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P. P. KONONCHUK

ONE-DIMENSIONAL MODEL OF THE DIFFUSION PROCESS WITH A MEMBRANE THAT IS DESCRIBED BY THE FELLER–WENTZEL CONJUGATION CONDITION

An integral representation of the operator semigroup that corresponds to the most general nonbreaking Feller process on a line that is pasted from two diffusion processes is constructed, by using analytical methods.

1. Basic notations and statement of the problem

Let $D_i = \{x : (-1)^i x > 0\}$, $i = 1, 2$ be two domains on the number axis, let $\partial D = \{0\}$ be the common boundary of these domains, and let $\overline{D_i} = D_i \cup \partial D$ be a closure of D_i. If D is a domain in $\mathbb{R}$, and $\overline{D}$ is its closure, then $C_b(\overline{D})$ means a Banach space of all continuous bounded functions $\varphi(x)$ on $\overline{D}$ with norm $\|\varphi\| = sup_{x\in\overline{D}}|\varphi(x)|$, and $C_0(\overline{D})$ denotes a space of bounded uniformly continuous functions on $\overline{D}$. For every function $\varphi \in C_b(\mathbb{R})$, we denote a section of φ on $\overline{D}_i$ by φ_i. By $C_2(\overline{D}_i)$, we denote a subset of $C_b(\overline{D}_i)$ that consists of all functions such that $\varphi_i, \varphi_i', \varphi_i'' \in C_0(\overline{D}_i)$, $i = 1, 2$.

Assume that the differential operators L_i, $i = 1, 2$, are given in domains D_i. Let they generate some diffusion processes on $C_2(\overline{D}_i)$:

$$L_i\varphi(x) = \frac{1}{2}b_i(x)\frac{d^2\varphi_i}{dx^2}(x) + a_i(x)\frac{d\varphi_i}{dx}(x), \quad i = 1, 2, \tag{1}$$

where $b_i(x)$ and $a_i(x)$ are bounded continuous functions on $\overline{D}_i$, and $b_i(x) \geq 0$. By $C_2(\mathbb{R})$, we denote a subset of $C_b(\mathbb{R})$ that consists of all functions $\varphi(x)$ such that $\varphi_i \in C_2(\overline{D}_i)$, $i = 1, 2$, and $L_1\varphi_1(0) = L_2\varphi_2(0)$. We also define an operator L that acts on $C_2(\mathbb{R})$ by the following rule:

$$L\varphi(x) = \begin{cases} L_1\varphi(x), & if x \in \overline{D}_1, \\ L_2\varphi(x), & if x \in \overline{D}_2. \end{cases} \tag{2}$$

We will write an additional conjugation condition at the point $x = 0$ that contracts the operator L (or its closure) to some infinitesimal operator of Feller semigroups on $C_0(\mathbb{R})$. This conditions is as follows:

$$L_0\varphi(0) \equiv \sigma L\varphi(0) + q_1\varphi'(0-) - q_2\varphi'(0+) + \int_{D_1\cup D_2} (\varphi(0) - \varphi(y))\,\mu(dy) = 0. \tag{3}$$

Here, σ, q_i, $i = 1, 2$, are nonnegative numbers, and $\mu(\cdot)$ is a nonnegative measure on $D_1 \cup D_2$ and such that

$$\int_{(D_1\cup D_2)\setminus D_\delta} |y|\mu(dy) < \infty, \quad \mu(D_\delta) < \infty, \tag{4}$$

where $D_\delta = \{x \in \mathbb{R} : |x| > \delta > 0\}$. In this case, the numbers σ, q_1, q_2, and $m = \mu(D_1\cup D_2)$ are not equal to zero simultaneously.

We note (see [1]–[3]) that if we consider only nonbreaking Markov processes on the line, then condition (3) describes various continuations of the given processes after reaching

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the point $x = 0$. In our case, the the right-hand side of equality (3) corresponds to possible continuations of the process such as their delay, partial reflection, and jumps into one of the domains D_1 or D_2.

In the present work, we consider the problem of existence of an operator semigroup $\{\mathcal{T}_t\}_{t\geq 0}$ that describes a Feller process (not necessarily continuous) on $\mathbb{R}$ such that, in the domains D_i, $i = 1, 2$, it coincides with diffusion processes controlled by the operators L_i, $i = 1, 2$, and its behavior at the point $x = 0$ is determined by the conjugation condition (3). We consider the case where σ from (3) is not equal to zero, i.e., where $\sigma > 0$. The considered problem is often called the problem of the pasting of two diffusion processes on a line (see, e.g., [1, 4, 5, 6, 7]).

