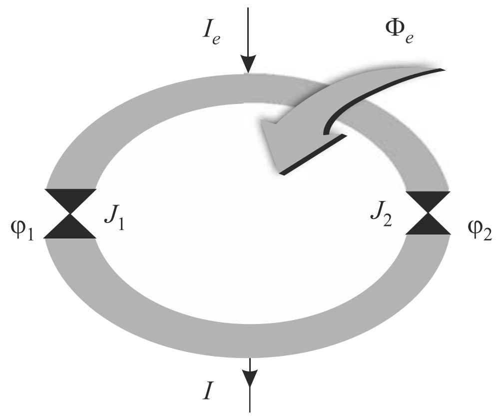

Fig. 1. Schematic presentation of dc SQUID.

In the case of Josephson junctions based on topological superconductors, the current-phase relation include additional fractional term [16–18]

$$I = I_{c0} f_m(\varphi) = I_{c0}\left[\sin\varphi + m\sin(\varphi/2)\right]. \quad (3)$$

The second term in Eq. (3) related with Majorana quasiparticles and dynamical detection of these particles seems very challenging in condensed matter physics. The discovery of Majorana fermions seems very interesting also from the point of fault-tolerant quantum computing [19]. The few papers devoted to the dynamical properties of single Josephson junctions with Majorana term (3) [20, 21]. In this study, we carried out the analysis of the critical current of the dc SQUID on Josephson junctions with unconventional current-phase relations (2), (3).

Basic equations

The dynamics of dc SQUID in the general case described by the system of equations [1] (see Fig. 1)

$$i_1 + i_2 = i_e$$

$$\varphi = \varphi_e - l(i_1 - i_2), \quad \varphi_e = \frac{2\pi\Phi_e}{\Phi_0}, \quad (4)$$

where Φ_0 is the quantum of magnetic flux. It is well known that [1] in the case of dc SQUID with identical Josephson junctions, the small total inductance of loop $l = 2\pi L I_c/\Phi_0 << 1$ and with current-phase relation (1) is equivalent to single Josephson junction with effective critical current $I_M = 2I_{c0}\cos[(\varphi_1 - \varphi_2)/2]$ and with effective phase $\varphi = (\varphi_1 + \varphi_2)/2$ (Fig. 1). The equation for the magnetic field can be written as

$$\varphi_1 - \varphi_2 = \varphi_e - 2l\sin\frac{\varphi_e}{2}\cos\varphi. \quad (5)$$

Taking into account Eq. (5), for external current i_e is the true relation

$$i_e = \frac{I_e}{I_{c0}} = 2\cos\frac{\varphi_e}{2}\sin\varphi + \frac{l}{2}\sin^2\frac{\varphi_e}{2}\sin 2\varphi. \quad (6)$$

It means that the inductance of the superconducting loop in dc SQUID causes additional electrodynamic anharmonicity to current-phase relation and should be taken into account [22, 23]. Using Eqs. (2), (3) for unconventional current-phase relation leads to final expressions for external current in symmetric dc SQUID

$$i = \frac{i_e}{2} = \cos\frac{\varphi_e}{2}\sin\varphi + \left(\frac{l}{4}\sin^2\frac{\varphi_e}{2} + \alpha\cos\varphi_e\right)\sin 2\varphi, \quad (7)$$

$$i = \frac{i_e}{2} = \cos\frac{\varphi_e}{2}\sin\varphi + \frac{l}{2}\sin^2\frac{\varphi_e}{2}\sin 2\varphi + m\cos\frac{\varphi_e}{4}\sin\frac{\varphi}{2}. \quad (8)$$

As followed from Eqs. (7) and (8), inclusion of inductance and unconventional terms in current-phase relation changes the amplitude of second harmonic.

