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Superfluid stars and Q-balls in curved spacetime

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Within the framework of the theory of strongly-interacting quantum Bose liquids, we consider a general relativistic model of self-interacting complex scalar fields with logarithmic nonlinearity taken from dense superfluid models. We demonstrate the existence of gravitational equilibria in this model, described by spherically symmetric nonsingular finite-mass asymptotically-flat solutions. These equilibrium configurations can describe both massive astronomical objects, such as bosonized superfluid stars or cores of neutron stars, and finite-size particles and non-topological solitons, such as *Q*-balls. We give an estimate for masses and sizes of such objects.

Keywords: quantum Bose liquid, superfluidity in cold stars, *Q*-ball, logarithmic scalar gravity.

1. Introduction

In the hierarchy of superdense stars, various objects exist, which occupy an intermediate place between neutron stars and black holes. To a distant observer, such objects would look almost like black holes; but they have no horizon, for which reason they are often aggregated under the name of compact stars (CS) and black hole mimickers (BHM). Widely known examples of such objects include geons, quark stars, gravastars, and the most popular of them, boson stars, which have long since been studied [1–5]. In this paper, we propose another type of CS/BHM-type bosonic objects — superfluid stars, which are modeled by the scalar field with logarithmic nonlinearity motivated by the theory of strongly-interacting dense superfluids. Furthermore, we demonstrate that our model can also describe composite particle-like objects, or *Q*-balls, which self-interact and curve spacetime.

Because superfluids are macroscopic wave-mechanical objects, the stability of superfluid stars against gravitational collapse is expected to be enhanced by the uncertainty principle, similar to boson star models. Moreover, superfluidity introduces an additional effect here. The inviscid flow, caused by suppression of dissipative fluctuations, makes the fluid parcels and volume elements more resistant to coming to a full stop and adhering to each other. Therefore superfluid stars are expected to have a larger degree of resistance to gravitational collapse than the conventional boson stars.

Logarithmic Bose liquids are nonlinear effective models, which have successfully been used to describe laboratory superfluids, such as the helium II phase [6–8]. Unlike models with quartic nonlinearity, such as the Gross–Pitaevskii one, logarithmic models go well beyond the two-body interaction approximation. In fact, logarithmic nonlinearity occurs in a leading-order approximation of any strongly-interacting many-body systems, which can be described in terms of collective degrees of freedom by a single wavefunction, and for which typical interparticle interaction potentials are much larger than kinetic energies [9]. This nonlinearity also occurs in a theory of superflow-induced spacetime and emergent gravity [10–12].

While the original logarithmic models are nonrelativistic, their Lorentz-symmetric analogues are straightforward to construct. Relativistic logarithmic scalar fields are known to possess a dilatation symmetry [13], which makes them universally usable over a large range of length and mass scales. This feature, together with the superfluid-enhanced stability against gravitational collapse, could result in the substantial increase of the maximal mass of logarithmic superfluid stars, which is a conjecture to be checked, among other things, in this paper.

2. The model

Adopting the units $c = 1$ and metric signature $(-+++)$, we write a classical Lagrangian of the model:

$$\mathcal{L} = \frac{R}{16\pi G} - \frac{1}{2}\nabla_\mu \phi^* \nabla^\mu \phi - V(\phi, \phi^*), \tag{1}$$

where the potential of a minimally coupled complex scalar field ϕ is given by

$$V(\phi, \phi^*) = -b|\phi|^2 \left[\ln\left(|\phi|^2 / a\right) - 1\right], \tag{2}$$

where a and b are constant parameters. These parameters have a different physical meaning to parameters in other relativistic scalar field theories, such as the ϕ^4 model [5].

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Their values are not determined by the physics of point particles, but by the properties of a macroscopic wave-mechanical object with collective degrees of freedom, which Bose liquids and condensates are [6].

