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THEORETICAL INVESTIGATIONS OF THE IDEALIZED MODEL FOR THE MUSHY REGION

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In this paper the theoretical analysis of the behaviour of the stream function, temperature, and local solid fraction for the model of ideal mushy layer is presented. In the case of steady free mush convection, explicit lower and upper estimates for the main characteristics of the process are found for the large values of the Rayleigh number. For the unsteady regime the one of explicit forms of these characteristics is obtained.

Keywords: mushy layer, steady and unsteady free mush convection, stream function, temperature, local solid and liquid fraction

Досліджено поведінку функцій потоку, температури та локальної твердої фракції для ідеальної моделі мішаного шару. У випадку стійкої вільної конвекції знайдено точні нижні та верхні оцінки зверху та знизу для основних функцій, які характеризують процес, що має місце при великих значеннях числа Релея. Для нестаціонарного режиму отримано також явний вид основних характеристик.

Ключові слова: мішаний шар, стійка та нестійка конвекція, функція потоку, температура, локальні тверда та рідка фракції

1. Introduction

A mushy layer, a two-phase medium of coexisting liquid and solid phases, arises as a result of morphological instability of solidification front, see [5,6]. It can be considered as a porous medium through which the residual liquid can flow [7,13]. Therefore, the permeability structure of the mushy layer has to be calculated simultaneously with solving the coupled equations of heat, mass, and momentum transport [13].

Most theoretical studies of mushy layers consider the process of solidification at horizontal boundaries, see [12] and references therein. However, in many cases the process of solidification takes place at vertical boundaries. For example, in magma chambers various aqueous solutions are cooled and solidified from a sidewall in confined spaces [10,9,8,4].

The problem of the lateral solidification of a semi-infinite mushy region influenced by the vertical interstitial melt was investigated in [3]. The authors considered a binary alloy releasing a buoyant residual fluid in the solidification process. The fluid was assumed to be pulled horizontally at the constant speed V past the heat exchanger maintaining the eutectic temperature T_E at the fixed vertical plane $x = 0$, see Figure. The material supplied at $x = +\infty$ had the solute composition C_0, and the temperature equals to its liquid temperature $T_L(C_0)$. A mushy region was considered in the semi-infinite region $x > 0$ and $z > 0$.

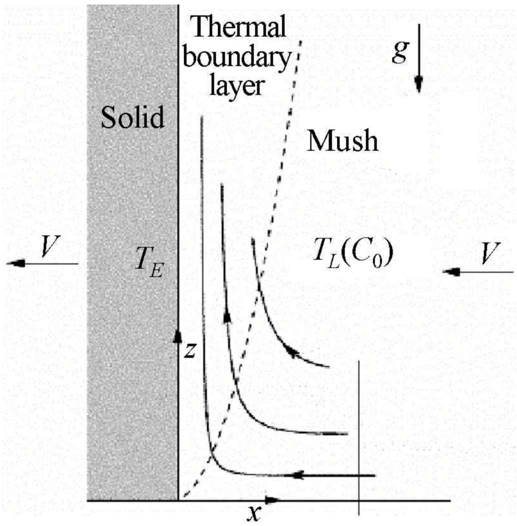

Fig. A semi-infinite mushy region of far-field temperature $T_L(C_0)$ solidifies laterally at fixed speed V to form a solid at the eutectic temperature T_E. The release of a buoyant residual is confined to a thermal boundary layer adjacent to the interface. Illustrative streamlines are shown relative to the (moving) solid phase, see [2]

In [3] assuming that the mushy region is ideal, Worster's model from [13] for description of the evolution of the dimensionless temperature θ and the local solid fraction ϕ in the domain $x > 0$ $z > 0$ is applied:

