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On the energy spectrum of the 3D velocity field, generated by an ensemble of vortex loops

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The energy spectra of a three dimensional velocity field, induced by a set of vortex loops with various configurations are reviewed. This problem is closely related to the actual question of whether a chaotic set of vortex filaments can reproduce the real hydrodynamic turbulence. In the paper we discuss several cases, which allow evaluating spectra in an exact form. The research was made for an ensemble of vortex rings of different sizes as well as for vortex loops with fractal Hausdorf dimension equal to 5/3, which corresponds to Flory's vortex model, the so-called self-avoid lines. The results obtained are discussed.

Keywords: superfluidity, vortices, quantum turbulence.

1. Introduction

In the numerical simulation experiments on turbulent flows in a classical fluid with Reynolds numbers about 100 and in studies on modeling the dynamics of quantum vortex lines in superfluid helium, similar patterns in the behavior of classical and quantum turbulence were found. As a result, the approach of describing classical turbulence in terms of chaotic vortex lines has acquired particular importance [1–4]. One of the main arguments supporting the idea of the semiclassical behavior of quantum turbulence is the dependence of the energy spectra $E(k)$ on the wave vector k, obtained in numerical simulations and in experiments. There are a quite number of works that demonstrate the function dependence $E(k)$, which is alike to the Kolmogorov spectrum: $E(k)\propto k^{-5/3}$. These works are based on both the vortex line method [5–8], and the Gross–Pitaevskii equation [9–11]. Presence of similar spectrum doubtlessly shows connection of chaotic vortex lines dynamics with classical turbulence and proves ununiform energy distribution in the wave numbers k space.

However, despite the great importance of this result and numerous discussions, the question of how the Kolmogorov spectrum is generated remains open. Therefore, it seems attractive to find a mechanism for the occurrence of a Kolmogorov-type spectrum based directly on the quantum vortex lines configuration. Earlier, energy spectra of three-dimensional velocity field were numerically and analytically studied. Method for energy spectrum calculation was proposed, the general expression for energy spectrum was obtained, research of energy spectrum of the 3D velocity field was carried out for a single smooth line, ring, lines with Kelvin waves, as well as for a single fractal vortex loop with various Hausdorf dimension [12–15].

This paper presents the results of calculating the spectral characteristics of the energy of superfluid helium motion generated by various series of objects: (i) a set of vortex loops of various structures; (ii) vortex rings of different sizes; and (iii) vortex loops with fractal Hausdorf dimension equal 5/3, which corresponds to Flory's vortex model, the so-called self-avoid lines. The results obtained are discussed.

2. Calculation method

The general formula for superfluid helium energy can be expressed in terms of vortex lines configuration as (see, [12, 13] for explanations and notations)

$$E=\left\langle \int \frac{\rho_s \mathbf{v}_s^2}{2} d^3\mathbf{r} \right\rangle =$$

$$=\int\limits_{\mathbf{k}} \frac{d^3\mathbf{k}}{4\pi k^2}\left\langle 2\pi\rho_s\kappa^2 \int\limits_0^L\int\limits_0^L \mathbf{s}'(\xi_1)\mathbf{s}'(\xi_2)\, e^{i\mathbf{k}(\mathbf{s}(\xi_1)-\mathbf{s}(\xi_2))} d\xi_1 d\xi_2 \right\rangle. \quad (1)$$

Here $\mathbf{s}(\xi)$ are the vortex line positions parameterized by the arc length ξ, running from 0 to the length of line L, $\mathbf{s}'(\xi)$ denotes the derivative with respect to arc length along the line (the tangent vector). Since we will be dealing with an ensemble of vortex loops, then, generally speaking, it is necessary to perform summation over all these loops. However, by default, we unify the summation over different loops $\sum_j$ and the integration procedure $\int_C$, i.e., $\int_C = \int_C \sum_j$. Brackets $\langle\ \rangle$ imply an averaging over all possible vortex loop configurations. Clearly, the integrand in (1) can be interpreted as the three-dimensional distribution of energy $dE/d^3\mathbf{k}$, and, accordingly, expression within the brackets $\langle\ \rangle$ is just scalar energy spectral density (see [12, 13]). When calculating the energy spectrum for an ensemble of vortex loops we will suppose that different vortex loops are statistically independent, so averages of cross loops disappear, i.e., $\langle \mathbf{s}'_i(\xi_i)\mathbf{s}'_j(\xi_j)\rangle = 0$. For the isotropic case, the spectral density depends on the absolute value of the wave number k. Integrating over solid angle lead to formula (see [14]):

$$E(k) = \left\langle \frac{\rho_s \kappa^2}{(2\pi)^2} \int_0^{L_i}\int_0^{L_j} \mathbf{s}'_j(\xi_i)\mathbf{s}'_j(\xi_j)d\xi_i d\xi_j \frac{\sin\left(k\left|\mathbf{s}(\xi_j)-\mathbf{s}(\xi_j)\right|\right)}{k\left|\mathbf{s}(\xi_i)-\mathbf{s}(\xi_j)\right|} \right\rangle. \tag{2}$$

