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Return current of dc SQUID based on tunnel Josephson junctions with unconventional current-phase relation

I. N. Askerzade

Department of Computer Engineering and Center of Excellence of Superconductivity Research, Ankara University Ankara 06100, Turkey
E-mail: imasker@eng.ankara.edu.tr

Institute of Physics of the Azerbaijan National Academy of Sciences, Baku AZ1143, Azerbaijan

R. Askerbeyli

Karabuk University, Department of Business Administration, Karabuk, Turkey

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We carried out the analysis of the return current of dc SQUID based on tunnel Josephson junction with unconventional current-phase relation. We analyzed two cases of current-phase relation with additional terms to the first harmonic sin φ: a case of the second harmonic sin 2φ and the case of the fractional term sin (φ/2). It is shown that the changing of the return current of dc SQUID on junctions with unconventional current-phase relation is determined by the amplitude of the second term in current-phase relation, geometrical inductance, and external magnetic field.

Keywords: dc SQUID, current-phase relation, return current, anharmonic and fractional terms.

Introduction

It is well known that the dynamics of Josephson junction for the case of harmonic current-phase relation $I = I_c \sin\varphi$ is given by the equation of simple resistive model [1, 2]

$$\beta\ddot{\varphi} + \dot{\varphi} + \sin\varphi = i_e, \tag{1}$$

where ie is normalized external current via Josephson junction in units of critical current I_c, dots over φ corresponds to the derivative in respect to dimensionless time $\Phi_0 / (2\pi I_c R_N)$, Φ_0 is the magnetic flux quantum. In Eq. (1) notation β is the McCumber parameter of Josephson junction $\beta = (2e/\hbar) I_c R_N^2 C$, which determines the size of hysteresis in current-voltage characteristic. It is well known, that the case of $\beta >> 1$ corresponds to tunnel junction [1, 2]. In the case of $\beta << 1$, hysteresis on current-voltage characteristic is absent and in Eq. (1) the first term can be neglected.

Sinusoidal current-phase relation is fulfilled with high accuracy for Josephson junctions based on low-temperature superconductors [2]. In the limit of tunnel junction (high capacitance limit) $\beta >> 1$, the numerical solution of Eq. (1) shows that the current-voltage characteristic has two separate branches: *S* (superconducting) and *R* (resistive) branches. The important parameter of the current-voltage characteristic is the return current I_R, at which switching from *R*-state to *S*-state arises [1, 2]. Calculation of return current I_R using the simple resistive model of Josephson junction for tunnel case $\beta >> 1$ leads to result [1],

$$I_R = \frac{4}{\pi\sqrt{\beta}} I_c \,. \tag{2}$$

This result is in good agreement for tunnel junction based on low-temperature superconductors [1]. The presence of return current on current-voltage characteristic leads to lowering of the clock frequency in latching logic circuits based on tunnel Josephson junctions [1, 2].

Very recently in Ref. 3, the analysis of the return current was carried out for a single tunnel junction with unconventional current-phase relation. Two types on the current-phase relation with additional terms to sinusoidal current were analyzed: the case of second harmonic $\sin 2\varphi$ (anharmonic case) and the case of the fractional term $\sin(\varphi/2)$. It can be stated that the return current of the Josephson junction decreased by increasing the amplitude of the second term in the current-phase relation. The case of the second harmonic $\sin 2\varphi$ in current-phase relation was experimentally observed in Josephson junctions based on high-temperature superconductors [4, 5]. The amplitude of the anharmonic term in current-phase relation α depends on the junction preparation technology. In general, anharmonicity in the current-phase relation for high temperature and Fe-based superconductors based junctions are associated with the *d*-wave behavior of the order parameter and many band character of superconducting state in new compounds [6–10]. Dynamical properties of single Josephson junctions with an anharmonic current-phase relation were previously studied in Refs. 11–14.

The fractional term in current-phase relation $\sin(\varphi/2)$ related with Majorana quasi-particles [15–17] and dynamical detection of these particles seems very challenging in condensed matter physics. The discovery of Majorana fermions seems very interesting also from the point of fault-tolerant quantum computing [18]. There are few papers devoted to dynamical properties of single Josephson junctions with the fractional term [19, 20]. Study of properties of dc SQUID on junctions with unconventional current-phase relation presented in Ref. 21. In this paper critical current of dc SQUID on junctions with unconventional current-phase relation is calculated. In this study, we carried out the analysis of the return current I_R of the dc SQUID on tunnel Josephson junctions with unconventional current-phase relation.