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\theta+u\cdot\nabla\theta=\nabla^2\theta+\Upsilon\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\phi, \quad (1)$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\left[(1-\phi)\theta+\Phi\phi\right]+u\cdot\nabla\theta=0, \quad (2)$$

$$\nabla^2\psi-\frac{1}{\Pi}\nabla\psi\cdot\nabla\Pi=-Ra\Pi\frac{\partial\theta}{\partial x}, \quad (3)$$

where t is dimensionless time, u is the volume flux (or Darcy velocity), ψ is the stream function defined by $u = (-\psi_z, \psi_x)$, and Π is the permeability. In this model the dimensionless constants are the Stefan number $\Upsilon = L/(c_p\Delta T)$, the compositional ratio $\Phi = (C_s - C_0)/\Delta C$, and the mush Rayleigh number $Ra = \beta\Delta C g\Pi_0/(\nu V)$, where $\Delta C = C_0 - C_E$, C_0, is the initial composition, C_E is the eutectic composition, C_s is the composition of the solid phase, L is the specific latent heat, c_p is the specific heat capacity, $\beta = \beta^* - \Gamma\alpha^*$, α^* and β^* being the thermal and solutal expansion coefficients, g is the gravity acceleration, and ν is the liquid kinematic viscosity.

Equations (1)–(3) are supplemented with the following boundary and initial conditions

$$\phi = \phi_0 \text{ at } t = 0, \quad \phi = \phi_\infty \text{ at } x = 0, \quad \phi \to \phi_\infty \text{ as } x \to \infty, \tag{4}$$

$$\theta = -1 \text{ at } x = 0, \quad \theta \to 0 \text{ as } x \to \infty \tag{5}$$

for $z \geq 0$, $t > 0$, where the function $\phi_0 = \phi_0(x, z)$ matches with ϕ_∞ at $x = 0$ and $x \to \infty$. Without loss of generality, we will assume that $\phi_\infty = 0$.

In the case of steady free mush convection, the boundary condition for ψ can be written as

$$\psi = 0 \text{ at } x = 0, \quad \frac{\partial \psi}{\partial x} \to 0 \text{ as } x \to \infty, \tag{6}$$

in the case of unsteady free mush convection, it can be given by

$$\psi = 0 \text{ at } z = 0, \quad \frac{\partial \psi}{\partial x} \to 0 \text{ as } z \to \infty \tag{7}$$

for $x \geq 0$, $t > 0$.

In this paper, we study the processes in a mushy region cooled from one side. In this model the flow occurs in a narrow thermal layer within the mushy region. The main aim of this paper is to study the qualitative asymptotic behaviour of self-similar solutions of the laminar boundary-layer flows in the steady case, describing essential physical properties of the process. We consider the behaviour of the stream function, temperature, and local solid fraction for unsteady situation too.

The present paper is organized as follows. In Section 2 we study the situation of steady free mush convection and obtain the explicit lower and upper estimates for a solution of this problem at $t = O(Ra^{-1})$ In the unsteady case, we find one of the set of explicit solutions of the problem for all $t > 0$. This result is contained in Section 3. Appendix contains some auxiliary routine calculations connected with Section 2.

2. Steady free mush convection

In this section, we consider a particular asymptotic regime where the thermal and flow are steady. This model was considered in [3], where the numerical approach was applied. We use the analytical methods for studying the asymptotic behaviour of the appropriate functions. Using scaling analysis of (1)–(3) (similar to [3]), we consider $\bar{\Upsilon}$, $\bar{\Phi}$, X, T, Ψ defined by

$$\Upsilon = Ra^{1/2}\bar{\Upsilon}, \quad \Phi = Ra^{1/2}\bar{\Phi}, \quad x = Ra^{-1/2}X, \quad t = Ra^{-1/2}T, \quad \psi = Ra^{1/2}\Psi. \tag{8}$$

Here $\bar{\Upsilon}$, $\bar{\Phi}$, X, T and Ψ are assumed to be $O(1)$ as $Ra \to \infty$. Substituting (8) into (1)–(3), taking the limit $Ra \to \infty$ and rearranging, we find that

$$\Omega\left(-\frac{\partial \Psi}{\partial z}\frac{\partial \theta}{\partial X} + \frac{\partial \Psi}{\partial X}\frac{\partial \theta}{\partial z}\right) = \frac{\partial^2 \theta}{\partial X^2}, \tag{9}$$

$$\frac{\partial\phi}{\partial T}-\frac{\partial\phi}{\partial X}=-\frac{1}{\Omega\bar{\Phi}}\frac{\partial^2\theta}{\partial X^2}, \tag{10}$$

$$\frac{\partial^2\Psi}{\partial X^2}=-\frac{\partial\theta}{\partial X}, \tag{11}$$

where $\Omega = 1+\bar{\Upsilon}/\bar{\Phi}$. In [3] the boundary-value problem represented by(9), (11), (5) and (6) was considered by using two different approaches: numerical and approximate one.