For anisotropic situations, formula (2) is understood as the angular average. Thus, for calculation of the energy spectrum $E(k)$ of the 3D velocity field, induced by the vortex filaments we need to know the exact configuration $\{s(\xi)\}$ of vortex lines. Further, we will use formulas (1), (2) to calculate the energy spectrum of the three-dimensional velocity created by a set of vortex rings, as well as by a set of vortex loops of various fractal dimensions.

3. Energy spectrum of the three-dimensional flow created by a set of vortex rings

Let's discuss first the spectra $E_{\text{ring}}(k)$ created by a single vortex ring. Vortex ring with the radius R is located in the xy plane (z = 0), and it is given by equation $s = (R\cos\varphi, R\sin\varphi, 0)$. From general formulas (1), (2) one can get an expression for spectral density of energy:

$$E(k) = \frac{\rho_s\kappa^2 R}{(2\pi)^2} \times$$

$$\times \int_0^{2\pi}\int_0^{2\pi} d\varphi_i d\varphi_j \frac{\cos(\varphi_j-\varphi_i)\sin\left(2kR\sin\dfrac{(\varphi_i-\varphi_j)}{2}\right)}{k\sin\dfrac{(\varphi_i-\varphi_j)}{2}}.$$

Calculation $E_{\text{ring}}(k)$ for R = 1 by formula (3) gives the results shown in Fig. 1 (see [15]), Quantity $E_{\text{ring}}(k)$ is reduced by factor $\rho_s\kappa^2/(2\pi)^2$.

The spectrum $E_{\text{ring}}(k)$ scales like $E_{\text{ring}}(k) \propto k^2$ for wave vectors much smaller than the inverse radius, $kR \ll 1$. This distribution is a consequence of the fact that closed vortex domain induces a far field flow scaling as $1/r^3$. For the large values (kR >> 1) energy density is inversely proportional to the wavenumber $E(k) \propto k^{-1}$, like for straight

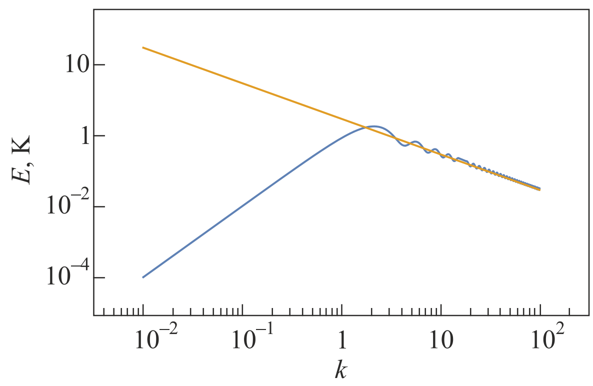

Fig. 1. Reduced energy spectral density $E_{\text{ring}}(k)$ for the single vortex ring (see text for detailed explanations).

line. Here and below, when calculating the spectra in accordance with formula (2), we omitted the factor $\rho_s\kappa^2/(2\pi)^2$. Straight line $\propto k^{-1}$ is plotted for reference.

Turning-point in spectral density behavior or maximum on the curve $E_{\text{ring}}(k)$ noccurs at the value of wavenumbers matching ring radius. It seems attractive to use this fact and choose an ensemble of rings with different lengths in order to investigate the behavior of the spectrum of the vortex rings set. Calculation of energy density for rings series gives energy spectrum values shown in Fig. 2. The transition from square law of energy spectrum $E(k) \propto k^2$ to $E(k) \propto k^{-1}$ is spread out in wavenumbers space. A transition region appears in which the spectral energy density depends on the distribution of the vortex rings in their radii space dN/dR.