We will use analytical methods to construct a semigroup. Within these methods (see [4, 5, 6, 7]), solving the problem is reduced to studying the corresponding conjugation problem for a second-order linear parabolic equation with discontinuous coefficients. The last problem consists in finding a function $u(t, x)$ $(t > 0, x \in \mathbb{R})$ that satisfies the following conditions:

$$\frac{\partial u}{\partial t}(t, x) = L_i u(t, x), \quad t > 0,\, x \in D_i,\, i = 1, 2, \tag{5}$$

$$u(0, x) = \varphi(x), \quad x \in \mathbb{R}, \tag{6}$$

$$u(t, 0-) = u(t, 0+), \quad t \geq 0, \tag{7}$$

$$L_0 u(t, 0) = 0, \quad t > 0, \tag{8}$$

where $\varphi \in C_b(\mathbb{R})$ is a given function.

We note that the conjugation condition (7) from problem (5)–(8) corresponds to the Feller property of the process, and equality (8) corresponds to the conjugation condition (3).

Under some additional assumptions on coefficients of the operator L_i, $i = 1, 2$, we first proved the classical solvability of problem (5)–(8) by the method of boundary integral equations, by using the ordinary fundamental solution of the parabolic equation, as well as the potentials generated by it.

We note that a similar problem was studied in [5] and [6] (also see [4]) under assumptions that $\sigma = 0$, $m = 0$ and $\sigma \neq 0$, $m = 0$, respectively, and in [7] in the case where $\sigma = q_1 = q_2 = 0$ and the measure $\mu(\cdot)$ is finite. We also mention paper [8] where a Markov process pasted from two diffusion processes was first obtained under assumptions that $\sigma = m = 0$ and $q_1 = q_2$ with the use of a somewhat different method (we call this case as the classical pasting of two diffusion processes).

In what follows, we use the following notations: T is a fixed positive number; $\mathbb{R}^2_T \equiv (0, T] \times \mathbb{R}$, $\mathbb{R}^2_\infty \equiv (0, \infty) \times \mathbb{R}$; (t, x) is a point in $\overline{\mathbb{R}}_\infty$, D^r_t and D^p_x are symbols of the r-th partial derivatives with respect to the variable t and the p-th partial derivative with respect to the variable x, respectively, where r and p are nonnegative integers; $C^{m,l}(\Omega)$ $(C^{m,l}(\overline{\Omega}))$, $m = 0, 1$, $l = 0, 1, 2$ $(C^{0,0}(\Omega) \equiv C(\Omega)$, $C^{0,0}(\overline{\Omega}) \equiv C(\overline{\Omega}))$ are sets of functions continuous in Ω (in $\overline{\Omega}$) with continuous derivatives D^r_t, D^p_x, $r \leq m$, $p \leq m$, in Ω $(\overline{\Omega})$, and Ω is a subset of $\mathbb{R}^2_\infty$; and $H^\alpha(\mathbb{R})$, $\alpha \in (0, 1)$, is a Hölder space as in [8]. By C and c, we denote some constants that do not depend on (t, x), and their exact values are of no interest to us.

2. Fundamental solution of the second-order parabolic equation and the potentials generated by it

Consider the parabolic operators $D_t - L_i$, $i = 1, 2$, in a domain R^2_∞. Without any loss of generality, we assume that coefficients of the operators are defined on $\mathbb{R}$ and satisfy the following conditions:

a) b_i, $a_i \in H^\alpha(\mathbb{R})$, $i = 1, 2$;

b) there exist positive constants b_{i0}, $i = 1, 2$, such that $b_i(x) \geq b_{i0}$ for all $x \in \mathbb{R}$.