We look for a similarity solution in the form of

$$\Psi = z^{1/2} f(\eta)/\Omega^{1/2}, \quad \theta = \theta(\eta), \quad \phi = \phi(\eta), \tag{12}$$

where $\eta = \Omega^{1/2}X/z^{1/2}$.

Then from (11) and (5), θ is given by

$$\theta(\eta) = -f'(\eta), \tag{13}$$

and from (9) and (6), f satisfies

$$f'''(\eta)+\frac{1}{2}f(\eta)f''(\eta)=0, \tag{14}$$

$$f(0)=0, \quad f'(0)=1, \quad f'(\eta)\to 0 \text{ as } \eta\to\infty. \tag{15}$$

The problem similar to (14), (15) appeared in papers by [2], where it was solved numerically only. Further, we study the behaviour of a solution of the problem (14), (15) and obtain the following proposition (see Appendix for proof):

Proposition 1. A solution of the problem (14), (15) satisfies the following estimates:

$$1.568 < f_\infty := f(\infty) < 2, \tag{16}$$

$$16(a^*)^3 \tanh\left(\frac{\eta}{8a^*}\right) \le f(\eta) \le \min\left[\eta, 2\tanh\left(\frac{\eta}{2}\right)\right] \tag{17}$$

for all $\eta \ge 0$, where $a^* \approx 0.461$.

From Proposition 1 it follows the qualified estimations of the main parameters of the initial problem (the stream function $\psi(x, z, t)$, the temperature $\theta(x, z, t)$, and local solid fraction $\phi(x, z, t)$) at the small time $t = O(Ra^{-1})$ only (see Appendix for details).

Proposition 2. A solution of the system (9)–(11) with the boundary conditions (4), (5) and (6) satisfies the following estimates:

$$\psi_{\min} := 16(a^*)^3\left(\frac{z\,Ra}{\Omega}\right)^{1/2}\tanh\left[\frac{x}{8a^*}\left(\frac{Ra\,\Omega}{z}\right)^{1/2}\right] \le$$

$$\le \psi_{\max} := 2\left(\frac{z\,Ra}{\Omega}\right)^{1/2}\tanh\left[\frac{x}{2}\left(\frac{Ra\,\Omega}{z}\right)^{1/2}\right], \tag{18}$$

$$\theta_{\min} := -1 + 2a \tanh\left[\frac{x}{2}\left(\frac{Ra\,\Omega}{z}\right)^{1/2}\right] \le \theta \le \theta_{\max} :=$$

$$:= -1 + 64\,(a^{*})^{3}\, a \tanh\left[\frac{x}{4}\left(\frac{Ra\,\Omega}{z}\right)^{1/2}\right], \tag{19}$$

$$\phi_{\min} := \left(\frac{Ra}{\Omega z}\right)^{1/2} \frac{a}{\Phi}\left\{1 - \tanh^{2}\left[\frac{x}{2}\left(\frac{Ra\,\Omega}{z}\right)^{1/2}\right]\right\} + \phi_{\infty} \le \phi \le$$

$$\le \phi_{\max} := \left(\frac{Ra}{\Omega z}\right)^{1/2} \frac{a}{\Phi}\left\{\tanh^{2}\left[\frac{x}{4}\left(\frac{Ra\,\Omega}{z}\right)^{1/2}\right] - 1\right\}^{2} + \phi_{\infty}, \tag{20}$$

at $t = O(Ra^{-1})$ for all $x \ge 0$, $z \ge 0$. Here $0 < a \le a^{*}$, a^{*} is from Proposition 1.