Results and discussions

The presence of the second harmonic in current-phase relations (2), (3) leads to the renormalization of critical current I_{c0}. The critical current of a Josephson junction is a maximum value of the superconducting current $I_c/I_{c0} = \max f_{\alpha,m}(\varphi)$ [see Eqs. (2), (3)]. The critical current I_c of Josephson junction with unconventional current-phase relation was found in [24] as an analytical solution for the maximum point of the functions $f_{\alpha,m}(\varphi)$ [Eqs. (2), (3)] similarly to [25]. Calculations leads to the expression for the renormalized critical current I_c at small parameters $\alpha << 1$ and $m << 1$ [24]

$$\frac{I_c}{I_{c0}} = \begin{cases} 1 + 2\alpha^2, & \text{anharmonic term} \\ 1 + \dfrac{m}{\sqrt{2}} + \dfrac{7}{64}m^2, & \text{Majorana term} \end{cases}. \quad (9)$$

The critical current of a Josephson junction with unconventional current-phase relation for arbitrary values of parameters α and m calculated in Ref. 24. Last result means that all physical parameters will be renormalized, replacing critical current I_c by $I_{c0}\max f_{\alpha,m}(\varphi)$.

For small geometrical inductance of symmetrical dc SQUID $l << 1$, from Eqs. (7), (8) under small parameters $\alpha << 1$ and $m << 1$, we can obtain final expressions for the normalized critical current of dc SQUID

$$i_c = \left\{1 + 2\frac{\left(\dfrac{l}{4}\sin^2\dfrac{\varphi_e}{2} + \alpha\cos\varphi_e\right)^2}{\cos^2\dfrac{\varphi_e}{2}}\right\}\cos\frac{\varphi_e}{2}, \quad (10)$$

$$i_c = \left\{1 + 2\frac{l^2}{16}\frac{\sin^4\dfrac{\varphi_e}{4}}{\cos^2\dfrac{\varphi_e}{2}} + 0.71\frac{m\cos\dfrac{\varphi_e}{4}}{\cos\dfrac{\varphi_e}{2}}\right\}\cos\frac{\varphi_e}{2}. \quad (11)$$

In derivation of last Eqs. (10), (11), we use above presented results [see Eq. (9)] for critical current for Josephson junctions with unconventional current-phase relation [24]. For the small values of geometrical inductance l and different amplitude of anharmonicity parameter α, the result of numerical calculations of critical current i_c presented in Fig. 2. In the case current-phase relation with Majorana term (3) similar calculations leads to Fig. 3. It is clear that for the case of $\alpha = 0$ and $m = 0$ we have analytical result corresponding to classical result $i_c = \cos(\varphi_e/2)$ [1]. Also it is useful to note that obtained results are symmetrical in respect to axes i_c and obtained the whole picture is a periodical in respect to external the magnetic field φ_e with the period 2π. As followed from Figs. 2 and 3, the inclusion of geometrical and intrinsic anharmonicity to consideration changes behavior at $i_c(\varphi_e)$. Near $\varphi_e = \pi$ the amplitude of the first harmonic in current-phase relation tends to zero

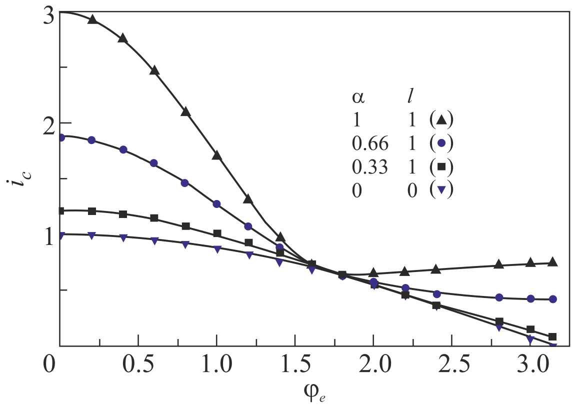

Fig. 2. Critical current of dc SQUID with small inductance l under parameters α from top to bottom.