Let $g_i(t, x, y)$ ($t > 0$, $x, y \in \mathbb{R}$) be the fundamental solution (f.s.) of the equation with the operators $D_t - L_i$, $i = 1, 2$ (see [4, Ch. II, §2] and [8, Ch. IV, §11]):

$$g_i(t, x, y) = g_{i0}(t, x, y) + g_{i1}(t, x, y), \quad i = 1, 2, \tag{9}$$

where

$$g_{i0}(t, x, y) = (2\pi b_i(y)t)^{-\frac{1}{2}} \exp\left(-\frac{(y-x)^2}{2b_i(y)t}\right), \tag{10}$$

g_{i1} are integral terms with a weaker singularity than g_{0i}, when $t \to 0$; in addition, $g_i(t, x, y) = 0$, when $t \leq 0$. We recall that the functions $g_i(t, x, y)$, $i = 1, 2$, are continuous and positive in all variables, satisfy Eq. (5) at a fixed y in the domain $(t, x) \in \mathbb{R}^2_\infty$, and are such that, for every $x \in \mathbb{R}$ and $\varphi \in C_b(\mathbb{R})$,

$$\lim_{t \downarrow 0} \int_{\mathbb{R}} g_i(t, x, y)\varphi(y)dy = \varphi(x), \quad i = 1, 2.$$

We present the estimations ($i = 1, 2$, x, $y \in \mathbb{R}$)

$$|D_t^r D_x^p g_i(t, x, y)| \leq Ct^{-\frac{1+2r+p}{2}} \exp\left(-c\frac{(y-x)^2}{t}\right), \ 2r + p \leq 2, \ t \in (0, T], \tag{11}$$

$$|D_t^r D_x^p g_{i1}(t, x, y)| \leq Ct^{-\frac{1+2r+p-\alpha}{2}} \exp\left(-c\frac{(y-x)^2}{t}\right), \ 2r + p \leq 2, \ t \in (0, T] \tag{12}$$

and equalities ($t > 0$, $x \in \mathbb{R}$)

$$\begin{aligned} \int_{\mathbb{R}} g_i(t, x, y)dy \ &= \ 1, \quad i = 1, 2, \\ \int_{\mathbb{R}} g_i(t, x, y)(y - x)dy \ &= \ \int_0^t d\tau \int_{\mathbb{R}} g_i(\tau, x, y)a_i(y)dy, \ i = 1, 2, \\ \int_{\mathbb{R}} g_i(t, x, y)(y - x)^2 dy \ &= \ \int_0^t d\tau \int_{\mathbb{R}} g_i(\tau, x, y)b_i(y)dy + \\ &+ \ 2\int_0^t d\tau \int_{\mathbb{R}} g_i(\tau, x, y)a_i(y)(y - x)dy, \ i = 1, 2. \end{aligned} \tag{13}$$

Consider the integrals

$$u_{i0}(t, x) \ = \ \int_{\mathbb{R}} g_i(t, x, y)\varphi(y)dy, \ i = 1, 2, \tag{14}$$

$$u_{i1}(t, x) \ = \ \int_0^t g_i(t - \tau, x, 0)V_i(\tau)d\tau, \ i = 1, 2. \tag{15}$$

Here, $\varphi(x)$ and $V_i(t)$, $i = 1, 2$, are given functions. In the theory of parabolic equations, the functions u_{i0} and u_{i1} are called the Poisson potential and the simple-layer potential, respectively. It follows from the definition and properties of f.s. g_i, $i = 1, 2$, that if a function $\varphi \in C_b(\mathbb{R})$, then $u_{i0}(t, x)$ are continuous in $\overline{\mathbb{R}^2_\infty}$, bounded in the variable x, and satisfy Eq. (5) in a domain $(t, x) \in \mathbb{R}^2_\infty$ and the initial condition (6). The following inequality holds in $\mathbb{R}^2_T$:

$$|D_t^r D_x^p u_{i0}(t, x)| \leq C\|\varphi\|t^{-\frac{2r+p}{2}}, \quad 2r + p \leq 2, \ i = 1, 2. \tag{16}$$

Let us assume that the functions $V_i(t)$, $i = 1, 2$, from (15) are bounded and continuous in $[0, \infty)$. Then the functions $u_{i1}(t, x)$ are continuous in $\overline{\mathbb{R}^2_\infty}$, bounded in the variable x, and satisfy Eq. (5) in domains $(t,\, x) \in (0, \infty) \times D_1$ and $(t,\, x) \in (0, \infty) \times D_2$ and the initial condition $u_{i1}(0,x) = 0$, $x \in \mathbb{R}$, $i = 1,2$. We mention another important property of the simple-layer potential $u_{i1}(t,x)$. It concerns with a behavior of the derivatives with respect to the variable x in a neighborhood of the point $x = 0$. It is well known (see [4, Ch. II, §5], [8, Ch. IV, §15]) that the so-called jump formula for the derivative of a simple-layer potential holds and has, in our case, the form

$$D_x u_{i1}(t, 0\pm) = \int_0^t D_x g_{i1}(t-\tau, 0, 0) V_i(\tau) d\tau \mp \frac{V_i(t)}{b_i(0)}, \quad t > 0, \, i = 1,2. \tag{17}$$

We note that the existence of the integral in (17) follows from inequality (12) at $r = 0$, $p = 1$, $x = y = 0$.

