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Magnetic vortices in media with spatially inhomogeneous exchange interaction

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In the framework of the classical equations for the magnetization dynamics and the collective variables approximation, the dynamics of magnetic vortices in two-dimensional ferromagnets with the easy-plane anisotropy and spatially inhomogeneous exchange interaction is considered. In the case of a wide straight line of the interface between magnetic media with different but slightly different exchange interaction, the dependence of the magnetic vortex velocity on its distance to the structure defect and position inside the interface domain is obtained.

Keywords: easy-plane ferromagnet, inhomogeneous magnetic media, magnetic defects, magnetic vortex, Thiele and Sonin equations.

Introduction

The physics of magnetic phenomena is a traditional area of condensed matter physics and, in particular, solid state physics. Recently, additional interest in magnets has arisen in connection with the synthesis of a large number of new compounds with a complex structure and unusual various properties, the so-called multiferroics. On the other hand, traditional magnets have attracted interest due to the possibility of their application in new areas of microelectronics, which use the physical ideas formulated in spintronics and magnonics [1]. The miniaturization of microelectronic elements (information transmission lines, switches, transistors, transmission, recording and reading elements) led to the study of the possibility of using magnetic systems as these elements. In this case, information can be recorded, transmitted, and read out using magnetic excitations of a topological nature: domain walls, magnetic vortices and skyrmions [2, 3]. The microsize of these objects and their stability associated with their topological nature make them promising for technological applications. Topological excitations represent the essentially nonlinear objects and their dynamics is also nonlinear. The nonlinear dynamics of magnets has been actively studied recently, both theoretically and experimentally. The most interesting objects of research are represented by magnetic solitons of various types, in particular, dynamic and topological ones [4–6]. The latter are represented, for example, by vortices in easy-plane [7, 8] and skyrmions [9, 10] in easy-axis ferromagnets. In connection with the possibility of their use for transmitting information in magnetic transmission lines [11], important problems arise in the interaction of vortices with defects in the magnetic structure and their movement in magnon waveguides. The question of the interaction of isolated vortices with defects was investigated in a number of works [12, 13]. As a rule, defects are considered point-like and the oscillations of a vortex in an impurity field or its rotation in isolated magnetic nanodots are studied (mainly numerically) [14]. In [15], the interaction of individual magnetic vortices and vortex pairs with a finite-size defect was analyzed analytically. For this defect a model of cylindrical inclusions of a magnetic phase with an exchange interaction different from the matrix one was proposed. The results were obtained regarding the rotation of vortices "trapped" by the defect and the scattering of vortex pairs on such a defect. But, as it was shown, the most interesting phenomena of the scattering and formation of pairs associated with a defect appear at small impact distances, when the distance from the vortices to the defect interface becomes of the order of the magnetic length. However, at such distances, the used approximation of vortex dynamics within the framework of the method of collective variables (Thiele equations [16]) lose its applicability. Similarly, when studying the motion of vortices along magnetic channels and along the interfaces of magnetic media, the most interesting features of such dynamics arise when vortices move near the boundaries of magnetic media. In this case, the conditions of applicability of the Thiele equations and their equivalent Sonin equations [8] are also violated. Therefore, this communication proposes a different model of the interface between magnetic media. In it, the transition area has a finite width, and the change in the parameters of the magnet in this domain of spatial inhomogeneity is considered small. These assumptions make it possible to consider the motion of vortices in the whole volume of the magnet, including the inner domain of the interface. A similar problem arises in the theory of dislocations in inhomogeneous media when calculating the shear stresses acting on dislocation [17, 18]. The study of this problem within the framework of the linear approach showed its inconsistency. With regard to magnetic media, the possibility of a consistent solution to the problem lies in the consistent consideration of the nonlinear properties of the magnet.

1. Formulation of the model and basic equations

First of all, we note that the problems of the structure and dynamics of vortices in a superfluid liquid and Bose–Einstein condensates, similar to the considered in this article, have been investigated in many works [19, 20], but for vortices in magnets, the problem reveals its own specificity. In a superfluid liquid, the inhomogeneity of the medium is related to the external field and does not affect the terms in the Hamiltonian, which depend on the field gradients. This leads to local external influences in dynamic equations and substantially simplifies the problem. In addition, in superfluids, boundaries can only be impermeable, and a number of questions about the structure of boundaries are removed. In the theory of dislocations, concerning to their structure in the case of coordinate-dependent spatially inhomogeneous elastic moduli, the formulation of the problem is close to that considered in this article. This problem in the theory of dislocations has not yet been resolved [17, 18], and some results of present paper can be used in this theory.

Let us consider a two-dimensional ferromagnet with magnetic anisotropy of the easy-plane type within the framework of the classical approach. We assume that the exchange interaction constant is inhomogeneous in the plane of the sample. This situation is most easily achieved when the thickness of a deposited magnetic film depends on the coordinate in its plane. In this case the effective magnetization M changes proportionally to the film thickness. Renormalization of this quantity leads to a coordinate dependence of the effective exchange interaction constant, which also becomes dependent on the film thickness. Simultaneously the effective single-ion anisotropy constant also changes. But since the magnetic anisotropy is significantly less than the exchange interaction, its weak variation can be neglected. Since below we assume that the spatial change in the exchange interaction is also small. The energy of the magnet in this case has the form [4, 5]

$$E=\int\varepsilon\,dxdy=\int\left(\frac{1}{2}J(x,y)a_0^2(\nabla\mathbf{M})^2+\frac{1}{2}\beta(\mathbf{Mn})^2\right)dxdy, \quad (1)$$

where $\mathbf{M}$ is the magnetic moment, $J(\mathbf{r})$ is the coordinate-dependent exchange interaction constant, and β is the single-ion anisotropy constant ($\beta>0$). The unit vector $\mathbf{n}$ is directed perpendicular to the "easy plane" XY along the axis Z. (Below we restrict ourselves to the two-dimensional case). As equations of motion of the magnetization vector, we use the Landau–Lifshitz equations without damping (LLE) [21]

$$\frac{\partial\mathbf{M}}{\partial t}=\frac{2\mu_0}{\hbar}\left[\mathbf{M},\frac{\delta\varepsilon}{\delta\mathbf{M}}\right], \quad (2)$$

where μ_0 is the Bohr's magneton. Since the magnitude of the vector $\mathbf{M}$ is conserved and equal $|\mathbf{M}|=M_0$, it is convenient to choose as variables the normalized Z component of the magnetization vector $m=M_z/M_0$ and the azimuthal angle $\varphi=\arctan(M_y/M_x)$ of its rotation in the XY plane. In an easy-plane ferromagnet, the value $m=0$ corresponds to the ground state. In this case, the azimuthal angle is arbitrary, and the ground state is infinitely degenerate in the XY plane. The variables (φ, m) play the role of canonically conjugate quantities for Hamiltonian (1) and the corresponding Hamilton equations have the form

$$\dot\varphi=-\frac{2\mu_0}{\hbar M_0}\frac{\partial\varepsilon}{\partial m},\quad \dot m=\frac{2\mu_0}{\hbar M_0}\frac{\partial\varepsilon}{\partial\varphi}, \quad (3)$$

and the Hamiltonian is written as

$$E=\frac{M_0^2}{2}\int\left(J(\mathbf{r})a_0^2\left(\frac{(\nabla m)^2}{1-m^2}+\left(1-m^2\right)(\nabla\varphi)^2\right)+\beta m^2\right)dxdy. \quad (4)$$