Basic Equations

It is well known that [1, 2] in the case of low inductance symmetrical dc SQUID $l = 2\pi(LI_c/\Phi_0) << 1$ (L is the total inductance) on junctions with sinusoidal current-phase relation is equivalent to the single junction with effective critical current $I_M = 2I_c\cos[(\varphi_1-\varphi_2)/2]$ and with effective phase $\varphi = (\varphi_1+\varphi_2)/2$ (Fig. 1). The equation for magnetic field can be written as

$$\varphi_1-\varphi_2 = \varphi_e - l\sin\frac{\varphi_e}{2}\cos\varphi, \quad \varphi_e = \frac{2\pi\Phi_e}{\Phi_0}. \tag{3}$$

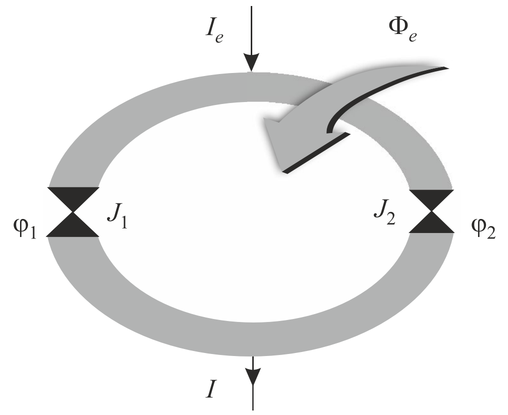

Fig. 1. Schematic presentation of dc SQUID.

Taking into account Eq. (3) leads to relation for external current in dc SQUID

$$i_e = \frac{I_e}{I_c} = 2\cos\frac{\varphi_e}{2}\sin\varphi + l\sin^2\frac{\varphi_e}{2}\sin 2\varphi. \tag{4}$$

It means that the inductance of the superconducting loop in dc SQUID causes additional electrodynamic anharmonicity to the current-phase relation and should be taken into account [22, 23]. Use of unconventional current-phase relation with the second term (in the form of $\sin 2\varphi$ (anharmonic) or $\sin(\varphi/2)$ (fractional) leads to corresponding final expressions for external current in a symmetric dc SQUID:

$$i = \frac{i_e}{2} = \cos\frac{\varphi_e}{2}\sin\varphi + \left(\frac{l}{2}\sin^2\frac{\varphi_e}{2} + \alpha\cos\varphi_e\right)\sin 2\varphi, \tag{5a}$$

$$i = \frac{i_e}{2} = \cos\frac{\varphi_e}{2}\sin\varphi + \left(\frac{l}{2}\sin^2\frac{\varphi_e}{2} + m\cos\frac{\varphi_e}{4}\right)\sin\frac{\varphi}{2}\,. \tag{5b}$$

For the calculation of return current I_R of dc SQUID based on tunnel Josephson junction with unconventional current-phase relation, we neglect damping effects in the junctions, i.e., β^{-1} could be considered as a small parameter [3]. In this approximation the energy of tunnel junction with anharmonic current-phase relation (5a), under small external current $i_e << 1$, can be written as

$$E = E_c\left\{\frac{\beta\dot\varphi^2}{2} + (1-\cos\varphi)\cos\frac{\varphi_e}{2} + \left(\frac{l}{4}\sin^2\frac{\varphi_e}{2} + \frac{\alpha}{2}(1-\cos 2\varphi)\right)\right\} \tag{6a}$$

with Josephson energy $E_c = \Phi_0 I_c/2\pi$. In the case of the fractional term $\sin(\varphi/2)$, a similar energy can be presented as

$$E = E_c\left\{\frac{\beta\dot\varphi^2}{2} + (1-\cos\varphi)\cos\frac{\varphi_e}{2} + \left[\frac{l}{4}\sin^2\frac{\varphi_e}{2} + 2m(1-\cos\frac{\varphi}{2})\right]\right\}. \tag{6b}$$

In the general case, the loss of energy by the resistance of the junction with unconventional current-phase relation, for the period of the phase T, can be calculated as [1, 3]

$$W_i = \frac{\Phi_0 I_c}{T}\int_0^T \dot\varphi(\varphi)d\varphi\,. \tag{7a}$$

The energy, flowing from the current source W_e for $I = I_R$, can be written as

$$W_e = \Phi_0 I_R\,. \tag{7b}$$

The dependence $\dot\varphi(\varphi)$ for different current-phase relation can be found from (6a) and (6b) correspondingly as

$$\dot{\varphi}(\varphi)=\left(\frac{2}{\beta}\right)^{1/2}\left\{\frac{E}{E_c}-(1-\cos\varphi)\cos\frac{\varphi_e}{2}-\left[\frac{l}{4}\sin^2\frac{\varphi_e}{2}+\frac{\alpha}{2}(1-\cos 2\varphi)\right]\right\}^{1/2}, \qquad (8a)$$

$$\dot{\varphi}(\varphi)=\left(\frac{2}{\beta}\right)^{1/2}\left\{\frac{E}{E_c}-(1-\cos\varphi)\cos\frac{\varphi_e}{2}-\left[\frac{l}{4}\sin^2\frac{\varphi_e}{2}+2m(1-\cos\frac{\varphi}{2})\right]\right\}^{1/2}. \qquad (8b)$$