We also mention that the stated properties of the simple-layer potential will hold under more general assumptions on the functions V_i, $i = 1,2$, from (15).

3. Solution of the parabolic conjugation problem

Consider problem (5)–(8). First, we prove the existence of a solution $u(t,x)$. We will try to find it as a sum of potentials:

$$u(t,x) = u_{i0}(t,x) + u_{i1}(t,x), \quad (t,x) \in [0,\infty) \times \overline{D_i}, \, i = 1,2, \tag{18}$$

where the functions u_{i0} and u_{i1} are defined by formulas (14) and (15), respectively. In addition, the densities V_i, $i = 1,2$, are unknown functions. To find them, we use the conjugation conditions (7) and (8). We denote, by $v(t)$, the value of the function $u(t,x)$ at $x = 0$ and substitute (18) in the conjugation condition (8). Using relations (9), (10), and (17), we obtain the equality

$$\begin{aligned} \frac{dv(t)}{dt} &= \sum_{j=1}^{2} (-1)^j \frac{q_i}{\sigma} \frac{\partial u_{j0}(t,0)}{\partial x} + \frac{1}{\sigma} \sum_{j=1}^{2} \int_{D_j} (u_{j0}(t,y) - u_{j0}(t,0)) \, \mu(dy) - \\ &- \sum_{j=1}^{2} \frac{q_j}{\sigma b_j(0)} V_j(t) + \sum_{j=1}^{2} \int_0^t V_j(\tau) d\tau \Bigg((-1)^j \frac{q_j}{\sigma} \frac{\partial g_{j1}(t-\tau,0,0)}{\partial x} + \\ &+ \frac{1}{\sigma} \int_{D_j} (g_j(t-\tau,y,0) - g_j(t-\tau,0,0)) \mu(dy) \Bigg), \quad t > 0. \end{aligned}$$

In view of the initial condition (7), we obtain

$$\begin{aligned} v(t) &= \sum_{j=1}^{2} \int_0^t \left((-1)^j \frac{q_j}{\sigma} \frac{\partial u_{j0}(\tau,0)}{\partial x} + \frac{1}{\sigma} \int_{D_j} (u_{j0}(\tau,y) - u_{j0}(\tau,0)) \mu(dy) \right) d\tau - \\ &- \sum_{j=1}^{2} \frac{q_j}{\sigma b_j(0)} \int_0^t V_j(\tau) d\tau + \sum_{j=1}^{2} \int_0^t V_j(\tau) d\tau \int_\tau^t \Bigg((-1)^j \frac{q_j}{\sigma} \frac{\partial g_{j1}(\beta-\tau,0,0)}{\partial x} + \\ &+ \frac{1}{\sigma} \int_{D_j} (g_j(\beta-\tau,y,0) - g_j(\beta-\tau,0,0)) \mu(dy) \Bigg) d\beta + \varphi(0), \quad t > 0. \end{aligned} \tag{19}$$

Now, we have obtained three different expressions for the function $v(t) = u(t,0)$: the first one is defined by Eq. (19), and two others are defined by formula (18), where we have to put $x = 0$ and use the conjugation condition (7). Equating the right-hand sides of the expressions for $v(t)$ and $u(t,0-)$ and then for $v(t)$ and $u(t,0+)$, we obtain the system of integral equations for V_1 and V_2,

$$\int_0^t g_i(t-\tau,0,0) V_i(\tau) d\tau + \sum_{j=1}^{2} \int_0^t K_{j0}(t-\tau) V_j(\tau) d\tau = \Phi_i(t), \quad t > 0, \quad i = 1,2, \tag{20}$$

where

$$\begin{aligned}
\Phi_i(t) &= \sum_{j=1}^{2}\int_0^t \left((-1)^j \frac{q_j}{\sigma}\frac{\partial u_{j0}(\tau,0)}{\partial x} + \frac{1}{\sigma}\int_{D_j}(u_{j0}(\tau,y)-u_{j0}(\tau,0))\mu(dy)\right)d\tau - \\
&- u_{i0}(t,0)+\varphi(0), \quad i=1,2,\\
K_{j0}(t-\tau) &= \frac{q_j}{\sigma b_j(0)} - \int_\tau^t \left((-1)^j\frac{q_j}{\sigma}\frac{\partial g_{j1}(\beta-\tau,0,0)}{\partial x} + \right.\\
&+ \left.\frac{1}{\sigma}\int_{D_j}(g_j(\beta-\tau,y,0)-g_j(\beta-\tau,0,0))\mu(dy)\right)d\beta, \quad j=1,2.
\end{aligned}$$