The static vortex solution of the Landau–Lifshitz equations for a vortex at a point $(x=X, y=Y)$ has the form

$$\varphi=q\arctan\frac{y-Y}{x-X}=\pm\arctan\frac{y-Y}{x-X},$$
$$m=m(r)=p\,f\left(\sqrt{(x-X)^2+(y-Y)^2}\right)=\pm f, \quad (5)$$

where the parameter $q=\pm1$ determines the topological charge of the vortex ("vortex" with $q=1$ and "antivortex" with $q=-1$), and the parameter $p=\pm1$ determines its polarization. The vortex core is localized in a region with a size of the order of magnitude of the "magnetic length" $l_0=a_0\sqrt{J/\beta}$ and its field has asymptotics [7, 8, 22]

$$m=p\left(1-c_1(r/l_0)^2\right),\quad r\to0;$$
$$m=pc_2\sqrt{l_0/r}\,\exp(-r/l_0),\quad r>>l_0 \quad (6)$$

with numerical constants $c_i\sim1$. Due to the exponential localization of Z projection of the magnetization in the region with a size of the order of l_0, in the rest of the magnetic area its influence can be neglected and the static configuration of the vortex can be described within the framework of the modified Laplace equation:

$$\operatorname{div}\left(J(\mathbf{r})\operatorname{grad}\varphi(\mathbf{r})\right)=J(\mathbf{r})\Delta\varphi+\nabla J(\mathbf{r})\cdot\nabla\varphi=0. \tag{7}$$

An important characteristic of a vortex is the vorticity density of the magnetization field in it

$$\gamma=\frac{\partial m}{\partial x}\frac{\partial\varphi}{\partial y}-\frac{\partial m}{\partial y}\frac{\partial\varphi}{\partial x}, \tag{8}$$

which determines the total vorticity of the vortex Γ and its gyrovector $\mathbf{G}$

$$\Gamma=\int\gamma(x,y)\,dxdy=-2\pi\,pq\,,\ \mathbf{G}=\mathbf{n}_z\Gamma\frac{\hbar M_0}{2\mu_0}. \tag{9}$$

If the distances between the vortices and the distance between them and the boundaries of the magnetic medium significantly exceed the magnetic length l_0 and the velocities of the vortices are much less than the velocity of magnons $c=(2\mu_0 M_0/\hbar)\sqrt{J\beta}$, then the dynamics of magnetic vortices can be described within the framework of the method of collective variables and simplified equations proposed by Thiele [16, 23] for the coordinates $\mathbf{R}=(X,Y)$ of the vortex centers. The Thiele equation has the form

$$\left[\frac{d\mathbf{R}_i}{dt_i},\mathbf{G}_i\right]+\frac{\partial E(\mathbf{R}_i)}{\partial\mathbf{R}_i}=0. \tag{10}$$

Equation (10) can be transformed into a slightly different, more convenient form proposed in [8]:

$$\frac{d\mathbf{R}_i}{dt}=-p_i\frac{2J(\mathbf{r})\mu_0 M_0}{\hbar}\nabla\varphi_{\text{ex}}(\mathbf{r}=\mathbf{R}_i), \tag{11}$$

where φ_{ex} is the magnetization field distribution in the easy plane at the point of location of vortex with the number i, induced by external sources of the magnetization field deformation. In particular, it can be a "self-action" field induced by this vortex itself due to inhomogeneities of the system parameters. (The sequential derivation of equation (11) is given in the Appendix).

The above systems of Eqs. (10) or (11) describe approximately the dynamics of a system of vortices in spatially homogeneous systems. In a spatially limited magnet, an isolated vortex has the energy $E_0=\pi JM_0^2\ln(l/d_0)$, where l is the distance to the nearest boundary of the magnetic area, and d_0 is the size of the vortex core, which coincides in order of magnitude with the magnetic length l_0.

In this paper, we will consider the dynamics of a magnetic vortex in spatially inhomogeneous systems consisting of two half-spaces with different exchange interactions. The main assumption will be that the exchange interactions in the two subsystems are slightly different: $(J_2-J_1)/J_1\ll 1$, and the width of the interface, in which the exchange changes from J_1 to J_2, significantly exceeds the magnetic length: $L/l_0\gg 1$.

2. Contact of two half-spaces with different values of exchange interaction

For this system, the geometry of the problem is shown in Fig. 1. First of all, consider the case when the exchange interaction constant depends only on one spatial coordinate X and takes values $J=J_1$ at $x\to\infty$ and $J=J_2$ in the limit $x\to-\infty$. For definiteness, we assume that. $J_2>J_1$. Since isolated vortices are considered below, we also assume that the vortex has the chirality $q=1$, polarization $p=1$ and it is located at a distance a from the center of the interface between the media. (Border coordinate: $x_S=-a$).

In the simplest case of an infinitely thin interface, when the distribution of the exchange constant has the form $J(x)=J_1+(J_2-J_1)\Theta(-x-a)$, where $\Theta(x)$ is the Heaviside function, and the vortex distance to the boundary is much greater than the magnetic length, the problem is simply solved by the "image method" [15, 24]: when the boundary conditions $\varphi_1|_S=\varphi_2|_S$ and $J_1 d\varphi_1/dn|_S=J_2 d\varphi_2/dn|_S$ are satisfied the magnetization field in half-space 1 is determined by the vortex field and the field of a fictitious nonquantized vortex with a "charge" $q_2=(J_2-J_1)/(J_2+J_1)$ at the point: $y=0$, $x=-2a$.