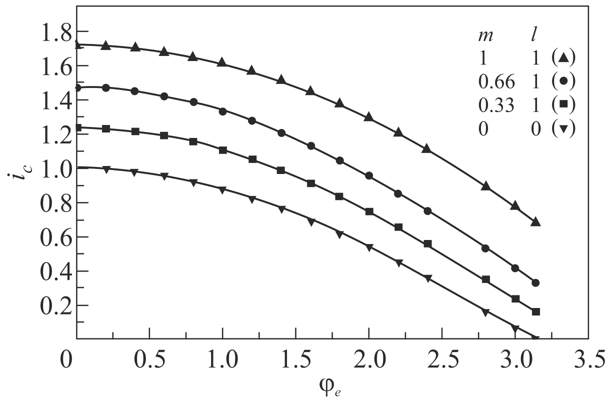

Fig. 3. Critical current of dc SQUID with small inductance l under parameters m from top to bottom.

and the critical current is determined by the amplitude of second harmonic $i_c \approx |l/4-\alpha|$. In the case of anharmonic current-phase relation (2), the increase of critical current arises at small external magnetic field $\varphi_e < \pi/2$ with increasing of anharmonicity parameter α, while for high magnetic field $\pi/2 < \varphi_e < \pi$ and high values of α we have "plateau-like" behavior in Fig. 2. In the case of current-phase relation (3), the dependence of critical current versus external magnetic field $i_c(\varphi_e)$ reveal more monotonic character similar to $i_c = \cos\varphi_e/2$ with higher amplitude in increasing of parameter m (Fig. 3). Calculations show that in all cases of current-phase relation [Eqs. (2) and (3)] the changing of the inductance of dc SQUID in the region $l < 1$ has small impact on presented results on Figs. 2 and 3. For the high values of inductance of dc SQUID $l >> 1$, Josephson inductance of junctions $\Phi_0/2\pi I_{c0}$ can be ignored in consideration of dynamical effects [1]. As a result, the phase of Josephson junctions on superconducting loop (Fig. 1) changes independently and in this limit is the true system of linear equations [1]

$$i_1 = \frac{i_e}{2} + \frac{\varphi_e}{l},$$
$$i_2 = \frac{i_e}{2} - \frac{\varphi_e}{l}. \quad (12)$$

If take into account the results of Ref. 24, the renormalization of critical current as $I_{c0} \max f_{\alpha,m}(\varphi)$ causes decreasing of Josephson inductance $\Phi_0/2\pi I_c$ approximately two times. It means that in dc SQUID with high geometrical inductance $l >> 1$, the unconventional effects in current-phase relations (2), (3) can be neglected.

Thus, in this study, the influence of unconventional current-phase relation of Josephson junction on the critical current i_c of dc SQUID was investigated. Renormalization of critical current in junctions with anharmonic and Majorana terms under external magnetic field was taken into account in the limit of the small geometrical inductance of dc SQUID. General expressions under external magnetic field were obtained in this limit $l < 1$, which coincide with the conventional current-phase relation under parameters $\alpha = 0$ and $m = 0$. In the opposite case of high geometrical inductance $l >> 1$, the unconventional effects in current-phase relation is negligibly small.

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Критичний струм dc джозефсонівського СКВІДа з нетрадиційним співвідношенням струм–фаза

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Представлено розрахунки критичного струму dc джозефсонівського СКВІДа з нетрадиційним струм-фазовим співвідношенням. Проаналізовано два випадки співвідношення струму й фази з'єднання: ангармонійним та майоранівським внесками. Показано, що зміна критичного струму у випадку малої геометричної індуктивності джозефсонівського СКВІДа постійного струму з незвичайним співвідношенням струм–фаза визначається амплітудою другого члена в цьому співвідношенні, геометричною індуктивністю цього СКВІДа та зовнішнім магнітним полем. У випадку високої індуктивності dc СКВІДа постійного струму незвичайні умови можна ігнорувати.

Ключові слова: dc СКВІД, співвідношення струму й фази, ангармонійні та майоранівські внески.