It is seen that the spectral density can both increase and decrease with wavenumbers, depending on the distribution of the loops. However, the degree of decrease and increase of the spectral function in the transition range is between the dependences $\propto k^2$ and $\propto k^{-1}$. The result shows that a set of ideal vortex rings does not induce a velocity with the Kolmogorov energy distribution spectrum $E(k) \propto k^{-5/3}$.

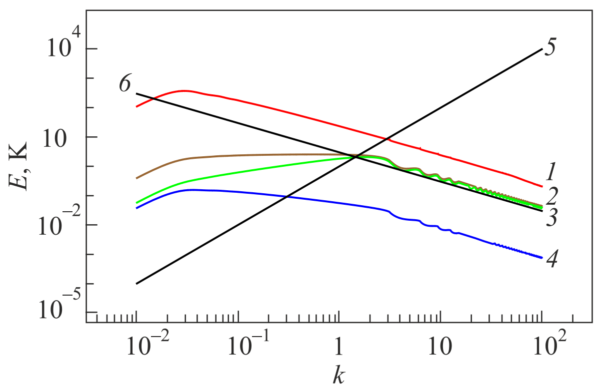

Fig. 2. Energy spectrum of vortex ring set. The distribution of the rings with the radii satisfies the relation $dN/dR \propto R^{-n}$, where n: 3/2 (*1*), 5/2 (*2*), 3 (*3*), 7/2 (*4*). Two straight lines corresponds to $\propto k^2$ (*5*) and $\propto k^{-1}$ (*6*). They are plotted for reference.

4. Energy spectrum of the three-dimensional flow created by an ensemble of Brownian loops of various Hausdorf dimensions

In this chapter, we will consider the case of an ensemble of Brownian loops of various fractal dimensions. To calculate the spectral characteristics based on formulas (2), (3) of a vortex tangle, we need one useful tool often used in statistical problems, the so-called characteristic (or generating) functional (see, for example, [16]). Following these works, we define the characteristic functional $W(\{\mathbf{P}_j(\xi_j)\})$ as average

$$W(\{\mathbf{P}_j(\xi_j)\}) = \left\langle \exp\left(i\int_0^L \mathbf{P}(\xi)\mathbf{s}'(\xi)d\xi \right) \right\rangle. \qquad (4)$$

Averaging can be performed using the probability distribution functional $\mathbf{P}(\mathbf{s}(\xi_j))$as a path integral

$$W(\{\mathbf{P}_j(\xi_j)\}) = \int D\mathbf{s}(\xi)P(\mathbf{s}(\xi))\exp\left(i\int_0^L \mathbf{P}(\xi)\mathbf{s}'(\xi)d\xi \right). \qquad (5)$$

The characteristic functional $W(\{\mathbf{P}(\xi)\})$ (5) allows calculating the average of any value depending on the configuration of vortex lines by simple functional differentiation. For instance, the average tangential vector $\langle \mathbf{s}'_{j\alpha}(\xi_j)\rangle$, or the correlation function between the tangential vectors of various elements of the vortex lines $\langle \mathbf{s}'_{j\alpha}(\xi_j)\mathbf{s}'_{j\beta}(\xi_j)\rangle$, can be easily expressed through the characteristic functional $W(\{\mathbf{P}_j(\xi_j)\})$ (4) in accordance with the rules:

$$\langle \mathbf{s}'_{j\alpha}(\xi_j)\rangle = \left.\frac{\delta W}{i\delta \mathbf{P}_j^{\alpha}(\xi_j)}\right|_{\text{all } \mathbf{P}=0}, \qquad (6)$$

$$\langle \mathbf{s}'_{j\alpha}(\xi_j)\mathbf{s}'_{j\beta}(\xi_j)\rangle = \left.\frac{\delta^2 W}{i\delta \mathbf{P}_j^{\alpha}(\xi_{j1})\ i\delta \mathbf{P}_j^{\beta}(\xi_{j2})}\right|_{\text{all } \mathbf{P}=0}. \qquad (7)$$

Combining (2) and (4) one concludes that the spectral density $E(k)$ is expressed via CF as follows (see [12, 17] for details):

$$E(k) = 2\pi\rho_s\kappa^2 \times$$
$$\times \int_0^L\int_0^L d\xi_1 d\xi_2 \left.\frac{\delta^2 W}{i\delta \mathbf{P}^{\alpha}(\xi_1)\, i\delta \mathbf{P}^{\alpha}(\xi_2)}\right|_{\mathbf{P}_i(\xi_i)=\mathbf{k}\theta(\xi-\xi_1)\theta(\xi_2-\xi)}. \qquad (8)$$

Here $\theta(\xi)$ is a unit-step function.