The system of equations (20) is a system of Volterra integral equations of the first kind, in which their right-hand sides, i.e. $\Phi_i(t)$, $i=1,2$, are continuous at $t\geq 0$ and continuously differentiable at $t>0$. Using the Holmgren method (see, e.g., [10]), we reduce the system of equations to a system of Volterra integral equations of the second kind. For this purpose, we define the integro-differential operator $\mathcal{E}$:

$$\hat{\Phi}_i(t) = \mathcal{E}(t)\Phi_i = \sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^t (t-s)^{-\frac{1}{2}}\Phi_i(s)ds, \quad t>0,\ i=1,2. \tag{21}$$

Since the functions $\Phi_i(t)$, $i=1,2$ are continuously differentiable at $t>0$, then it is easy to verify that the right-hand side of Eq. (21) can be re-arranged into the form

$$\hat{\Phi}_i(t) = \frac{1}{\sqrt{2\pi}}\int_0^t (t-s)^{-\frac{3}{2}}(\Phi_i(t)-\Phi_i(s))ds + \sqrt{\frac{2}{\pi}}\Phi_i(t)t^{-\frac{1}{2}}, \quad i=1,2,\ t>0. \tag{22}$$

Using formula (22), inequality (16), and condition (4), we prove that the functions $\hat{\Phi}_i(t)$, $i=1,2$, satisfy the following estimate in every domain $t\in(0,T]$:

$$|\hat{\Phi}_i(t)| \leq C\|\varphi\| t^{-\frac{1}{2}}, \quad i=1,2. \tag{23}$$

Applying the operator $\mathcal{E}$ from (21) to both sides of each of the equations in system (20), we obtain, after easy transformations, an equivalent system of Volterra integral equations of the second kind

$$V_i(t) = \sum_{j=1}^{2}\int_0^t K_{ij}(t-\tau)V_j(\tau)d\tau + \Psi_i(t), \quad t>0, \quad i=1,2, \tag{24}$$

where

$$\begin{aligned}
\Psi_i(t) &= \sqrt{b_i(0)}\hat{\Phi}_i(t) = \sqrt{b_i(0)}\mathcal{E}(t)\Phi_i, \quad i=1,2,\\
K_{ii}(t-\tau) &= \sqrt{\frac{2b_i(0)}{\pi}}(R_i(t-\tau)+K_i(t-\tau)), \quad i=1,2,\\
K_{ij}(t-\tau) &= \sqrt{\frac{2b_i(0)}{\pi}}K_j(t-\tau), \quad i,j=1,2, \quad i\neq j,\\
R_i(t-\tau) &= \frac{1}{2}\int_\tau^t (t-s)^{-\frac{3}{2}}(g_{i1}(s-\tau,0,0)-g_{i1}(t-\tau,0,0))ds - \\
&- (t-\tau)^{-\frac{1}{2}}g_{i1}(t-\tau,0,0), \quad i=1,2,\\
K_j(t-\tau) &= -\frac{q_j}{\sigma b_j(0)}(t-\tau)^{-\frac{1}{2}} + \frac{1}{\sigma}\int_\tau^t (t-s)^{-\frac{1}{2}}\left((-1)^j q_j \frac{\partial g_{j1}(s-\tau,0,0)}{\partial x} + \right.\\
&+ \left.\int_{D_j}(g_j(s-\tau,y,0)-g_j(s-\tau,0,0))\mu(dy)\right)ds, \quad j=1,2.
\end{aligned}$$