$$\varphi_1=\arctan\frac{y}{x}+q_2\arctan\frac{y}{x+2a}. \tag{12}$$

The second term in this expression can be considered as an "external field" φ_{ex} acting on the vortex from the side of its image and appearing in Eq. (11). In this case, the dependence of the vortex velocity on the distance to the interface between the media is determined by the following equation (below, to simplify the formulas, we set $\mu_0 M_0/\hbar=1$):

$$V_y=-J_1 q_2\frac{1}{a},\quad V_x=0. \tag{13}$$

A vortex placed in half-space 2 moves in the same direction with the velocity $V_y=-J_2q_2/a$. The given dependence of the vortex velocity on its position in the system is

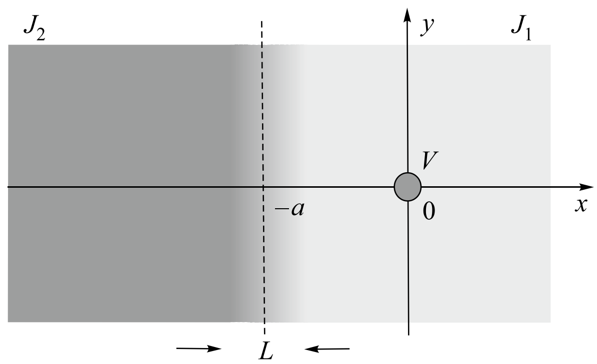

Fig. 1. Magnet with spatially inhomogeneous distribution of exchange interaction: contact of two magnetic half-spaces.

shown in Fig. 2(a). It should be borne in mind that the results obtained, as well as expression (12), are applicable only at distances greater than the magnetic length from the boundary [dashed lines in Fig. 2(a)].

We specially note that the decrease of the velocity is inversely proportional to the distance to the interface between the media: $V \sim 1/a$. This property, obviously, should be preserved for any interfaces with a sufficiently fast decrease of the interface characteristics in the depth of both half-spaces, since this is a manifestation of the nonlocal nature of the magnetic vortex fields.

To consider the motion of a vortex at any point in the system, including the region inside the interface, we construct a magnet in which the exchange interaction constant smoothly changes from the value J_2 to the value J_1 at distances $L >> l_0$ (Fig. 1). In addition, we assume that the total change in the value of the exchange integral is small: $J_2 - J_1 << J_2$ (or $q_2 << 1$). This assumption allows us to use the perturbation theory based on the smallness of this parameter. In this case, the solution for the vortex field in an inhomogeneous medium can be approximately written in the form

$$\varphi(\mathbf{r}) \approx \varphi_0(\mathbf{r}) + \varphi_{ex}(\mathbf{r}), \quad (14)$$

where φ_0 is the vortex solution $\varphi_0 = \arctan(x/y)$ in a homogeneous medium with an exchange constant J_1, and φ_{ex} is the weak self-action field of the vortex due to the presence of exchange inhomogeneity. The magnetostatic equation (7) can be approximately written in the form

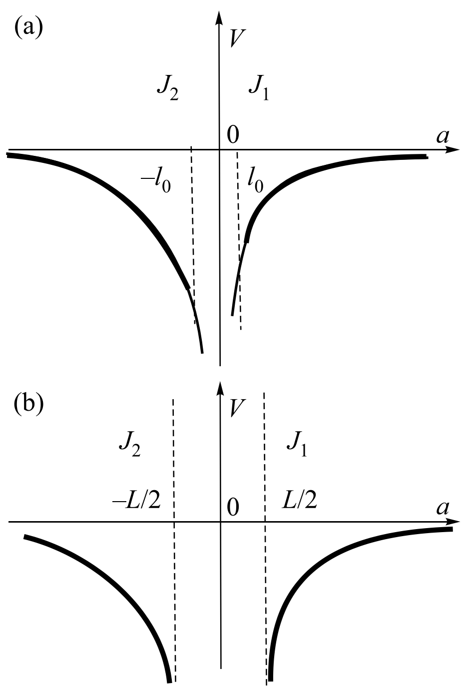

Fig. 2. Dependence of the vortex velocity on its distance to the interface: (a) in the model of the atomic thickness boundary, (b) in the boundary model (26) for a vortex with a singular core.

$$J_1 \Delta\varphi_{ex} + \nabla J(\mathbf{r}) \cdot \nabla\varphi_0 = 0. \quad (15)$$

Just in this form the problem of the dislocation field in a similar spatially inhomogeneous geometry was traditionally considered [17, 18].

We represent the coordinate dependence of the exchange integral in the form

$$J(x) = J_1 + (J_2 - J_1)F(x), \; \Phi(x) = \frac{dF}{dx}, \quad (16)$$

where the functions $F(x)$ and $\Phi(x)$ have a qualitative form, presented in Figs. 3(a) and 3(b).

Equation (15) can then be rewritten as

$$\Delta\varphi_{ex}(\mathbf{r}) = -2q_2\Phi(x)\frac{\partial\varphi_0(\mathbf{r})}{\partial x}, \; \varphi_0 = \arctan\frac{y}{x}. \quad (17)$$

It is important to note the following essential circumstance. The vortex position is assumed at the coordinate origin $x = y = 0$, and the function $\Phi(x)$ appearing on the right-hand side of Eq. (17) is defined in the entire space of the magnetic medium. In [17], the value of this function $\Phi(0)$ was substituted into equation (17) at the point where the vortex is located. This led to an incorrect result: the spatial derivatives of the addition to the field variable had a singularity at the center of the vortex. (The work [17] dealt with a formally similar problem of stress fields around a dislocation in an inhomogeneous elastic medium with spatially varying elastic moduli). In fact, due to the slowness of the decay of the fields around the vortex (or dislocation), the entire volume of the medium contributes to the self-action field, and there is no singularity in the field gradients at the center of the vortex (dislocation).

We use the Green's function of the two-dimensional Laplace equation [25] and represent the solution of the Laplace equation with the right-hand side of (17) in the form:

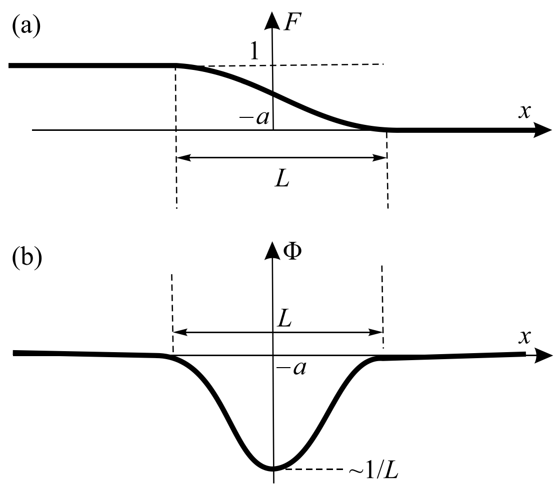

Fig. 3. Coordinate dependence of the exchange integral at the interface domain (a) and the derivative of this dependence (b).