Correspondingly, nonlinear coupling b is a linear function of the wave-mechanical temperature T_Ψ, which is defined, in a non-relativistic regime, as a thermodynamic conjugate of the Everett–Hirschman's information entropy,

$$S_\Psi = -\langle \Psi \,|\, \ln(|\,\Psi\,|^2/\bar{\rho}) \,|\, \Psi \rangle = -\int |\,\Psi\,|^2 \ln(|\,\Psi\,|^2/\bar{\rho}) d^3\mathbf{x},$$

where $\Psi = \Psi(\mathbf{x}, t)$ is condensate wavefunction, constant $\bar{\rho}$ is the decoupling density (a non-relativistic analogue of a, which compensates dimensionality of condensate wavefunction inside the logarithm), and the integral is taken over the volume occupied by the liquid, further details can be found in Ref. 14.

Thus, b is not *a priori* assumed parameter of the model, but a value related to the wave-mechanical thermodynamical properties of the system and its environment

$$b \sim T_\Psi - T_\Psi^{(0)}, \tag{3}$$

where $T_\Psi^{(0)}$ is a reference or critical value [8, 9, 15].

Since the potential (2) is defined up to an overall sign, we choose this sign so that, under our signature and curvature conventions, the scalar field equation would have a static Gaussian solution in the Minkowski spacetime limit [16]. This would correspond to selecting the non-topological soliton sector of the model. A detailed discussion of topological structure can be found in Refs. 8, 11, 15. We assume that $b > 0$ in what follows.

Furthermore, field equations can be derived from the Lagrangian (1) in a standard way:

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = 8\pi G T^\mu_\nu, \tag{4}$$

where the energy-momentum tensor is determined by complex scalar field:

$$T^\mu_\nu = \frac{1}{2} g^{\mu\sigma}\left(\nabla_\sigma \phi^* \nabla_\nu \phi + \nabla_\sigma \phi \nabla_\nu \phi^*\right) - $$
$$-\frac{1}{2}\delta^\mu_\nu \left(g^{\alpha\beta}\nabla_\alpha \phi^* \nabla_\beta \phi + 2V(\phi, \phi^*)\right). \tag{5}$$

The complex scalar field equation can be extracted from these equations via Bianchi identities or by varying Lagrangian (1) with respect to scalar field, and written in a form

$$\nabla_\mu \nabla^\mu \phi = 2\frac{\partial}{\partial \phi^*} V(\phi, \phi^*) = -2b \ln(|\,\phi\,|^2/a)\phi. \tag{6}$$

This equation is a Lorentz-covariant analog of the logarithmic Schrödinger equation, which was extensively studied: to mention only a few very recent works [17–22]. Relativistic wave equations with logarithmic nonlinearity have also been extensively studied in the past, assuming fixed Minkowski spacetime [10, 11, 16, 23, 27].

Let us consider now spherically symmetric and time-independent solutions of Einstein field Eq. (4). The line element can be written in static coordinates:

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{7}$$

and, due to a symmetry of the problem, we assume our scalar to be spherically symmetric and stationary

$$\phi(r, t) = \mathrm{e}^{-i\omega t}\Phi(r), \tag{8}$$

where $\Phi(r)$ is a real-valued function. Correspondingly, field Eqs. (4) and (6) reduce to a system of three ordinary differential equations:

$$\frac{A'}{A^2 x} + \frac{A-1}{Ax^2} = 2\left[\frac{\Omega^2}{B} + 1 - \ln(\sigma^2/k)\right]\sigma^2 + \frac{(\sigma')^2}{A}, \tag{9}$$

$$\frac{B'}{ABx} - \frac{A-1}{Ax^2} = 2\left[\frac{\Omega^2}{B} - 1 + \ln(\sigma^2/k)\right]\sigma^2 + \frac{(\sigma')^2}{A}, \tag{10}$$

$$\sigma'' + \left(\frac{2}{x} - \frac{A'}{2A} + \frac{B'}{2B}\right)\sigma' + 2A\left[\frac{\Omega^2}{B} + \ln(\sigma^2/k)\right]\sigma = 0, \tag{11}$$

where $x = r/L$, $L = 1/\sqrt{b}$, $\sigma = (4\pi G)^{1/2}\Phi$, $k = 4\pi G a$, $\Omega = \omega/\sqrt{2b}$, and a prime denotes d/dx. If we define

$$A(x) = \left[1 - 2\mathcal{M}(x)/x\right]^{-1}, \tag{12}$$

we can replace Eq. (9) with

$$\mathcal{M}'(x) = x^2\left\{\left[\frac{\Omega^2}{B} + 1 - \ln(\sigma^2/k)\right]\sigma^2 + \frac{(\sigma')^2}{2A}\right\}, \tag{13}$$

and deal with a dimensionless mass function $\mathcal{M}(x)$ in actual computations.