As followed from last Eqs. (8a) and (8b), if $l = 0$, $m = 0$, and $\alpha = 0$ the infinite motion of Josephson phase φ is possible for the case $E/E_c = 2$, which corresponds to the return current I_R.

The expression for the normalized return current i_R of dc SQUID on Josephson junction with unconventional current-phase relation taking into account (8a), and (8b) can be written as

$$i_R=\frac{I_R}{I_c}=\frac{1}{T}\int_0^T \dot{\varphi}(\varphi)d\varphi, \qquad (9)$$

similar to results that have been obtained in Ref. 3. In calculations of normalized return current i_R with anharmonic current-phase relation (5a), we use a period of phase $T = 2\pi$, for the case of fractional term (5b) period will be $T = 4\pi$.

Results and Discussions

Integral in last Eq. (9) with (8a) and (8b) for low inductance symmetrical dc SQUIDs $l << 1$, can be calculated analytically for small parameters $\alpha << 1$ and $m << 1$. In the limit $\alpha << 1$ and $m << 1$, the ratio E/E_c can be considered equal to 2, as in the case of harmonic current-phase relation. Then we obtain final expressions for the normalized return current i_R for small values of the amplitude of additional term $\alpha < 0.5$ and $m < 0.5$.

$$i_R=\sqrt{\left|\cos\frac{\varphi_e}{2}\right|}\left\{1-\frac{l}{6}\frac{\sin^2\left(\frac{\varphi_e}{2}\right)}{\cos\left(\frac{\varphi_e}{2}\right)}-\frac{\alpha}{3}\frac{\cos(\varphi_e)}{\cos\left(\frac{\varphi_e}{2}\right)}\right\}, \qquad (10a)$$

$$i_R=\sqrt{\left|\cos\frac{\varphi_e}{2}\right|}\left\{1-\frac{l}{6}\frac{\sin^2\left(\frac{\varphi_e}{2}\right)}{\cos\left(\frac{\varphi_e}{2}\right)}-0.55m\cos\left(\frac{\varphi_e}{4}\right)\right\}. \qquad (10b)$$

It is clear that at the fixed external magnetic field, for low inductance l and small amplitude of the second term, the return current i_R is decreased. For the high values of the amplitude of the second term ($\alpha > 0.5$ or $m > 0.5$), proper of the corrected value for E/E_c are given in the following relations [3]:

$$\frac{E}{E_c}=\begin{cases}1+\alpha+\dfrac{1}{4\alpha}, & \text{anharmonic term, } \alpha \ge 0.5,\\[2mm] 2\left(1+\dfrac{m}{2}\right)^2, & \text{fractional term, } m \ge 0.5.\end{cases} \qquad (11)$$

Substitution of (11) in (8a) and (8b) made it possible to determine the normalized reverse current i_R by numerical integration in (9) i_R, the results are presented in Figs. 2 and 3. In calculations of return current i_R, in addition to the renormalized value of threshold energy E/E_{c0}, we also take into account the increase of the McCumber parameter $\beta = (2e/\hbar)I_c R_s^2 C$ with renormalized critical current I_c [24].

For the different values of geometrical inductance l and the typical amplitude of anharmonicity parameter $\alpha = 0.7$, the results are of numerical calculations of return current i_R

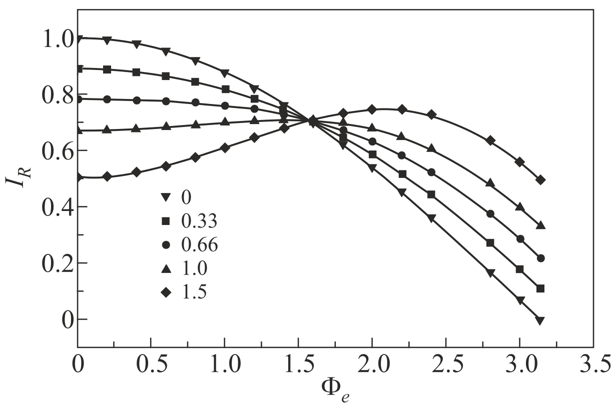

Fig. 2. Return current of dc SQUID versus external magnetic field for different values of inductance l from top to bottom. ($\alpha = 0.7$).