$$\varphi_{\text{ex}}(x,y)=$$
$$=-\frac{q_2}{\pi}\int\Phi(x'+a)\frac{\partial\varphi_0(x',y')}{\partial x'}\ln\left(\sqrt{(x-x')^2+(y-y')^2}\right)dx'dy', \quad (18)$$

where the integration is carried out over the entire volume of the magnet. Using Eq. (11), it is easy to obtain an expression for the velocity of motion of the vortex (remind that the vortex is located at the point $x=0$, and the coordinate of the center of the interface is $x=-a$):

$$\frac{\partial\varphi_{\text{ex}}(x,y)}{\partial y}=$$
$$=\frac{q_2}{\pi}\int\Phi(x'+a)\frac{y'}{x'^2+y'^2}\frac{y-y'}{(x-x')^2+(y-y')^2}dx'dy', \quad (19)$$

$$V=\frac{2J(0)q_2}{\pi}\int\Phi(x'+a)\left(\frac{y'}{x'^2+y'^2}\right)^2dx'dy'. \quad (20)$$

Integration over y' is easily performed, which gives us the final result for the dependence of the velocity of the vortex along the interface on its distance from the center of the boundary:

$$V=J(0)q_2\int\frac{\Phi(x'+a)}{|x'|}dx'. \quad (21)$$

This expression is written in a system of coordinates centered at the point where the vortex is located. In the frame of reference centered in the middle of the interface ($x=-a$), expression (21) will be rewritten as

$$V(a)=J(a)q_2\int\frac{\Phi(x')}{|x'-a|}dx', \quad (22)$$

where a is the distance of the vortex from the center of the interface.

For the boundary of atomic thickness with $F(x)=\Theta(-x)$ and $\Phi=-\delta(x)$, expression (22) reduces to result (13). (Note that, as can be seen from (16), for $a>0$ the exchange constant is $J(0)=J_1$, and at $a<0$ the exchange is equal to $J(0)=J_2$). But it should be kept in mind that formula (13) is valid for any values of the parameter q_2, and the expression for the velocity, which follows from formula (21), which corresponds to the perturbation theory, is valid only under the condition $q_2 \ll 1$. Substituting the expression $\Phi=-\delta(x+a)$ into formula (19), easy to get for $x>0$:

$$\frac{\partial\varphi_{\text{ex}}}{\partial y}=q_2\frac{(x+2a)}{y^2+(x+2a)^2}, \quad (23)$$

and a similar expression for $\partial\varphi_{\text{ex}}/\partial x$:

$$\frac{\partial\varphi_{\text{ex}}}{\partial x}=-q_2\frac{y}{y^2+(x+2a)^2}. \quad (24)$$

Formula (12) follows from (23) and (24), but it is applicable only for $q_2 \ll 1$.

Let us consider another example with a strict spatial localization of the parameters of the defect domain, for which the space outside this region is strictly homogeneous. But, in contrast to the previous example, we will consider the boundary domain as a finite width area. In the simplest case, we choose the following distribution of the exchange constant (see Fig. 4):

$$J(x)=\frac{J_2+J_1}{2}-\frac{J_2-J_1}{L}x,\ -L/2<x<L/2. \quad (25)$$

For such a spatial distribution of the exchange constant

$$F(x)=(1-2x/L)/2,\ \Phi=-1/L,\ -L/2<x<L/2. \quad (26)$$

In this case, the expression for the vortex velocity (22) is reduced to the following:

$$V(a)=\frac{J(a)q_2}{L}\ln\left(\frac{a-L/2}{a+L/2}\right),\ a>L/2. \quad (27)$$

At large distances from the defect, the asymptotes has the form (13), as well as for the interface of the atomic size, i.e., $V=-J_1q_2/a$, but the singularity at the boundaries of the separation region is also preserved, although it becomes weaker: $V\sim\ln(a-L/2)$ as it is shown in Fig. 2(b). This behavior (a decrease in velocity, inversely proportional to the distance from the defect and a singularity at the interface) is retained for any profiles of the parameters of the boundary strictly localized in a finite interval. The infinite increase in the velocity at the interface is associated with the divergence of the kernel $\sim 1/|x'-a|$ in expression (22). It is a consequence of taking into account only the distribution of the vortex field in the easy plane of the magnet. Until now, we have not taken into account the nonlinear properties of the medium, which lead to the removal of the singularity in the vortex core. The size of the vortex core is

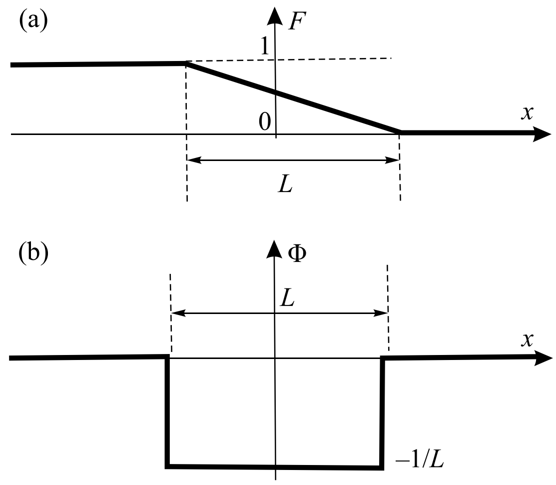

Fig. 4. Characteristics of the interface between the media in the model (25).

of the order of magnitude of the exchange length l_0 (6). Let us introduce in formula (22) the cutoff of the integral near the singularity at a distance of l_0, which will be justified below.

$$V(a) = J(a)q_2 \int_{-\infty}^{a-l_0} \frac{\Phi(x')}{a-x'}dx' + J(a)q_2 \int_{a+l_0}^{\infty} \frac{\Phi(x')}{x'-a}dx'. \tag{28}$$

For example, let us consider a wide (with $L >> l_0$) but strictly localized boundary with a fast (exponential) decreesing of the function Φ outside the volume of the interface:

$$J(x) = \frac{J_2+J_1}{2} - \frac{J_2-J_1}{2}\tanh\left(\frac{x}{L}\right),\ F(x) = \frac{1}{2}\left(1-\tanh\left(\frac{x}{L}\right)\right),$$

$$\Phi(x) = -\frac{1}{2L}\text{sech}^2\frac{x}{L}. \tag{29}$$

It can be seen from formula (28) that, in the considered model, the vortex velocity is limited and reaches a maximum value at the center of the boundary of the order of magnitude

$$V_{\max} \approx (J_1+J_2)q_2 \ln(l_0/L)/2L. \tag{30}$$

At large distances $a >> L$, the second integral in (28) is approximately reduced to an integral exponential function with known asymptotes at $l_0 << L$ and is equal to $-(2Jq_2/L)\exp(-2a/L)\ln(2\gamma l_0/L)$, where γ is the Euler's constant. It exponentially depends on the distance to the border. The first integral in (28) at $a/L >> 1$ has the form

$$V(a) = -\frac{J_1q_2}{2L}\int_{l_0/L}^{\infty}\frac{dz}{z\,ch^2(z-a/L)} \approx -\frac{J_1q_2}{a}. \tag{31}$$