In comparison with the paper [3], our results describe completely the asymptotic behaviour of solution of system (9)–(11), which has the explicit (not numerical) representation. This is very important for concrete physical interests. It is significant that $\psi_{\infty}(z)$ is included in estimates (18), (19) and (20). That is, $\psi_{\infty}(z)$ has an influence on the estimation from below of the temperature $\theta(x, z, t)$ and the estimations from above of the local solid fraction $\phi(x, z, t)$ and the stream function $\psi(x, z, t)$. Thus, this influence is essential and cannot be ignored.

3. Unsteady free mush convection

In this section, we look for solutions of unsteady equations (1)–(3) for any Ra and t. As far as we concerned, this interesting situation was not considered before. We are succeeded in finding the explicit solution of system (1)–(3) (perhaps not unique). However, this solution characterizes the real physical behaviour of the mushy layer. In fact, there is obtained a family of solutions of the problem.

Since the convection into the mushy region is directed along the axis x then it seems very natural to seek for solution of system (1)–(3) in the form of a travelling wave. Let $\xi = x + t$. We will seek this solution of the problem in the view $\theta = \theta(\xi, z)$, $\psi = \psi(\xi, z)$ and $\phi = \phi(\xi, z)$. Then we arrive at the following system:

$$\Delta_{\xi,z}\theta = 0, \tag{20}$$

$$-\frac{\partial\psi}{\partial z}\frac{\partial\theta}{\partial\xi} + \frac{\partial\psi}{\partial\xi}\frac{\partial\theta}{\partial z} = 0, \tag{21}$$

$$\Delta_{\xi,z}\psi = -Ra\frac{\partial\theta}{\partial\xi}, \tag{22}$$

with boundary conditions on the flow and thermal fields:

$$\theta(\xi, z) = -1 \text{ at } \xi = 1, \quad \theta(\xi, z) \to 0 \text{ as } \xi \to \infty, z \ge 0, \tag{23}$$

$$\psi(\xi, z) = 0 \text{ at } z = 0, \xi \ge 0, \quad \frac{\partial \psi(\xi, z)}{\partial \xi} \to 0 \text{ as } \xi \to \infty, z \ge 0. \tag{24}$$

Proposition 3. System (1)–(3) with the boundary conditions (4), (5) and (7) has the explicit solution:

$$\theta = -1 + \frac{2}{\pi} \arctan\left(\frac{x}{z}\right), \tag{25}$$

$$\psi = \frac{Ra}{\pi} z + \frac{C_1 z}{z^2 + x^2} - \frac{Ra}{2\pi} z \ln(z^2 + x^2), \tag{26}$$

$$\phi(x, z, t) = \phi_0(x + t, z) \tag{27}$$

for all x, z, t such that $z^2 + x^2 = C_2^2$, where $\phi(x, z, 0) = \phi_0(x, z)$ due to condition (4), and $C_i \in \mathbb{R}^1$. Here, (27) means that the solid fraction is transmitted to the solid state.

The Proposition 3 has clear physical meaning, namely, the local solid fraction into a mushy region decreases in time under the temperature and the stream function which do not change in time. Below we show that Proposition 3 holds. Indeed, it is easy to check that the function

$$\theta(\xi, z) = -1 + \frac{2}{\pi} \arctan\left(\frac{\xi - t}{z}\right) \tag{28}$$

is an explicit solution of the boundary problem (20), (23).

First we derive an equation for the function $\psi(\xi, z)$. Let

$$\xi = r\cos\varphi + t, \quad z = r\sin\varphi,$$

then from equation (22) with $(\xi, z) \mapsto (r, \varphi)$ we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \varphi} = -Ra\frac{2}{\pi}\frac{2\sin\varphi}{r}.$$

We are looking for solutions of this equation in the following form

$$\psi(r, \varphi) = \kappa(r)\sin\varphi.$$

After simple computation we obtain the equation for function $f(r)$:

$$r^2\kappa''(r) + r\kappa'(r) - \kappa(r) = -\frac{2Ra}{\pi} r.$$

Solving this equation we find

$$\kappa(r) = \tilde{C}_1 r + \frac{\tilde{C}_2}{r} - \frac{Ra}{\pi} r \ln r \quad \forall \tilde{C}_i \in \mathbb{R}^1.$$