3. Numerical analysis

We numerically solve Eqs. (9)–(13) using the shooting method [3, 5]. According to this method, values of parameters are chosen in such a way that our scalar field vanishes at spatial infinity, and has no nodes and singularities. Then, nonsingular finite-mass solutions of Eqs. (9)–(13) are expected to describe the lowest-energy bound states. Thus, Eqs. (9)–(13), together with the initial conditions $\sigma(0) = \sigma_0$, $\sigma'(0) = 0$, $\mathcal{M}(0) = 0$, and $B(0) = B_0$, are regarded as an eigenvalue problem for Ω and B_0. Once these are computed, the total mass can be derived from an asymptotic value of $\mathcal{M}(x)$: $M = \mathcal{M}(\infty)L/G = \mathcal{M}(\infty)/(G\sqrt{b})$.

Computations indicate that: when parameter k goes above a certain critical value $\tilde{k}$, then a level splitting occurs at $\sigma_0 \geqslant \tilde{\sigma}_0$, where $\tilde{\sigma}_0$ is a critical value of the central scalar field. We have numerically established that

$$\tilde{k} \approx 0.066, \tag{14}$$

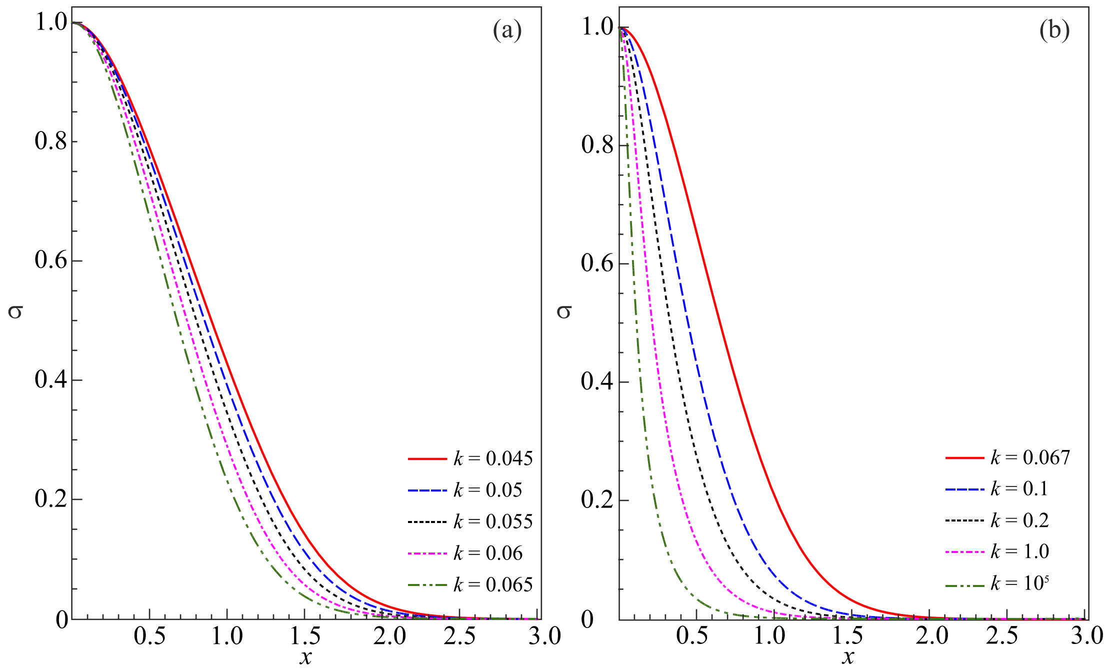

Fig. 1. Function $\sigma(x)$ at $\sigma_0 = 1$ for different values of k: $k < \tilde{k}$ (a), $k > \tilde{k}$ (b).

whereas the value $\tilde{\sigma}_0$ is different for each k. Therefore, in our computations, we restrict ourselves to the region $\sigma_0 < \tilde{\sigma}_0$, which is sufficient for the purposes of this paper.