Thus, despite the local character of the dependence of the parameter J at the interface, the decrease in velocity with distance is not exponential, but power-law, since it is an integral quantity. The asymptotes of the velocity at large distances have a standard form, as for a wall of atomic thickness. However, there is no infinite increase in velocity at the interface. But, from expression (30) it can be seen that the maximum velocity is determined by the thickness of the interface, and the expression for it diverges with a decrease in this thickness. The dependence of the vortex velocity on its position relative to the center of the interface is shown in Fig. 5(a).

Let us also present the dependence of the vortex velocity on its position with respect to the interface between the media in the case of the boundary model described by functions (26) (see Fig. 4). In this example, all dependencies are obtained in a simple analytical form:

$$V = -\frac{J_2q_2}{L}\ln\frac{2a-L}{2a+L},\ a < -\frac{L}{2}-l_0, \tag{32}$$

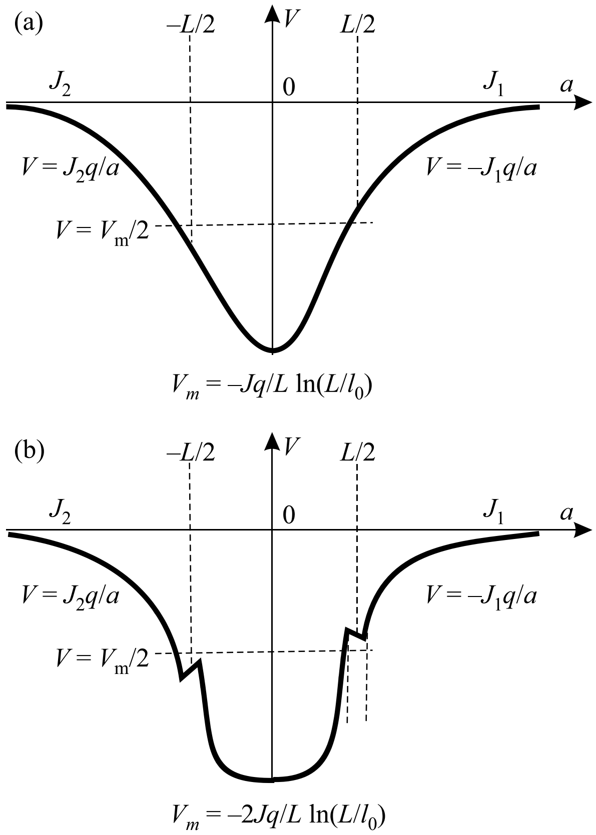

Fig. 5. Dependence of the vortex velocity on its distance from the center of the interface for (a) the model shown in Fig. 3, see formulas (29), and (b) for the model shown in Fig. 4, see formulas (26).

$$V = -\frac{J(a)q_2}{L}\ln\frac{L-2a}{2l_0},\ -\frac{L}{2}-l_0 < a < -\frac{L}{2}+l_0, \tag{33}$$

$$V = -\frac{J(a)q_2}{L}\ln\frac{L^2-4a^2}{4l_0^2},\ -\frac{L}{2}+l_0 < a < \frac{L}{2}-l_0, \tag{34}$$

$$V = \frac{J_1q_2}{L}\ln\frac{2a-L}{2a+L},\ a > \frac{L}{2}+l_0. \tag{35}$$

This dependence is shown in Fig. 5(b). It has qualitatively the same form as in the case of the model with a smooth dependence of the exchange integral on the coordinate. The maximum vortex velocity inside the interface is

$$V_{\max} \approx 2Jq_2\ln(l_0/L)/L, \tag{36}$$

those, it has the same order as in the previously considered model.

3. Influence of the vortex core structure

Another approach to the problem consists in calculating the vortex energy in the field of the interface between the media and taking into account the removal of the singularity of the field when the magnetization vector goes out of the easy plane. Let us return to expression (4) for the total energy of the magnet. It is easy to estimate the first term in parentheses. Using the approximation (6) for the spatial distribution of magnetization in the vortex core and integrating over the region of $r < l_0$, we find the contribution of this term to the energy of the system and the vortex velocity, which is calculated within the framework of the Thiele equation (10), which in this case has the form $V_y = -(\partial E / \partial X)/G$ with $G = -\pi$:

$$E_1 = 2\pi c_1 M_0^2 J(X),\ V_1 = 4c_1 J_1 q_2 \Phi(a). \qquad (37)$$

Note that this contribution to the vortex energy does not depend on its size l_0 and remains finite for a point-like vortex. In an example with a local interface of the form (29), this contribution to the vortex velocity reads

$$V_1 = -2c_1 \frac{J_1 q_2}{L} \operatorname{sech}^2\left(\frac{a}{L}\right), \qquad (38)$$

and at large distances it is exponentially small: $V_1 \sim \exp(-2a/L)$. The maximum value of this contribution to the vortex velocity is of the order of $V_{1\max} \sim Jq_2/L$, i.e., it is significantly less than the main contribution, which is of the order of $Jq_2 \ln(L/l_0)/L$. The contribution to the vortex velocity from this term of the energy corresponds to the change in the exchange interaction at the interface. It does not take into account the integral contribution of the magnetization distribution caused by the presence of the boundary. It arises from the second term in energy (4). Let us calculate this, taking into account that the multiplier $(1-m^2)$ in this term can be considered nonzero only in the region of $r > l_0$.

$$E_2 = \frac{M_0^2 a_0^2}{2} \int_{r>l_0} J(\mathbf{r})(\nabla\varphi)^2\, dxdy \approx \frac{M_0^2}{2} \int_{r>l_0} \frac{J(x+a)}{x^2+y^2} dxdy. \qquad (39)$$

The contribution to the vortex velocity from this part of the energy is as follows:

$$V_2 \approx \frac{J_1 q_2}{\pi} \int_{r>l_0} \frac{\Phi(x+a)}{x^2+y^2} dxdy. \qquad (40)$$

Figure 6 shows the areas of integration of expression (30) (the gray filling indicates the area of the interface between the media).