Then

$$\psi(\xi, z) = \tilde{C}_1 z + \frac{\tilde{C}_2 z}{z^2 + (\xi - t)^2} - \frac{Ra}{2\pi} z \ln\left(z^2 + (\xi - t)^2\right) \quad \forall \tilde{C}_i \in \mathbb{R}^1, \tag{29}$$

and the conditions (24) were satisfied. Substituting (28) and (29) i. e.

$$\psi_z = \tilde{C}_1 + \tilde{C}_2 \frac{(\xi - t)^2 - z^2}{\left(z^2 + (\xi - t)^2\right)^2} - \frac{Ra}{2\pi} \ln\left(z^2 + (\xi - t)^2\right) - \frac{Ra}{\pi} \frac{z^2}{z^2 + (\xi - t)^2},$$

$$\psi_\xi = -\tilde{C}_2 \frac{2z(\xi - t)}{\left(z^2 + (\xi - t)^2\right)^2} - \frac{Ra}{\pi} \frac{z(\xi - t)}{z^2 + (\xi - t)^2},$$

$$\theta_z = -\frac{2}{\pi} \frac{\xi - t}{z^2 + (\xi - t)^2}, \quad \theta_\xi = \frac{2}{\pi} \frac{z}{z^2 + (\xi - t)^2}$$

into (21), we obtain

$$\tilde{C}_1 - \frac{\tilde{C}_2}{z^2 + \left(\xi - t\right)^2} - \frac{Ra}{\pi} \ln\left(z^2 + (\xi - t)^2\right) = \frac{Ra}{\pi}. \tag{30}$$

Choosing $\tilde{C}_1 = \frac{Ra}{\pi}$, we find from (30) that

$$-\tilde{C}_2 \frac{\pi}{Ra} = \left(z^2 + (\xi - t)^2\right) \ln\left(z^2 + (\xi - t)^2\right). \tag{31}$$

As the function $\Phi(v) = v \ln v$ is monotone then there exists an inverse function $\Phi^{-1}(.)$. Using this fact, we obtain from (31) that

$$z^2 + \left(\xi - t\right)^2 = \Phi^{-1}\left(-\frac{2\pi\tilde{C}_2}{Ra}\right) = C^2, \tag{32}$$

where C is an arbitrary constant. Thus, the equality (16) is valid if the variables ξ and z are satisfied to the relation (32). Using changing of variables $(\xi, z) \mapsto (x, z, t)$ in (28), (29) and (32) we obtain the following explicit solution of system (1)–(3) with conditions (4), (5) and (6). Thus, Proposition 3 is proved completely.

4. Convergence to the travelling wave

Assume that $\Pi = 1$ in (3), and $\phi \equiv \phi_0$ is a positive constant. Changing variables $(x, z, t) \mapsto (\xi, z, t)$ in (1)–(3), where $\xi = x + t$, we obtain the following system

$$\theta_t + \mathbf{u} \cdot \nabla_{\xi,z} \theta = \Delta_{\xi,z} \theta, \tag{33}$$

$$(1 - \phi_0)\theta_t + \mathbf{u} \cdot \nabla_{\xi,z} \theta = 0, \tag{34}$$

$$-\Delta_{\xi,z} \psi = Ra\theta_\xi, \tag{35}$$

where $\mathbf{u} = (-\psi_z, \psi_\xi)$. We can reduce the system (33)–(35) to the one

$$\phi_0 \theta_t = \Delta_{\xi,z} \theta, \tag{36}$$

$$-\Delta_{\xi,z}\psi = Ra\theta_\xi \tag{37}$$

in the half-space $R_+ := \{\xi \geq 0, z \geq 0\}$. Note that if the solid fraction is absent, i.e. $\phi_0 = 0$, then we obtain that the solution (25), (26) is unique.