A profile of the scalar field is shown in Fig. 1, assuming the central field value to be equal to one. The field rapidly decays at spatial infinity (in the flat-spacetime limit it would have an exact Gaussian shape), it has no nodes and singular points.

Profiles of the dimensionless mass function $\mathcal{M}(x)$ and relativistic $2\mathcal{M}(x)/x$, are shown in Figs. 2 and 3, respectively. Unlike the scalar field, their behavior pattern changes as k goes across the critical value $\tilde{k}$. However, in both cases, it indicates that the solutions remain nonsingular, finite-mass, and horizon-free.

The effective compactness of our solution can be defined as the ratio $\mathcal{M}_{99}/x_{99}$, where $\mathcal{M}_{99} = 0.99\mathcal{M}(\infty)$, and

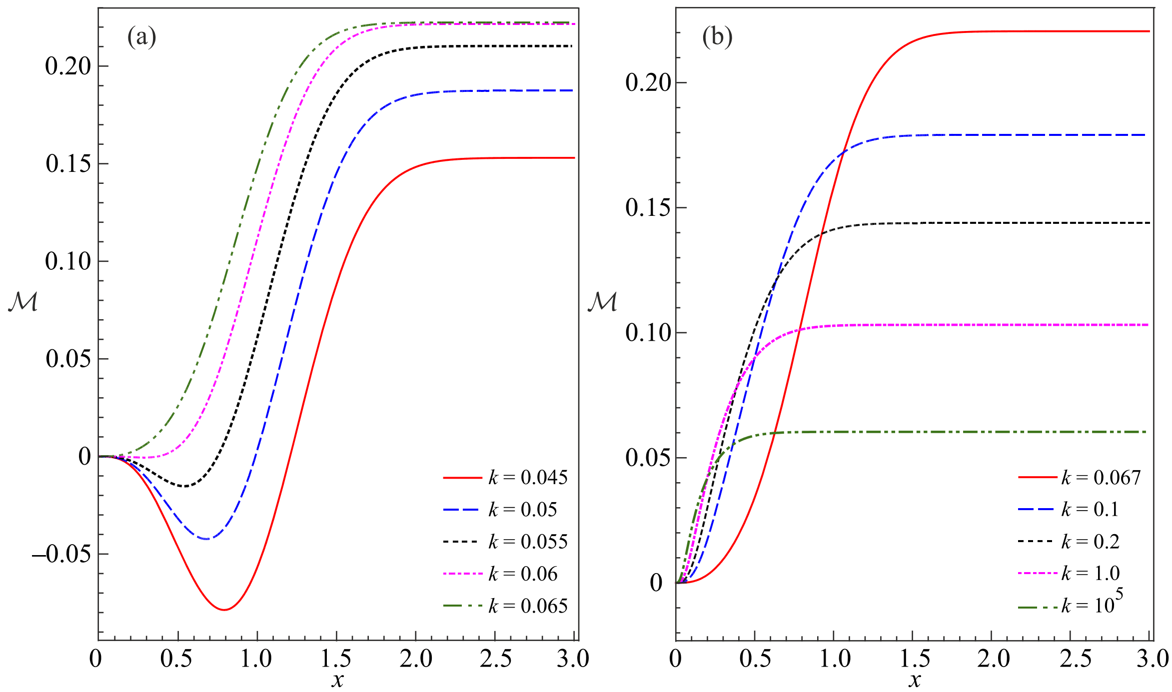

Fig. 2. Function $\mathcal{M}(x)$ at $\sigma_0 = 1$ for different values of k: $k < \tilde{k}$ (a), $k > \tilde{k}$ (b).