Using inequalities (11) and (12) and condition (4), we obtain an estimate for the functions $K_{ij}(t-\tau)$, $i, j = 1,2$ $(0 \le \tau < t \le T)$:

$$|K_{ij}(t-\tau)| \le C(t-\tau)^{-1+\frac{\alpha}{2}}. \tag{25}$$

Since the kernels of each equation from system (24) are weakly integrable, we find $V_i(t)$, $i = 1,2$ within the method of successive approximations. As a consequence, we obtain that the functions $V_i(t)$, $i = 1,2$ are continuous at $t > 0$ and satisfy estimation (23) in the domain $t \in (0,T]$. Estimation (23) for V_i, $i = 1,2$, together with estimation (11) give us the existence of the simple-layer potentials (18) and the same inequality as that for Poisson potentials (see estimation (16)):

$$|u_{i1}(t,x)| \le C\|\varphi\|, \quad i = 1,2,\ (t,x) \in \overline{\mathbb{R}^2_T}. \tag{26}$$

Hence, the existence of a classical solution of the parabolic conjugation problem (5)–(8) is proved. As concerns the conclusion of its uniqueness, we only notice that the computed function $u(t,x)$ can be interpreted as the unique solution of the first parabolic boundary-value problem in every domain $t > 0$, $x \in \overline{D_1}$ and $t > 0$, $x \in \overline{D_2}$:

$$\begin{gathered}\frac{\partial u}{\partial t} = L_i u, \quad t > 0, x \in D_i, i = 1,2,\\ u(0,x) = \varphi(x), \quad x \in \overline{D_i}, i = 1,2,\\ u(t,0) = v(t), \quad t \ge 0,\end{gathered}$$

where the function $v(t)$ is defined by relation (19).

The results obtained in this part of the study can be formulated as the following theorems.

Theorem 3.1. *Assume that the coefficients of operators L_i, $i = 1,2$ from (1) satisfy conditions a) and b). The parameters σ, q_i, $i = 1,2$, from (3) are such that $\sigma > 0$, $q_i \ge 0$, $i = 1,2$, and the measure μ satisfies condition (4). Then problem (5)–(8) has the unique solution*

$$u \in C(\overline{\mathbb{R}^2_\infty}) \cap C^{1,2}((0,\infty) \times D_i), \quad i = 1,2, \tag{27}$$

for any function $\varphi \in C_b(\mathbb{R})$, and

$$|u(t,x)| \le C\|\varphi\|, \quad (t,x) \in \overline{\mathbb{R}^2_T}. \tag{28}$$

Theorem 3.2. *The solution of problem (5)–(8) from class (27) is of the form of the sum of potentials (14) and (15), where the densities V_i, $i = 1,2$ from simple-layer potentials are determined by the solution of the system of integral equations (24).*

4. Construction of the process

Theorems 3.1 and 3.2 imply that, using the solution of problem (5)–(8), we can determine a family of linear operators $(\mathcal{T}_t)_{t\ge 0}$ that act on the space $C_b(\mathbb{R})$. For $t > 0$, $x \in \mathbb{R}$, and $\varphi \in C_b(\mathbb{R})$, we set

$$\mathcal{T}_t\varphi(x) = \int_{\mathbb{R}} g_i(t,x,xy)\varphi(y)dy + \int_0^t g_i(t-\tau,x,0)V_i(\tau,\varphi)d\tau, \quad x \in \overline{D_i},\ i = 1,2, \tag{29}$$

where $g_i(t,x,y)$, $i = 1,2$, are f.s. of the equation with the operators $D_t - L_i$, $i = 1,2$, and $V_i(t,\varphi) \equiv V_i(t)$, $i = 1,2$ is a solution of the system of Volterra integral equations of the second kind (24). In this case, $\mathcal{T}_0 = I$, where I is the identity operator, and inequality (28) holds for $\mathcal{T}_t\varphi(x)$ in the domain $(t,x) \in \overline{\mathbb{R}^2_T}$.

The existence of an integral representation for the operator family $(\mathcal{T}_t)$ makes it possible to easily verify the following statements:

1) if $\varphi_n \in C_b(\mathbb{R})$, $n \in \mathbb{N}$, $\sup_n \|\varphi_n\| < \infty$, and, for all $x \in \mathbb{R}$, $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$, where $\varphi \in C_b(\mathbb{R})$, then, for every $t \geq 0$, $x \in \mathbb{R}$, the following relation holds:

$$\lim_{n\to\infty} \mathcal{T}_t\varphi_n(x) = \mathcal{T}_t\varphi(x);$$

2) for all $t_1 \geq 0$, $t_2 \geq 0$, the following relation holds:

$$\mathcal{T}_{t_1+t_2} = \mathcal{T}_{t_1} \cdot \mathcal{T}_{t_2};$$

3) $\mathcal{T}_t\varphi(x) \geq 0$ for all $t \geq 0$, $x \in \mathbb{R}$, if $\varphi \in C_b(\mathbb{R})$ and $\varphi(x) \geq 0$;
4) $\|\mathcal{T}_t\| \leq 1$ for all $t \geq 0$.