Supposing that the probability functional $P(\mathbf{s}(\xi_j))$ is a Gaussian one, calculation of the CF can be readily made by the full square procedure to give the result (here we consider only the isotropic case)

$$W(\{\mathbf{P}(\xi)\}) = \exp\left(-\int_0^L\int_0^L d\xi' d\xi'' \mathbf{P}^{\alpha}(\xi') N_{\alpha\beta}(\xi'-\xi'')\mathbf{P}^{\beta}(\xi'') \right). \qquad (9)$$

From the definition (7) of the CF it follows that the function $N_{\alpha\beta}(\xi'-\xi'')$ coincides with the correlation function $\langle \mathbf{s}'_{\alpha}(\xi_1)\mathbf{s}'_{\alpha}(\xi_j)\rangle$ between tangent vectors. Substituting (9) in expression for energy (8), and supposing an isotropic situation [i.e., $N_{\alpha\beta}(\xi'-\xi'') = N(\xi'-\xi'')$], one can obtain [12, 17]:

$$E = 4\pi\rho_s\kappa^2 \int_0^L\int_0^L d d\xi_2 \int_0^{\infty} dk \left[3N(\xi_1-\xi_2) - 4\left(k\int_0^{\xi_2-\xi_1} N(x)dx \right)^2 \right] \times$$
$$\times \exp\left(-3k^2 \int_{\xi_1}^{\xi_2}\int_{\xi_1}^{\xi_2} N(\xi'-\xi'')d\xi' d\xi'' \right). \qquad (10)$$

This formula can be used to directly calculate the energy spectrum for various functions $N(x)$, representing the configuration of the vortex loops.

Earlier, the energy spectrum of a single vortex line has been calculated in accordance with the formula (2) (see, for example [12, 17]). In Fig. 3 on a logarithmic scale, the results of calculating the energy spectrum for vortex loops with a nominal length of 100 are presented. The structure of the loops is described by the power-law correlation function $N(x)=cx^{\lambda}$. The length L of such curves increases with its 3D size D as $L \propto D^2/(\lambda+2)$, which implies that an average loop is a fractal object having the Hausdorf dimension equal to $H_d = 2/(\lambda+2)$.

The upper dotted curve corresponds to the power-law dependence $N(x)$ with exponent $\lambda = 0$, i.e., smooth line with dimension 1. It is easy to see that dependence $E(k) \sim k^2$ changes to dependence $E(k) \sim k$ in the region $k \sim 0{,}01 = 100^{-1}$ that is quite expected, because dimension of the line is equal to 1, and the length is equal to 100 in 3D space.

The middle solid line is the energy spectrum for a vortex line, described by the Flory model (see, e.g., [18]), a self-avoid line, for which $N(x)$ is expressed by a power function with exponent $\lambda = -4/5$. For it, a change in the behavior of the energy spectrum from $E(k) \sim k^2$ to the

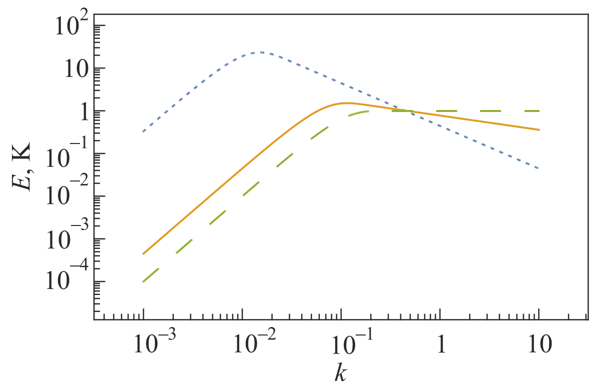

Fig. 3. Energy spectra of vortex lines of various fractal dimensions (see text for explanations).