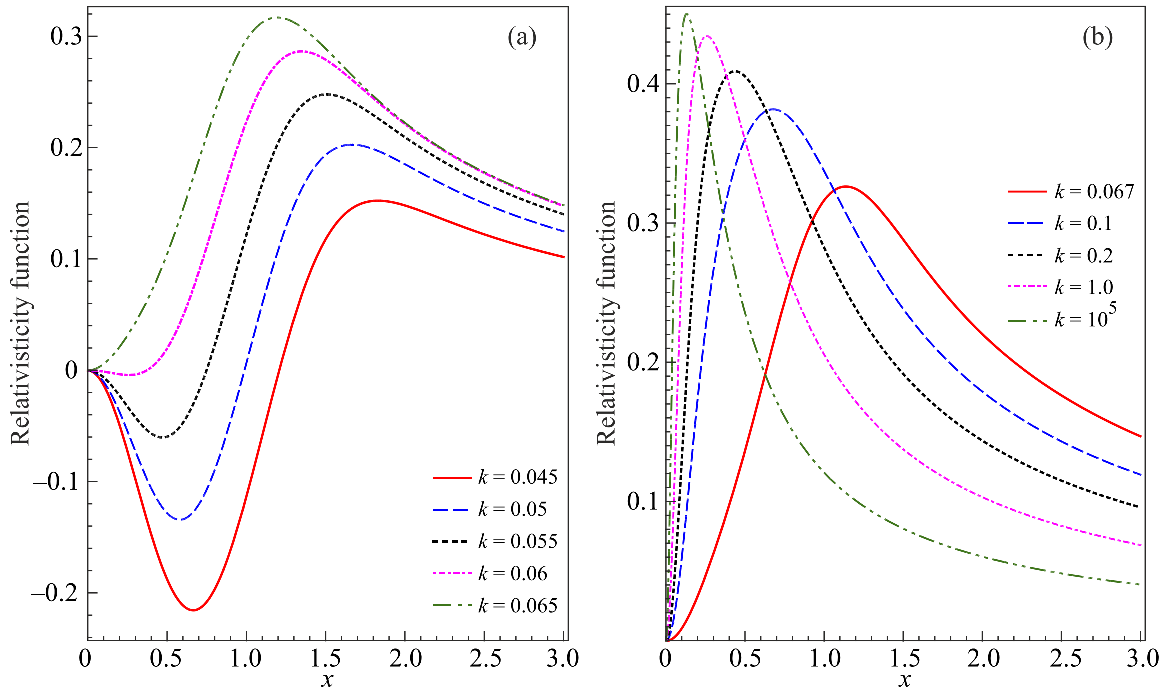

Fig. 3. Relativisticity function at $\sigma_0 = 1$ for different values of k: $k < \tilde{k}$ (*a*), $k > \tilde{k}$ (*b*).

x_{99} is the dimensionless radius containing $\mathcal{M}_{99}$. Its profile, shown in Fig. 4, is similar to that of boson stars formed by strongly self-interacting scalar fields [24]. The figure suggests that maximum compactness is achieved in the models in which k approaches the $\tilde{k}$ value.

From these figures one can estimate that most of the field's energy and star's mass are localized inside the radius

$$R \approx \alpha L \sim b^{-1/2}, \tag{15}$$

where α is a number of order one (its exact value depends on k and σ_0). Therefore, in general relativity, logarithmic superfluid tends to form lumps whose dimensions scale as $b^{-1/2}$. What about the bounds on their mass?

Profiles of the asymptotic values of the asymptotic $\mathcal{M}(\infty)$ as a function of central field value and frequency are given, respectively, in Figs. 5 and 6. Furthermore, Fig. 7 is a summarizing plot of the maximum values of $\mathcal{M}(\infty)$.