Let us denote by $v := \theta - \theta_{st}$. Then from (36) and (20) we obtain for v the following equation

$$\phi_0 v_t = \Delta_{\xi,z} v, \quad v(\xi, z, 0) = v_0(\xi, z) := \theta(\xi, z, 0) - \theta_{st}(\xi, z). \tag{38}$$

Problem (38) has the solution

$$v(\xi, z, t) = \frac{\phi_0}{4\pi t} \int_0^{+\infty}\int_0^{+\infty} e^{-\frac{\phi_0\left((\bar{\xi}-\xi)^2 + (\bar{z}-z)^2\right)}{4t}} v_0(\bar{\xi}, \bar{z}) d\bar{\xi} d\bar{z} \to 0 \text{ as } t \to +\infty. \tag{39}$$

From (39) it follows that $\theta \to \theta_{st}$ as $t \to +\infty$. Hence, it follows from this and (37) that $\psi \to \psi_{st}$ as $t \to +\infty$.

Conclusion

In this paper we consider situations of steady and unsteady free mush convection. For the steady regime the qualified estimates for the stream function, temperature, and local solid fraction are found for large values of the Rayleigh number and small time. For the unsteady case the precise behavior of these main characteristics is established. At that the behaviour of the temperature and stream function depend on the measured vertical upwards. The local solid fraction decreases under stationary behaviour of the temperature and the stream function.

We use the non-similar solution technique giving us possibility to establish the qualified estimates of the main characteristics of the process. A detailed analysis of solidification in mushy region is provided under the assumption that the permeability of the mush is uniform.

Appendix

Proof of Proposition 1

We consider the following auxiliary Cauchy problem for the problem (14), (15):

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0, \tag{40}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -a, \tag{41}$$

where $a > 0$. Now we show that there exists a parameter a such that the solution of problem (40), (41) satisfy the conditions

$$f'(\eta) \to 0 \text{ as } \eta \to \infty; \ f(\eta) \geq 0 \text{ is uniformly bounded.} \tag{42}$$

By (40) and (41) we deduce that

$$f''(\eta) = -a\exp\left\{-\frac{1}{2}\int_0^\eta f(z)\mathrm{d}z\right\}, \tag{43}$$

$$f'(\eta) = 1 - a\int_0^\eta \exp\left\{-\frac{1}{2}\int_0^y f(z)\mathrm{d}z\right\}\mathrm{d}y, \tag{44}$$

$$f(\eta) = \eta - a\int_0^\eta\int_0^y \exp\left\{-\frac{1}{2}\int_0^v f(z)\mathrm{d}z\right\}\mathrm{d}v\mathrm{d}y. \tag{45}$$

From (43) it follows that f' decreases and f is concave. Hence, in view of $f'''(\eta) = -\frac{1}{2}f(\eta)f''(\eta) \ge 0$, we arrive at

$$f'(\eta) \text{ is convex and } 0 \le f'(\eta) \le 1 \quad \forall\eta \ge 0. \tag{46}$$

It follows from (45) that $f(\eta) \le \eta$, whence

$$f(\eta) \le \eta - a\int_0^\eta\int_0^y \exp\left(-\frac{v^2}{4}\right)\mathrm{d}v\mathrm{d}y, \tag{47}$$

and

$$f'(\eta) = 1 - a\int_0^\eta \exp\left(-\frac{1}{2}\int_0^y f(z)\mathrm{d}z\right)\mathrm{d}y \le 1 - a\int_0^\eta \exp\left(-\frac{y^2}{4}\right)\mathrm{d}y. \tag{48}$$

In view of (48), we find that

$$0 = f'(\infty) \le 1 - a\int_0^\infty \exp\left(-\frac{y^2}{4}\right)\mathrm{d}y,$$

whence it follows that

$$0 \le a \le a_1 = \frac{1}{\int_0^\infty \exp\left(-\frac{y^2}{4}\right)\mathrm{d}y} = \frac{1}{\sqrt{\pi}} \approx 0.5641895. \tag{49}$$

The last inequality guarantees that conditions (42) is valid for any a satisfying (49). Moreover, using (47), from (45) we deduce

$$f(\eta) \le \eta - a\int_0^\eta\int_0^y \exp\left\{-\frac{v^2}{4} + \frac{a}{2}\int_0^v\int_0^z \exp\left(-\frac{w^2}{4}\right)\mathrm{d}w\mathrm{d}z\right\}\mathrm{d}v\mathrm{d}y. \tag{50}$$