The contribution of region *1* to the integral is

$$V_{21} \approx 4GJ_1 q_2 \Phi(a)/\pi \approx 1.17 J_1 q_2 \Phi(a), \qquad (41)$$

where $G \approx 0.916$ is the Catalan's constant, i.e., it is of the order of V_1. In region *2*, the corresponding contribution is

$$V_{22} \approx J_1 q_2 \int_{l_0}^{\infty} \frac{\Phi(x+a)}{x} dx\ , \qquad (42)$$

which for model (29) at $a >> L$ gives the value $V_{22} \approx -(2J_1 q_2/L)\exp(-2a/L) Ei(-2l_0/L) \approx (2J_1 q_2/L) \times \exp(-2a/L)\ln(2\gamma l_0/L)$, where γ is the Euler's constant.

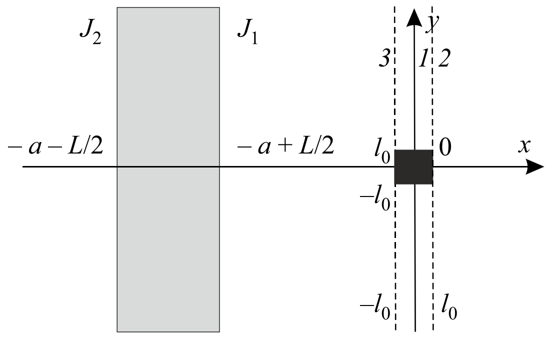

Fig. 6. Integration domains of calculating in the integral in expression (39).

In the same model, the contribution of region *1* is equal to $V_{21} \approx -(2.35 J_1 q_2/L)\exp(-2a/L)$, i.e., it is less according to $\ln(l_0/L)$. Finally, the contribution of region *3* to the expression for the velocity has the form

$$V_{23} \approx J_1 q_2 \int_{l_0}^{\infty} \frac{\Phi(x-a)}{x} dx \qquad (43)$$

and the integration corresponds to the interface region. In this case, for model (29) at $a >> L$ we obtain the expression: $V_{23} \approx -J_1 q_2 / a$. This expression coincides with the result (13) for the interface of the atomic thickness. Thus, the contribution of this region to the vortex velocity at large distances is the main. It depends on the integral characteristic of the interface, does not depend on the coordinate-dependent exchange constant at the vortex position. Note that in a similarly formulated paper [19], in which the dynamics of a vortex in the Bose–Einstein condensate was considered, the authors obtained a result of which is followed that the vortex velocity depends on the local value of the field at the vortex point, since in the condensate the field inhomogeneity is local characteristic.

Finally, let us calculate the maximum velocity of the vortex in the middle of the interface between the media. It is easy to find that it equals to $V_{\max} \approx (2J_1 q_2/L)\ln(l_0/L)$, which coincides in order of magnitude with the $V_{\max} \approx (J_1 q_2/L)\ln(l_0/L)$ obtained in the previous approach (28). Thus, it is possible to propose an interpolation formula for the vortex velocity in the form $V = -\left(J_1 q_2 / \sqrt{L^2+a^2}\right) f(l_0/L,\ a/L)$ in which the dimensionless function f increases at the interface to a measure of the parameter $\ln(l_0/L)$ and tends to unity at large distances from the boundary.

The total dependence of the vortex velocity on the distance from the center of the interface is given by the formula

$$V \approx J_1 q_2 \left(C\Phi(a) + \int_{l_0}^{\infty} \frac{\Phi(x+a)}{x} dx + \int_{l_0}^{\infty} \frac{\Phi(x+a)}{x} dx \right), \qquad (44)$$

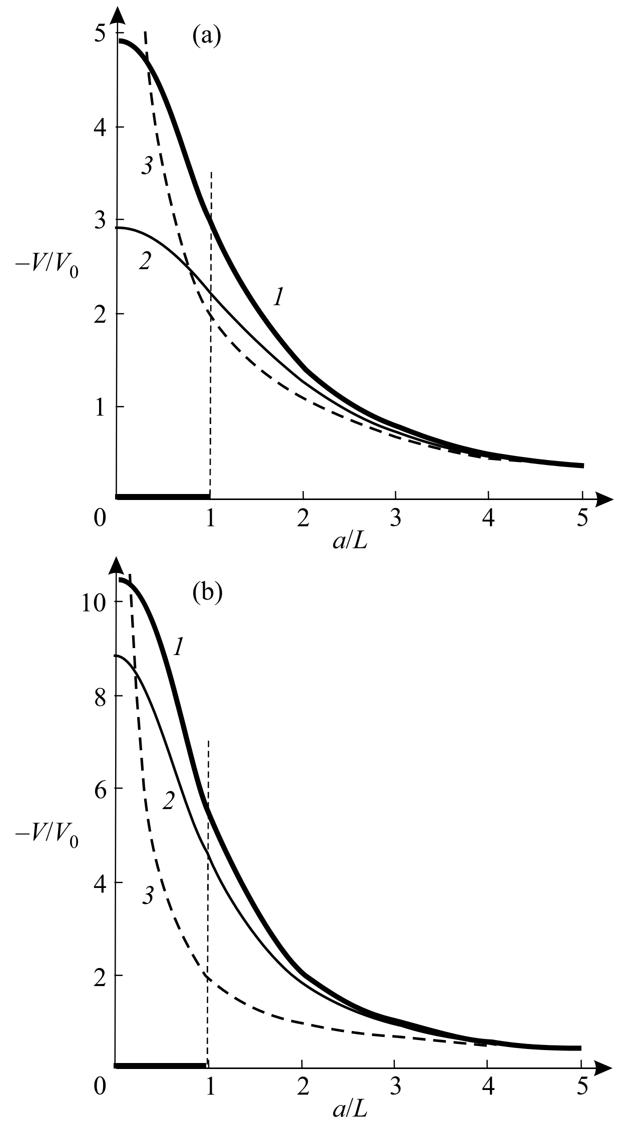

Fig. 7. Dependence of the vortex velocity on its distance from the center of the interface between media with different exchange interactions for $l_0 / L = 0.2$ (a) and $l_0 / L = 0.01$ (b). The area occupied by the interface between the media is highlighted in the figures.

where $C = 4G/\pi + c_1/2$. The first term in parentheses distinguishes two different approaches to the problem. In the first simplified approach, this term was absent. However, we have shown that in the asymptotes of the solution outside the region of the boundary, this term decreases exponentially in comparison with the power-law decrease of the main contribution to the velocity, and inside the region it is small in comparison with the main contribution to the measure of the small parameter $1/\ln(L/l_0)$. This justifies the use of a simplified approach within the framework of Eqs. (14), (15), (18) with cutting off the resulting integrals on the dimensions of the vortex core to solve problems on the dynamics of vortices in inhomogeneous systems with more complex geometry. For interface model (29), expression (44) is reduced to the following:

$$V \approx -V_0 \left(\frac{2}{ch^2(a/L)} + \int_{l_0/L}^{\infty} \frac{1}{z\,ch^2(z - a/L)} dz + \int_{l_0/L}^{\infty} \frac{1}{z\,ch^2(z + a/L)} dz \right), \tag{45}$$

where $V_0 = Jq_2/2L$. Since the exact value of the constant c_1 is unknown, we set it equal to one. The expression in parentheses depends on the value of a/L and on the parameter l_0/L. It is easy to find it numerically. Figure 7 shows the dependence $V = V(a/L)$ for the values $l_0/L = 0.2$ (a) and $l_0/L = 0.01$ (b). We recall that we assumed that this value was small.