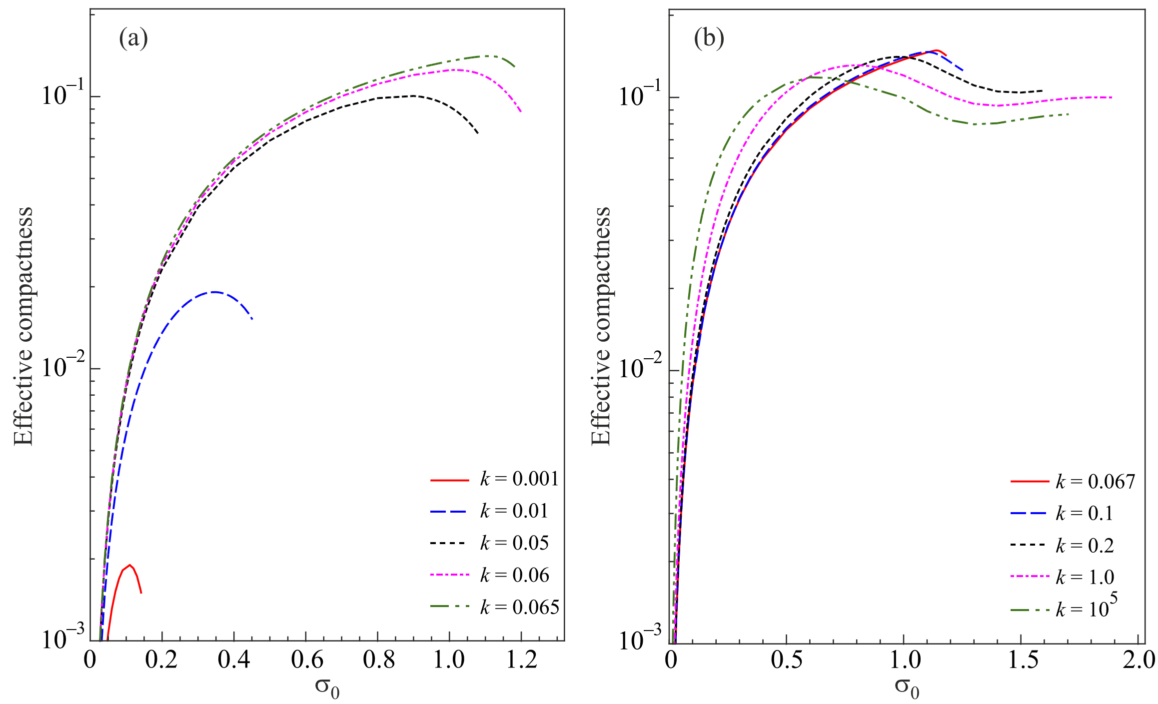

Fig. 4. Effective compactness versus central field σ_0 for different values of k: $k < \tilde{k}$ (*a*), $k > \tilde{k}$ (*b*).

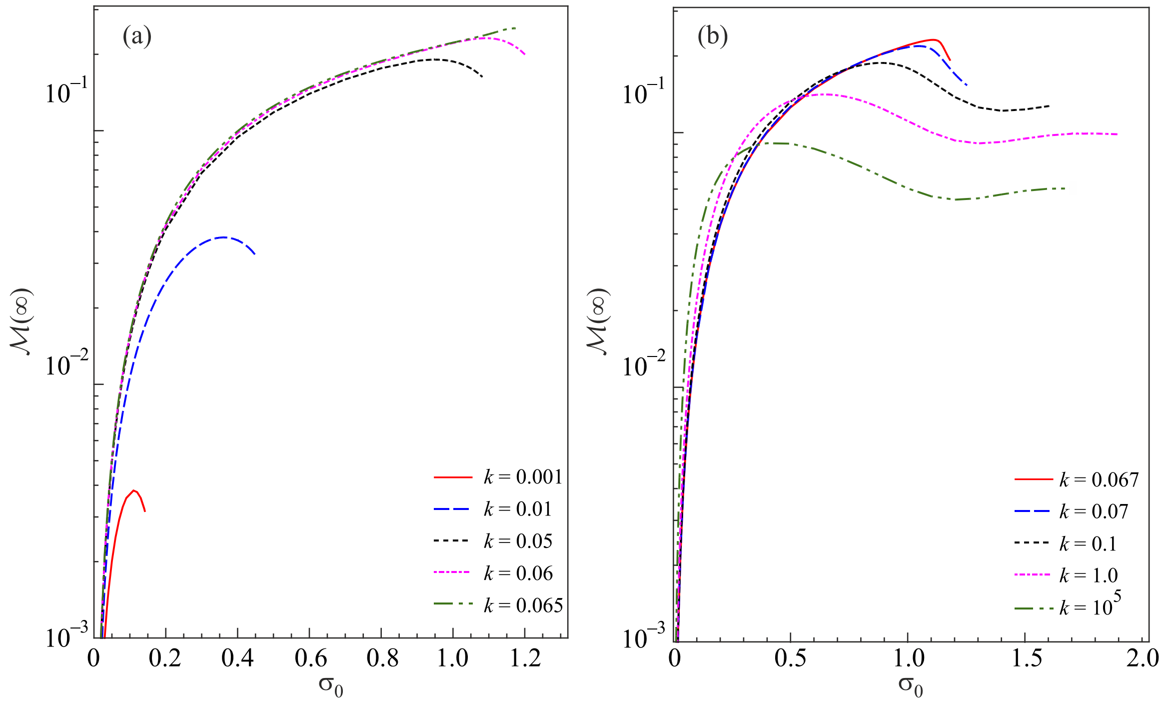

Fig. 5. Asymptotic value $\mathcal{M}(\infty)$ versus σ_0 for different values of k: $k < \tilde{k}$ (*a*), $k > \tilde{k}$ (*b*).