Analogously to (49), we find from

$$0 = f'(\infty) \le 1 - a\int_0^\infty \exp\left\{-\frac{y^2}{4} + \frac{a}{2}\int_0^y\int_0^z \exp\left(-\frac{w^2}{4}\right)\mathrm{d}w\mathrm{d}z\right\}\mathrm{d}y,$$

that

$$0 \le a \le a_2 = 0.45342952. \tag{51}$$

Continuing the same iteration procedure, we can find the sharp upper bound a_∞ for the desired a, i.e. $a \le a_\infty$.

Now we show the estimate from below of the solution to problem (40), (41). It follows from (45) that

$$f(\eta) \ge \eta - a\int_0^\eta\int_0^y \mathrm{d}v\mathrm{d}y = \eta - \frac{a}{2}\eta^2.$$

We denote by f_1 the function on the right-hand side of the last inequality:

$$f_1(\eta) := \eta - \frac{a}{2}\eta^2, \quad 0 \le x \le \frac{2}{a}.$$

Taking into account that $(f_1)'_\eta = 1 - a\eta = 0$ for $\eta_{\max} = \frac{1}{a}$, we see that

$$f_1\left(\frac{1}{a}\right) = \frac{1}{a} - \frac{a}{2}\frac{1}{a^2} = \frac{1}{2a}$$

Hence,

$$f(\eta) \ge \eta - \frac{a}{2}\eta^2, \quad 0 \le x \le \frac{1}{a}.$$

From the decreasing of f' it follows that $f(\eta) \ge \frac{1}{2a} = A$ for $x \ge \frac{1}{a}$. Using (45) we obtain that

$$f(\eta) \ge \eta - a\int_0^\eta\int_0^y \exp\left(-\frac{A}{2}v\right)\mathrm{d}v\mathrm{d}y = \eta - a\int_0^\eta \frac{2}{A}\left\{1 - \exp\left(-\frac{A}{2}y\right)\right\}\mathrm{d}y =$$

$$= \eta - a\frac{2}{A}\left\{\eta - \frac{2}{A}\left[1 - \exp\left(-\frac{A}{2}\eta\right)\right]\right\} \ge \frac{4a}{A^2}\left[1 - \exp\left(-\frac{A}{2}\eta\right)\right],$$

where $1 - \frac{2a}{A} \ge 0$. Then $A \ge 2a$, $a^2 \le \frac{1}{4}$, whence $0 \le a \le \frac{1}{2}$. Let us denote by f_2 the following function

$$f_2(\eta) = 16a^3\left[1 - \exp\left(-\frac{\eta}{4a}\right)\right].$$

Then

$$f(\eta) \ge 16a^3\left[1 - \exp\left(-\frac{\eta}{4a}\right)\right], \ \eta \ge \frac{1}{a}.$$

For the continuity we suppose that $f_1\left(\frac{1}{a}\right) = f_2\left(\frac{1}{a}\right)$:

$$\frac{1}{2a} = 16a^3\left[1-\exp\left(-\frac{1}{4a^2}\right)\right],$$

$$\exp\left(-\frac{1}{4a^2}\right) = 1-\frac{1}{32a^4} > 0 \Rightarrow a^4 > \frac{1}{32} \Rightarrow a > 2^{-5/4}.$$

Then for $2^{-5/4} \approx 0.42045 < a \le 1/2$ we have

$$4a^2 \ln\left(1-\frac{1}{32a^4}\right)+1 = 0 \Rightarrow a^* = 0.46106906.$$

Finally, we derive

$$f(\eta) \ge \begin{cases} \eta - \dfrac{a^*}{2}\eta^2, & 0 \le \eta \le \dfrac{1}{a^*}, \\ 16(a^*)^3\left[1-\exp\left(-\dfrac{\eta}{4a^*}\right)\right], & \eta \ge \dfrac{1}{a^*}, \end{cases}$$

where $a^* = 0.46106906$. This means that

$$f(\eta) \ge 16(a^*)^3\left[1-\exp\left(-\frac{\eta}{4a^*}\right)\right] \ge f_{\min}(\eta) := 16(a^*)^3 \tanh\left(\frac{\eta}{8a^*}\right) \quad \forall\, \eta \ge 0, \quad (52)$$

where $f_{\min}(\infty) \approx 1.568259$. Due to the estimate (52), we need that $0 \le a \le a^*$. As $a^* > a_\infty$ then (52) is the lower estimate.