In Fig. 7, the bold lines represent the dependence for the complete expression (45), and the thin lines of the dependence (45) without the “local” first term, i.e., when using the approximation (15), (17), (18), (28). It can be seen that with an increase in the thickness of the boundary, i.e., decreasing the parameter l_0/L, these dependences are close. The lower curve *2* in Fig. 7 corresponds to the qualitative graph in Fig. 5(a). For large values of the parameter a/L, the asymptotics with high accuracy has the form $V/V_0 = 2L/a$. It is shown by a dotted line (curve *3*) in Fig. 7. It can be seen that at small distances the divergence of the dependences increases with the thickness of the boundary, which is natural, since the asymptotics $V/V_0 = 2L/a$ corresponds to an infinitely thin boundary.

Conclusion

In the paper we investigate the motion of a magnetic vortex along the contact boundary of two magnetic media with different exchange interactions. In the case of a thin boundary of atomic size, the study of such dynamics within the framework of the usual approach using the Thiele equations is impossible in the region near the boundary, where the vortex velocity tends to infinity, exceeding the spin waves velocity. However, in the case when the spatial change of the exchange integral is small in comparison with its value, and the width of the boundary is much larger than the size of the vortex core, an approximate solution of the problem is possible.

It allows you to find the vortex velocity at any distance, including the area inside the boundary. An approach to solving the problem is developed that takes into account the nonlocality of the magnetization fields of the magnetic vortex. It is shown that at large distances from the boundary, the dependence of the velocity on the distance retains a power-law form, as well as for the boundary of the atomic size. However, there is no infinite increase in the velocity inside the interface between the media. The maximum vortex velocity reaches the value $V_{\max} \sim c \frac{\delta J}{J} \frac{l_0}{L} \ln \frac{l_0}{L}$, where c is the magnon velocity, δJ is the variation of the exchange integral, L is the characteristic range of the exchange constant, and l_0 is the magnetic length (the size of the vortex core).

Appendix

It was shown in [8] that in the presence of a uniform helicoidal distribution of the azimuthal angle field $\varphi = \mathbf{kr}$, where $\mathbf{k}$ and $\mathbf{r}$ are two-dimensional vectors in the *xy* plane (x, y), it follows from conservation of the total field momentum of the system that the vortex moves uniformly along the vector $\mathbf{k}$ with a velocity

$$\mathbf{V} = \frac{2\mu_0}{\hbar} JM_0 p\mathbf{k}, \qquad (A1)$$

where p is the polarization of the vortex. The given formula corresponds to the “frozen-in” vortex in the so-called “spin flow”. This phenomenon is completely analogous to the freezing-in of a hydrodynamic vortex into a fluid flow, since $\mathbf{k} = \partial\varphi / \partial\mathbf{r}$ and the azimuthal angle of magnetization plays the role of the velocity potential in hydrodynamics. Since formula (A1) includes the polarization of the vortex, it means that, its local characteristic associated with a z component that is nonzero at a distance from the center of the vortex of the order of the magnetic length, relation (A1) can be rewritten as

$$\mathbf{V} = \frac{2\mu_0}{\hbar} JMp\frac{\partial\varphi}{\partial\mathbf{r}}. \qquad (A2)$$

Let us show that this relation is equivalent to the Thiele equation and follows from it. Consider an isolated vortex located at a point with coordinates (X, Y) in a field of general form:

$$\varphi = q\arctan\frac{Y-y}{X-x} + \phi(x,y) = \varphi_0(\mathbf{R}-\mathbf{r}) + \phi(\mathbf{r}). \qquad (A3)$$

The energy of interaction of a vortex with an external field is

$$E_{\text{int}} = JM_0^2 \int_{r>0} \left(1-m_0^2\right)\left(\frac{\partial\varphi_0}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial\varphi_0}{\partial y}\frac{\partial\phi}{\partial y}\right) dxdy. \qquad (A4)$$

The change in energy due to the coordinate of the vortex changes reeds

$$\frac{\partial E_{\text{int}}}{\partial X} = JM_0^2 \int_{r>0} \frac{\partial}{\partial X}\left(\left(1-m_0^2\right)\frac{\partial\varphi_0}{\partial x}\right)\frac{\partial\phi}{\partial x} dxdy +$$

$$+ JM_0^2 \int_{r>0} \frac{\partial}{\partial X}\left(\left(1-m_0^2\right)\frac{\partial\varphi_0}{\partial y}\right)\frac{\partial\phi}{\partial y} dxdy. \qquad (A5)$$

Taking into account that in the first term $\partial / \partial X = -\partial / \partial x$, integrating the expression by parts with respect to the variable x and using the relation $\partial^2\phi / \partial x^2 = -\partial^2\phi / \partial y^2$, we transform the relation (A5) to the form

$$\frac{\partial E_{\text{int}}}{\partial X} = -JM_0^2 \int_{r>0} \left(\left(1-m_0^2\right)\frac{\partial\varphi_0}{\partial x}\right)\frac{\partial^2\phi}{\partial y^2} dxdy +$$

$$+ JM_0^2 \int_{r>0} \frac{\partial}{\partial X}\left(\left(1-m_0^2\right)\frac{\partial\varphi_0}{\partial y}\right)\frac{\partial\phi}{\partial y} dxdy. \qquad (A6)$$