One can see that the absolute maximum for $M(\infty)$ is achieved at $k \to \tilde{k}$, which yields the relation

$$M_{\max} \equiv \max[\mathcal{M}(\infty)]L/G \approx (4G\sqrt{b})^{-1}. \qquad (16)$$

Therefore, masses of equilibrium configurations of the logarithmic model scale as an inverse the square root of nonlinear coupling, similarly to the sizes, cf. Eq. (15).

4. Conclusions

It is shown that in general relativity, self-interacting logarithmic scalar fields can form equilibria, which are described by nonsingular, finite-mass, and horizon-free solutions. According to Eqs. (15) and (16), their characteristic scales of mass and size are determined by the nonlinear coupling: $M \sim b^{-1/2}$ and $R \sim b^{-1/2}$. Since this coupling

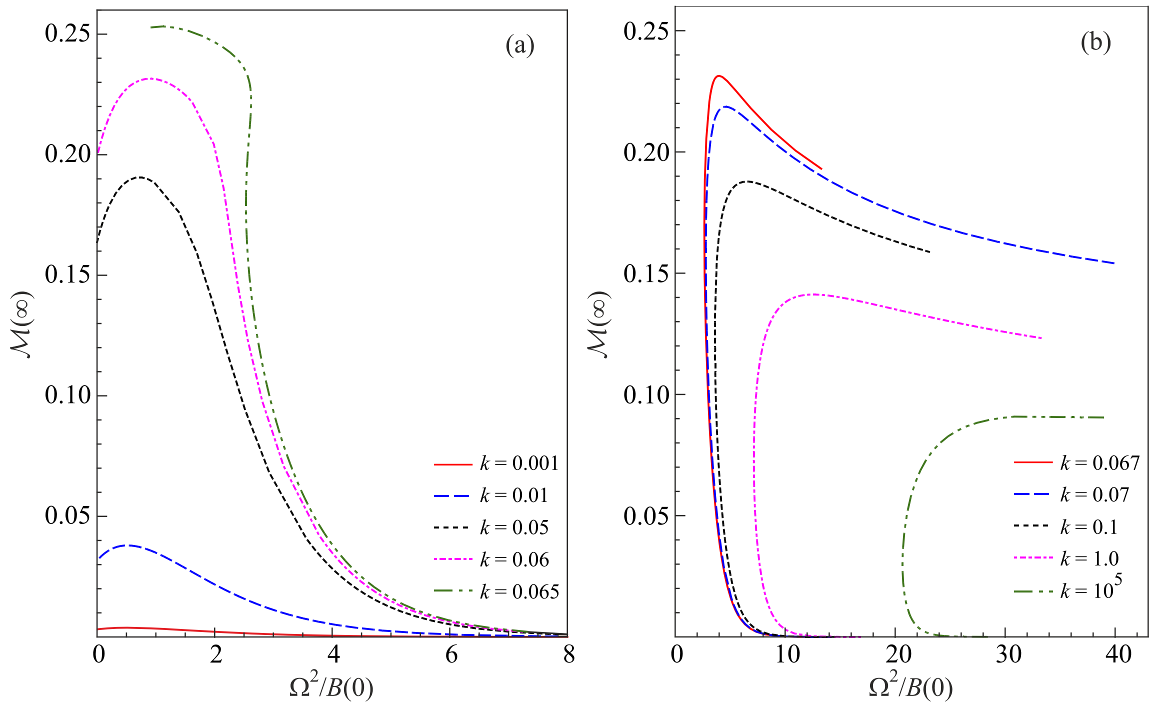

Fig. 6. Asymptotic value $\mathcal{M}(\infty)$ versus $\Omega^2 / B(0)$, for different values of k: $k < \tilde{k}$ (*a*), $k > \tilde{k}$ (*b*).