We now describe briefly a scheme of the proof of the above-stated properties. Property 1) is a corollary of the obvious equality for the solution of the system of integral equations (24) $\lim_{n\to\infty} V_i(t, \varphi_n) = V_i(t, \varphi)$, $t > 0$, $i = 1, 2$ and the Lebesgue theorem on a limiting transition under the sign of integral. The second property also known as the semigroup property of the operators $\mathcal{T}_t$ follows from the statement of Theorem 3.1 of the uniqueness of the solution of problem (5)–(8).

Assuming that $\varphi \in C_b(\mathbb{R})$ and $\varphi(x) \geq 0$ for all $x \in \mathbb{R}$, we now prove that $\mathcal{T}_t\varphi(x) \geq 0$ for all $t > 0$, $x \in \mathbb{R}$. With regard for statement 1), it is sufficient to consider the case where the function φ is finite. Assume the contrary. Let, for some $T > 0$, we have

$$\inf_{(t,x)\in\overline{\mathbb{R}^2_T}} \mathcal{T}_t\varphi(x) = \gamma < 0. \tag{30}$$

We recall that the function $\mathcal{T}_t\varphi(x)$ satisfies Eq. (5) in the domains $(t, x) \in (0, \infty) \times D_i$, $i = 1, 2$ and the initial condition $\lim_{t\downarrow 0} \mathcal{T}_t\varphi(x) = \varphi(x)$. In addition, since φ is the function with a compact carrier, we obtain that $\mathcal{T}_t\varphi(x) \to 0$ as $|x| \to \infty$. This fact and the maximum principle for parabolic equations imply (see [11]) that, if φ is nonnegative and condition (30) holds, then there exists such $t_0 \in (0, T]$ that $u(t_0, 0) = \gamma$.

Since the function $\mathcal{T}_t\varphi(x)$ obviously is not constant, there exists a neighborhood U of the point $(t_0, 0)$ such that $\mathcal{T}_{t_0}\varphi(x) > \gamma$ for $(t, x) \in U \cap \{(0, T] \times D_i\}$, $i = 1, 2$. But then the following relations must hold:

$$\frac{\partial \mathcal{T}_{t_0-}}{\partial x} \leq 0, \quad \frac{\partial \mathcal{T}_{t_0+}}{\partial x} \geq 0, \quad \frac{\partial \mathcal{T}_{t_0}}{\partial x} = 0.$$

Moreover, it follows from Theorem 14 [11, Ch. II, §5] that the equality signs in two last inequalities are eliminated. Hence, if (30) holds, then, for some $t_0 > 0$ and $x = 0$, we have

$$\frac{\partial \mathcal{T}_{t_0-}}{\partial x} < 0, \quad \frac{\partial \mathcal{T}_{t_0+}}{\partial x} > 0, \quad \frac{\partial \mathcal{T}_{t_0}}{\partial x} = 0. \tag{31}$$

From (31) and the inequality $\mathcal{T}_{t_0}\varphi(0) - \mathcal{T}_{t_0}\varphi(x) < 0$, $x \in D_1 \cup D_2$, that is also a simple corollary of the maximum principle, we conclude that none of the possible versions of the conjugation condition (8) holds at $t = t_0$. This contradiction follows from assumption (30), which means that $\gamma > 0$, and property 3) is proved.

Using property 3), it is possible to prove property 4). Indeed, it follows from (29) and (24) that $\mathcal{T}_t\varphi_0(x) = 1$ for all $t > 0$, $x \in \mathbb{R}$, if $\varphi_0(x) \equiv 1$. Second, since $\varphi(x) \leq \|\varphi(x)\|$ for all $x \in \mathbb{R}$, we obtain $\mathcal{T}_t\varphi(x) \leq \|\varphi\|$ for all $t > 0$, $x \in \mathbb{R}$, according to property 3). By changing φ to $-\varphi$ in the last inequality, we obtain $\mathcal{T}_t\varphi(x) \geq -\|\varphi\|$ for all $t > 0$, $x \in \mathbb{R}$. So, property 4) is proved.