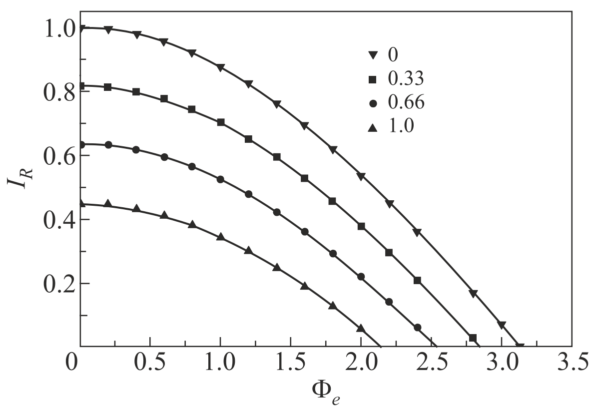

Fig. 3. Return current of dc SQUID versus external magnetic field for different values of inductance l from top to bottom ($m = 0.7$).

are presented in Fig. 2. In the case of fractional current-phase relation with the amplitude of this term $m = 0.7$, results of the similar calculations are shown in Fig. 3. It is clear that for the case of $\alpha = 0$ and $m = 0$ we have analytical results corresponding to the case $i_R = \sqrt{|\cos(\varphi_e/2)|}$. Also, it is useful to note that obtained results are symmetrical with respect to axes i_R and obtained the whole picture is the periodical in respect to external the magnetic field φ_e with the period 2π. As follows from Fig. 2, the inclusion of geometrical inductance to consideration, drastically changes behavior i_R. At small inductance $l < 1$ return current i_R reveals monotonically decreasing character in increasing of external magnetic field φ_e. At inductance $l > 1$ return current i_R firstly grows with increasing of external magnetic field $\varphi_e < \pi/2$ and monotonically decreases in magnetic fields $\varphi_e > \pi/2$ with no peak. As follows from Fig. 3, the inclusion of geometrical inductance l into consideration does not change the general decreasing character of return current $i_R(\varphi_e)$. It is clear that the return current decreases in the whole region of the external magnetic field φ_e. The geometrical inductance suppresses the amplitude of changing of return current. Similar numerical calculations were conducted in Ref. 25.

For the high inductance dc SQUID $l >> 1$, Josephson inductance of junctions $[\Phi_0/(2\pi I_{c1,2})]$ can be ignored in consideration of dynamical effects [1]. In this limit, the phase of Josephson junctions in the superconducting loop (Fig. 1) changes independently. If take into account the results of Ref. 24, the renormalization of critical current causes decreasing of Josephson inductance approximately by two times. It means that in dc SQUID with high geometrical inductance $l >> 1$ the unconventional effects in current-phase relations can be neglected.

In the case of dc SQUID with overdamped Josephson junctions $(\beta \geq 1)$ above presented calculations are not valid. In this case, the numerical methods described in [14] is applicable. In Ref. 14, the Eq. (1) with unconventional current-phase relation was solved numerically, and the return current I_R of Josephson junction was calculated as a point where an averaged voltage is vanished on the junction. The case of junction shunted by different values of shunt inductance L_s and resistance R_s were considered.

In summary, in the present paper, the return current I_R of dc SQUID having tunnel Josephson junctions $(\beta >> 1)$ with unconventional current-phase relation was investigated. The changes of the return current of dc SQUID caused by the amplitude of the second term in current-phase relation, geometrical inductance, and external magnetic field were studied.

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Зворотній струм СКВІДа постійного струму на тунельних джозефсонівських переходах з нетрадиційним співвідношенням струм–фаза

I. N. Askerzade, R. Askerbeyli

Проведено аналіз зворотнього струму СКВІДа постійного струму на базі тунельних джозефсонівських переходів з нетрадиційним співвідношенням струм–фаза. Проаналізовано два випадки співвідношення струм–фаза при додаванні до гармонійного доданка sin φ додаткових: з подвоєною sin 2φ та дробовою sin (φ/2) фазою. Показано, що зміна струму вертання СКВІДа постійного струму на переходах з нетрадиційним співвідношенням струм–фаза визначається амплітудою доданка в ньому, геометричною індуктивністю та зовнішнім магнітним полем.

Ключові слова: СКВІД постійного струму, співвідношення струм–фаза, ток вертання, ангармонічні та дробові складові.