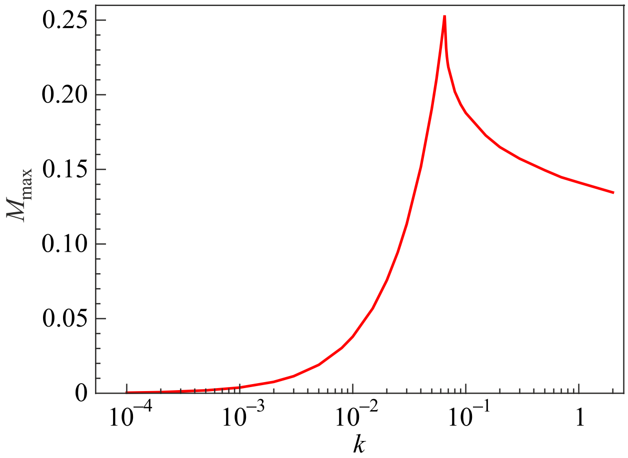

Fig. 7. Profile of $M_{\max} = \max[\mathcal{M}(\infty)]$ versus k.

has no known bounds other than being positive-definite, see discussion in between Eqs. (2) and (4), both mass and size of the equilibrium configurations have no upper and lower bounds. Therefore, our model can be used to describe objects with lengths and masses of a wide scale-range, which agrees with the dilatation symmetry mentioned above.

If b is small, then our model refers to astronomical-scale CS/BHM objects, such as superfluid stars and cores of neutron stars [25, 26]. For example, let us assume that our superfluid star has the maximum mass which is equal to the mass of Sun, $M_\odot \approx 2 \cdot 10^{33}$ g. Then from Eq. (16), we obtain

$$b_\odot \equiv b(M_{\max} = M_\odot) = (4GM_\odot)^{-2} \approx 2.8 \cdot 10^{-12}\, \text{cm}^{-2}.$$

Correspondingly, Eq. (15) gives us an estimate for the radius of such a star:

$$R_{M_\odot} = \alpha / \sqrt{b_\odot} \approx \alpha (2.8 \cdot 10^{-12})^{-1/2}\, \text{cm} \sim \alpha (6 \cdot 10^5)\, \text{cm} \lesssim 10\, \text{km}.$$

When b is large, then the model describes composite particle-like objects of finite size, such as Q-balls, which self-interact and curve spacetime (logarithmic Q-balls in fixed flat spacetime were studied in Refs. 16, 11, 27, 28).

Furthermore, if we assume wave-mechanical temperature to be proportional to the thermal one, $T_\Psi \propto T$, then Eqs. (3) and (16) suggest that mass of logarithmic superfluid stars and Q-balls scales as an inverse square of temperature, $M \sim (T - T_c)^{-1/2}$, where $T_c \propto T_\Psi^{(0)}$. It means that temperature of these objects should depend on their mass as

$$T \sim M^{-2}, \qquad (17)$$

up to an additive constant T_c. If this constant is close to absolute zero, then massive superfluid stars alone must be cold objects with low evaporation rate by thermal radiation, whereas logarithmic Q-balls have a longer lifetime in thermally hot and dense environments.

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Надплинні зірки та *Q*-кулі у криволінійному просторі

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За допомогою теорії сильно взаємодіючих квантових бозе-рідин розглянуто загальну релятивістську модель самовзаємодіючих комплексних скалярних полів з логарифмічною нелінійністю, яку взято з щільних надплинних моделей. Показано існування гравітаційних рівноваг у цій моделі, які описані сферично-симетричними несінгулярними асимптотично плоскими рішеннями кінцевої маси. Ці рівноважні конфігурації можуть описувати як масивні астрономічні об'єкти, такі як бозонні надплинні зірки або ядра нейтронних зірок, так і частинки кінцевих розмірів, а також нетопологічні солітони, такі як *Q*-кулі. Надано оцінку мас та розмірів таких об'єктів.

Ключові слова: квантова бозе-рідина, надплинність у холодних зірках, *Q*-куля, логарифмічна скалярна гравітація.