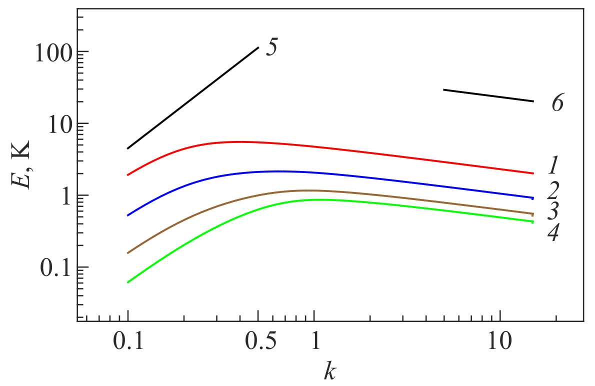

Fig. 4. Energy spectrum for set of vortex loops with Hausdorf dimension 5/3. The distribution with loops lengths satisfies relation $dN/dL \propto R^{-p}$, where p: 0 (*1*), 3/2 (*2*), 5/2 (*3*), 7/2 (*4*). Curves are normalized per unit length of the vortex line. Two straight lines corresponds to $\propto k^2$ (*5*) and $\propto k^{-1}$ (*6*). They are plotted for reference.

dependence $E(k) \sim k^{-1/3}$ in the region $k \sim 100^{-3/5}$ is also observed, this is in good agreement with general considerations, since the fractal dimension of such a line is equal to 5/3. The lower dashed curve shows the energy spectrum of the Brownian vortex line, for which the correlation function is a delta function. The squared dependence $E(k) \sim k^2$ alters into a plateau, i.e., $E(k) \sim k^0$, in the region $k \sim 1/\sqrt{100}$, as expected given that formally λ = 0, and Hausdorf dimension is equal 2.

In present work, we consider some preliminary results on the energy spectrum obtained for an ensemble of loops described by the Flory model. In Fig. 4 the results of calculating the energy spectrum for vortex loops with 5/3 Hausdorff dimension are presented on a logarithmic scale. The curves are normalized per unit length of the vortex line.

Straight lines have a slope of k^2 and $k^{-1/3}$, and they are plotted a reference. The upper line corresponds to a set of loops with a fixed length, and its spectrum naturally coincides with the spectrum of a single Flory line. The next (from top to bottom) lines have a distribution of lengths L satisfies relation, $dN/dL \propto R^{-p}$, with p equal to 0, 3/2, 5/2, 7/2, respectively. It can be seen that the area of slope changing from k^2 to $k^{-1/3}$ is extended and shifted towards large k, as well as this area is slightly stretched. Nothing dramatic happens, though. The result obtained shows that a set of fractal lines does not induce a velocity with the Kolmogorov energy distribution spectrum $E(k) \propto k^{-5/3}$.

5. Conclusion

Summarizing, it can be resumed that the 3D energy spectrum $E(k)$ generated by an ensemble of vortex loops consists of several parts. The first one appears at small k, associated with the large scales, of the order of the size of the system and the spectrum behaves as $E(k) \propto k^2$. This behavior is general, it does not depend on loops shape, this is the consequence of the asymptotic behavior ($1/r^3$ at $r \to \infty$) of the velocity field. For large wave numbers spectrum $E(k)$ should be close to k^{-1} for smooth lines, and again regardless of the specific model. The spectrum $E(k)$ in the region of intermediate k depends on the detailed structure of loops constituting the vortex tangle. It depends also on the distribution of loops in space of their lengths dN/dL. The spectral density can both increase and decrease with wavenumbers k depending on the distribution of loops and their shape. However, the degree of decrease and increase of the spectral function in the transition region is between the dependences $\propto k^2$ and $\propto k^{-1}$, or between the dependences $\propto k^2$ and $\propto k^{-1/3}$ in the case of Flory loops. Thus the variation of the distribution of lengths in the ensemble of vortex loops lines cannot create 3D flow, which has the Kolmogorov spectrum $E(k) \propto k^{-5/3}$ [19]. It appears that a stronger singular velocity field is required to obtain the Kolmogorov spectrum. It is obvious that the model of vortex loops built within the framework of the local approximation fails to create the necessary singularity, and further research should be concentrated on studying the velocity field of colliding interacting vortex loops.

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Енергетичний спектр 3D-поля швидкості, яке індуковано набором вихрових петель

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Розглянуто енергетичні спектри тривимірного поля швидкості, індукованого набором вихрових петель з різною конфігурацією. Ця проблема тісно пов'язана з актуальним питанням, чи може хаотичний набір вихрових ниток відтворити реальну гідродинамічну турбулентність. Наведено кілька випадків, що дозволяють точно оцінити спектри. Дослідження проведено для ансамблю вихрових кілець різного розміру, а також для вихрових петель із фрактальною розмірністю Хаусдорфа, що дорівнює 5/3 та відповідає моделі полімерів Флорі. Обговорено отримані результати.

Ключові слова: надплинність, вихори, квантова турбулентність.