It follows from 1)–4) that the semigroup $\mathcal{T}_t$, $t \geq 0$, constructed by formulas (29) and (24) defines a homogeneous Feller process in $\mathbb{R}$ (see [4]). We denote its transition probability by $P(t, x, dy)$, so that

$$\mathcal{T}_t\varphi(x) = \int_{\mathbb{R}} P(t, x, dy)\varphi(y). \tag{32}$$

The existence of an integral representation of the semigroup $\mathcal{T}_t\varphi$ makes it possible to easily calculate its infinitesimal generator denoted by $\mathcal{A}$. We recall that, by definition (see [4]),

$$\mathcal{A}\varphi(x) = \lim_{t\downarrow 0} \frac{\mathcal{T}_t\varphi(x) - \varphi(x)}{t} \tag{33}$$

under condition that the limit exists (in a topology generated by the norm). We denote the domain of definition of the operator $\mathcal{A}$ (e.g., a set of all $\varphi \in C_b(\mathbb{R})$, for which the limit exists) by $D_{\mathcal{A}}$. By calculating the limit on the right-hand side of (33), we obtain that a bounded continuous function $\varphi(x)$ belongs to $D_{\mathcal{A}}$ if and only if $\varphi \in C_2(\mathbb{R})$ and $L_0\varphi(0) = 0$. At the same time, $\mathcal{A}\varphi(x) = L\varphi(x)$, where the operator L is defined by relation (2).

So, we have proved the following theorem.

Theorem 4.1. *The operator semigroup that is defined by formulas (29) and (24) generates a homogeneous Feller process in $\mathbb{R}$ that coincides with the given diffusion processes defined by the operators L_1 and L_2 at inner points of the domains D_1 and D_2, respectively, and its behavior on the boundary of these domains ∂D_i, $i = 1, 2$, is determined by the conjugation condition (3).*

Analyzing the constructed process, we note that the presence of the integral term in the conjugation condition (3) means that, generally speaking, trajectories of the process are discontinuous. On the other hand, the fulfillment of the condition $\sigma > 0$, $q_i \geq 0$, $i = 1, 2$, for the coefficients σ, q_i, $i = 1, 2$, guarantees the existence of a positive probability of the fact that continuations of the considered diffusion processes can have continuous trajectories after their falling at the point $x = 0$. In this connection, there appears a question about the influence of the given parameters and the measure $\mu(\cdot)$ on a local behavior of the particle that is diffusing along the trajectory of a continuous continuation of processes at the point $x = 0$. Calculating the local diffusion characteristics gives us a partial answer to the question. The existence of the characteristics is verified under the assumption that the measure $\mu(\cdot)$ from (3) satisfies the additional conditions

$$\int_{D_1\cup D_2} |y|\mu(dy) < \infty, \quad \int_{D_1\cup D_2} y^2\mu(dy) < \infty. \tag{34}$$

Under conditions (34) with the use of equalities (13), the direct computations result in that the transition function $P(t, x, dy)$ satisfies the relations

$$\begin{aligned} \lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}} (y - x)P(t, x, dy) &= a(x), \\ \lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}} (y - x)^2 P(t, x, dy) &= b(x), \end{aligned} \tag{35}$$

where

$$a(x) = \begin{cases} a_i(x), & x \in D_i,\ i = 1, 2, \\ \frac{q_2 - q_1 + m_1}{\sigma}, & x = 0, \end{cases}$$

$$b(x) = \begin{cases} b_i(x), & x \in D_i,\ i = 1, 2, \\ \frac{m_2}{\sigma}, & x = 0, \end{cases}$$

$$m_1 = \int_{D_1\cup D_2} y\mu(dy), \quad m_2 = \int_{D_1\cup D_2} y^2\mu(dy).$$

Equalities (35) mean that, for the constructed process with the transition function $P(t, x, dy)$, there exist the ordinary diffusion coefficient equal to $b(x)$ and the drift coefficient equal to $a(x)$.

Then, in addition to Theorem 4.1, we obtain the following proposition.

Theorem 4.2. *If, for the coefficients of the operators L_i, $i = 1, 2$, from (1) and the numerical parameters that determine the operator L_0 from (2), the conditions of Theorem 1 hold, and the measure $\mu(\cdot)$ from (2) satisfies condition (34), then the operator semigroup constructed by formulas (29) and (24) describes a Feller process on $\mathbb{R}$, and its transition probability satisfies relation (35).*

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Ivan Franko National University of Lviv, Department of Higher Mathematics, 1, Universytetska Str., Lviv 79000, Ukraine

E-mail address: `p.kononchuk@gmail.com`