Coming back to equation (40) and using (48), we have

$$\left(f'(\eta)+\frac{1}{4}f^2(\eta)\right)'' = \frac{1}{2}\left(f'(\eta)\right)^2 \le \frac{1}{2}.$$

Integrating this inequality with (41), we deduce that

$$f'(\eta)+\frac{1}{4}f^2(\eta) \le \frac{\eta^2}{4} - a\eta + 1.$$

As $\frac{\eta^2}{4} - a\eta + 1 \le 1$ for $0 \le \eta \le 4a$ then, solving $f'(\eta)+\frac{1}{4}f^2(\eta) \le 1$ with $f(0) = 0$, $f'(0) = 1$, we find that

$$f(\eta) \le f_{\max}(\eta) := 2\tanh\left(\frac{\eta}{2}\right) \quad (53)$$

for all $\eta: 0 \le \eta \le 4a$. As the graph of the right-hand side of (50) lies under the one of $2\tanh\left(\frac{\eta}{2}\right)$ and $f''_{\max}(0) = 0$, then choosing $a \le a_\infty$ we obtain that the estimate (53) is valid for all $\eta \ge 0$.

Thus, there is the the parameter point $a \in (0, a^*)$ such that the problem (14), (15) has an unique solution, and the following estimates hold:

$$f_{\min} \le f \le f_{\max}.$$

Moreover $1.568 \le f(\infty) \le 2$. These estimates provide reliable analytical information about the behaviour of solution.

Proof of Proposition 2

The estimates (18) is a simple corollary from (17) that is,

$$16(a^*)^3\left(\frac{Raz}{\Omega}\right)^{1/2}\tanh\left(\frac{\eta}{8a^*}\right) \le \psi \le 2\left(\frac{Raz}{\Omega}\right)^{1/2}\tanh\left(\frac{\eta}{2}\right). \quad (54)$$

From the equalities (44) and (13), in view of estimate (17), we deduce

$$-1+2a\tanh\left(\frac{\eta}{2}\right) \le \theta(\eta) \le -1+64(a^*)^3 a\tanh\left(\frac{\eta}{4}\right). \quad (55)$$

Let us obtain estimations of function $\phi(\eta)$. From (10) we have

$$\phi(\eta) = \frac{\theta'(\eta)}{\Omega^{1/2}\bar{\Phi}z^{1/2}} + \phi_\infty, \quad (56)$$

where ϕ_∞ is defined by (4). Then, taking into account (43), we obtain from (56) that

$$\frac{a}{\Phi}\left(\frac{Ra}{z\Omega}\right)^{1/2}\exp\left(-\int_0^\eta \tanh\left(\frac{y}{2}\right)dy\right) + \phi_\infty \le \phi(\eta) \le$$

$$\le \frac{a}{\Phi}\left(\frac{Ra}{z\Omega}\right)^{1/2}\exp\left(-8(a^*)^3\int_0^\eta \tanh\left(\frac{y}{8a^*}\right)dy\right) + \phi_\infty. \quad (57)$$

Inequalities (19) and (20) follow from estimates (55), (57).

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ТЕОРЕТИЧЕСКИЕ ИССЛЕДОВАНИЯ ИДЕАЛЬНОЙ МОДЕЛИ «MUSHY REGION»

В данной работе исследуется поведение функции потока, температуры и локальной твердой фракции для идеальной модели смешанного слоя. В случае устойчивой свободной конвекции найдены точные нижние и верхние оценки основных функций, характеризующих процесс, при больших значениях числа Рэлея. Для нестационарного режима также найден явный вид этих основных характеристик.

Ключевые слова: смешанный слой, устойчивая и неустойчивая конвекция, функция потока, температура, локальные твердые и жидкие фракции