After re-integrating by parts of the first term with respect to the variable y and using the relation $\partial / \partial y = -\partial / \partial Y$, we transform the formula (A6) in this way:

$$\frac{\partial E_{\text{int}}}{\partial X} =$$

$$= -JM_0^2 \int_{r>0} \frac{\partial\phi}{\partial y}\left[\frac{\partial}{\partial Y}\left(\left(1-m_0^2\right)\frac{\partial\varphi_0}{\partial x}\right) - \frac{\partial}{\partial X}\left(\left(1-m_0^2\right)\frac{\partial\varphi_0}{\partial y}\right)\right] dxdy, \qquad (A7)$$

which is easy to convert to expression

$$\frac{\partial E_{\text{int}}}{\partial X} = -JM_0^2 \int_{r>0} \frac{\partial\phi}{\partial y}\left[\frac{\partial m_0^2}{\partial Y}\frac{\partial\varphi_0}{\partial X} - \frac{\partial m_0^2}{\partial X}\frac{\partial\varphi_0}{\partial Y}\right] dxdy. \qquad (A8)$$

Since $m_0 = m_0(|\mathbf{R}-\mathbf{r}|) = m_0(\rho)$, taking into account the form of the vortex solution (A3), expression (A8) is reduced to the form:

$$\frac{\partial E_{\text{int}}}{\partial X} = JM_0^2 q \int_{r>0} \frac{\partial\phi}{\partial y}\frac{dm_0^2(\rho)}{d\rho} d\chi d\rho. \qquad (A9)$$

Z component of the vortex solution is localized in the region of the vortex core, so the expression $\partial\phi / \partial y$ can be replaced by $\partial\phi / \partial Y$ and removed from the integral sign. In this case, we finally get

$$\frac{\partial E_{\text{int}}}{\partial X} = -2\pi JM_0^2 q\frac{\partial\phi}{\partial Y}. \qquad (A10)$$

Using this relation, Thiele equation (10) is reduced to the equation

$$\frac{dY}{dt} = -p\frac{2J\mu_0 M_0}{\hbar}\frac{\partial\phi}{\partial Y}, \qquad (A11)$$

which coincides with (11) and is the generalization of relation (A1) to the case of an arbitrary external field distribution.

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1. I. Zutic, J. Fabian, and S. Das Sarma, *Spintronics: Fundamentals and Applications*, Rev. Mod. Phys. **76**, 323 (2004).
2. S. Bohlens, B. Krüger, A. Drews, M. Bolte, G. Meier, and D. Pfannkuche, *Current Controlled Random-Access Memory Based on Magnetic Vortex Handedness*, Appl. Phys. Lett. **93**, 142508 (2008).
3. Naoto Nagaosa and Yoshinory Tokura, *Topological Properties and Dynamics of Magnetic Skirmions*, Nat. Nanotechnol. **8**, 899 (2013).
4. A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, *Nonlinear Waves of Magnetization. Dynamical and Topological Solitons*, Naukova Dumka, Kiev (1983) [in Russian].
5. A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, *Magnetic Solitons*, Phys. Rep. **194**, 117 (1990).

6. A. B. Borisov and V. V. Kiselev, *Nonlinear Waves, Solitons and Localized Structures in Magnets,* UrO RAN, Yekaterinburg (2011), Vol. 2 [in Russian].
7. A. M. Kosevich, V. P. Voronov, and I. M. Manzhos, *JETP* **84**, 148 (1983).
8. A. V. Nikiforov and E. B. Sonin, *JETP* **58**, 373 (1983).
9. A. N. Bogdanov and D. A. Yablonskii, *JETP* **68**, 101 (1989).
10. A. N. Bogdanov and U. K. Rossler, *Phys. Rev. Lett.* **87**, 037203 (2001).
11. A. Fert, V. Gros, and J. Sampaio, *Skyrmions in the Trac*, *Nat. Nanotechnol.* **8**, 152 (2013).
12. A. R. Pereira L. A. S. Mól, S. A. Leonel, P. Z. Coura, and B. V. Costa, *Vortex Bexavior near a Spin Vacancy in Two-dimensional XY Magnets*, *Phys. Rev. B* **68**, 132409 (2003).
13. G. M. Wysin, *Vortex-vacancy Interaction in Two-dimensional Easy-plane Magnets*, *Phys. Rev. B* **68**, 184411 (2003).
14. A. S. Kovalev, F. G. Mertens, and H. J. Schnitzer, *Cycloidal Vortex Motion in Easy-plane Ferromagnets due to Interaction with Spinwaves*, *Eur. Phys. J. B* **33**, 133 (2003).
15. A. S. Kovalev and Y. E. Prilepskii, *Magnetic Vortices Interaction with Defects*, *Fiz. Nizk. Temp.* **44**, 847 (2018) [*Low. Temp. Phys.* **44**, 663 (2018)].
16. A. A. Thiele, *Phys. Rev. Lett.* **30**, 230 (1973).
17. D. H. Barnett, *On the Screw Dislocation in an Inhomogeneous Elasticmedium*, *Int. J. Solids Struct.* **8**, 651 (1972).
18. H. Kirshner, *The Force on the Elastic Singurarity in a Non-homogeneous Medium*, *J. Mech. Phys. Solids* **47**, 993 (1999).
19. B. I. Rubinstein and L. M. Pismen, *Vortex Motion in the Spatially Inhomogeneous Conservative Ginzburg–Landau Model*, *Physica D* **78**, 1 (1994).
20. T. N. Zueva, *Dissipative Motion of Vortices in Spatially Inhomogeneous Bose–Einstein Condensates*, *Fiz. Nizk. Temp.* **45**, 78 (2019) [*Low. Temp. Phys.* **45**, 67 (2019)].
21. L. D. Landau and E. M. Lifshitz, *Phys. Z. Sowjetunion* **8**, 153 (1935).
22. M. E. Gouvea, G. M. Wysin, A. R. Bishop, and F. G. Mertens, *Phys. Rev. B* **39**, 11840 (1989).
23. D. L. Huber, *Phys. Rev. B* **26**, 3758 (1982).
24. L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press (1984).
25. F. M. Mors and G. Feshbax, *Methods of Theoretical Physics*, Inostr. Lit., Moskow (1958), Vol. 1 [in Russian].

Магнітні вихори в середовищах з просторово неоднорідною обмінною взаємодією

A. S. Kovalev

У межах класичних рівнянь динаміки намагніченості в наближенні колективних змінних розглянуто динаміку магнітних вихорів у двовимірних феромагнетиках з анізотропією типу «легка площина» і просторово неоднорідною обмінною взаємодією. У випадку прямої межі розподілу магнітних середовищ з різною, але дещо іншою обмінною взаємодією, отримано залежність швидкості магнітного вихору від його відстані до дефекту структури та розташування всередині цієї межі розподілу.

Ключові слова: феромагніт типу легкої площини, неоднорідні магнітні середовища, магнітні дефекти, магнітний вихор, рівняння Тіле та Соніна.