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V. V. KONAROVSKII

THE MARTINGALE PROBLEM FOR A MEASURE-VALUED PROCESS WITH HEAVY DIFFUSING PARTICLES

A mathematical model of the joint motion of diffusing particles with mass, which influences the coefficient of diffusion, is considered. Particles start from some set of points on a line, move independently until the time of collision, and then are stuck, with their masses added. It is shown that the measure-valued process describing the given model is the unique solution of the martingale problem in the introduced space of integer-valued measures.

1. Introduction

Let us consider a mathematical model of diffusing particles on a line which begin to move from a countable set of points. Each particle moves independently of the others until the time of collision. Two collided particles are stuck and then move as a single particle with a mass equal to the sum of masses. We suppose that the coefficient of diffusion σ of any particle with mass m is equal to

$$\sigma^2 = \frac{1}{m}.$$

Similar systems were studied in works by R.A. Arratia [1], A.A. Dorogovtsev [6], H. Wang [13], [14], D.A. Dawson [2], [4] et al. In works [1] and [6] the model of Brownian particles is studied, which start from every point of the real axis, move independently until the time of collision, and then are stuck. Since the particles are Brownian, the sticking does not influence the diffusion coefficient. In our model, the diffusion of a separate particle depends on the behavior of all others. We note that such an interaction of particles complicates significantly the study of the system. It is worth to mention work [4], where it was assumed that the masses of diffusing particles vary by a certain law, though their diffusion coefficients are constant. So, the term "particle's mass" has different meaning in [4] and in the present work. According to work [4], the particles only transfer some masses. But, in our case, the mass and the diffusion coefficient are connected with each other. Roughly speaking, a heavier particle moves more slowly.

The goal of the present work is a mathematical description of our system in terms of the evolution of a random measure that is a distribution of particles on the real axis. Since the interaction is singular, we cannot construct our process with the help of some stochastic differential equation, as it was made, for example, for systems with a regular interaction in [3]–[5], [5]. We will proceed in the following way. We define a random process, being a mathematical description of the given model, as the unique solution of the martingale problem in the space of locally finite integer-valued measures. In the process, we will find a generator, for which this martingale problem is posed. In this approach, the following problem arises. To verify that a continuous process, which is a solution of the martingale problem, describes the evolution of the mass of the given

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collection of particles, we should derive a system of continuous processes in $\mathbb{R}$ from such measure-valued process, that would define the trajectories of these particles. But we are not able to do this. Therefore, we are forced to somewhat change the classical definition of a solution of the martingale problem. According to the ideas proposed in monograph [5], we will consider the space of sequences as the phase space for the trajectories of particles along with the space of measures.

2. System of heavy diffusing particles

In this section, we define the set of processes which determine the trajectories of particles. Despite the fact that such a system was constructed and studied in works [9] and [10], we will discuss the idea of its construction once more. This will allow us to obtain some new properties of the system which will be used in what follows.

Theorem 1. [10] *Let $\{x_k,\ k \in I \subseteq \mathbb{Z}\}$ be a nondecreasing sequence of real numbers such that one of the following conditions is satisfied:*

1*) $I = \{1, \dots, n\}$*;*

2*) $I = \mathbb{Z}$, *and there exist a sequence $\{n_i,\ i \in \mathbb{Z}\}$ and a constant $C > 0$ such that, for any $i \in I$, $x_{n_i+1} - x_{n_i} \geq C$.*

Then there exists a system of random processes $\{x_k(t),\ k \in I,\ t \geq 0\}$ such that

1°) $x_k(\cdot)$ *is a continuous square-integrable local martingale relative to*
$$(\mathcal{F})_{t\geq 0} = (\sigma(x_l(s),\ s \leq t,\ l \in I))_{t\geq 0}\,;$$

2°) $x_k(0) = x_k,\ k \in I$;

3°) $x_k(t) \leq x_l(t)$ *for arbitrary* $k, l \in I$, $k < l$, *and* $t \geq 0$;

4°) $\langle x_k(\cdot)\rangle_t = \int_0^t \frac{ds}{m_k(s)}$, *where*
$$m_k(t) = \#\{l \in I :\ \exists\, s \leq t,\ x_k(s) = x_l(s)\};$$

5°) *the common characteristic*
$$\langle x_k(\cdot), x_l(\cdot)\rangle_t \mathbb{I}_{\{t\leq \tau_{k,l}\}} = 0,$$
where $\tau_{k,l} = \inf\{t :\ x_k(t) = x_l(t)\}$.

The distribution of $(x_k(\cdot))_{k\in I}$ in the space $\big((C(\mathbb{R}^+))^I,\ \mathcal{B}((C(\mathbb{R}^+))^I)\big)$ is uniquely determined by conditions 1°)–5°).

The set of processes $\{x_k(\cdot),\ k \in I\}$, which was constructed in Theorem 1, describes the joint behavior of diffusing particles on a line. According to condition 5°), the particles move independently until the time of collision, then are stuck, and change their mass, according to 4°), (the masses are added), which influences the coefficient of diffusion. Since a positive martingale after reaching zero remains there [9], the sticking effect is set by conditions 1°)–5°).

Below, we describe the idea of the proof of the theorem. Let $\{w_k,\ k \in \mathbb{Z}\}$ be a system of standard independent Wiener processes. We assume that $I = \{1, \dots, n\}$. In this case, there exists a mapping
$$F_n : C(\mathbb{R}^+)^I \to C(\mathbb{R}^+)^I$$
such that the system of processes $\{x_k(\cdot),\ k \in I\}$, where
$$(x_1(\cdot), \dots, x_n(\cdot)) = F_n(x_1 + w_1, \dots, x_n + w_n),$$
satisfies properties 1°)–5°). The structure of the mapping F_n is given in [10]. We note that F_n sticks functions in a certain way. Therefore, it is easy to verify the measurability of this mapping and the stochastic continuity of the process $F_n(x_1 + w_1, \dots, x_n + w_n)$ in $(x_1, \dots, x_n)$ in the space $C(\mathbb{R}^+)^n$.

If $I = \mathbb{Z}$, then condition 2*) ensures the existence of a limit of the sequence $\{x_k^n(\cdot)\}_{n\geq |k|}$, $k \in \mathbb{Z}$, where
$$(x_{-n}^n(\cdot), \dots, x_n^n(\cdot)) = F_{2n+1}(x_{-n} + w_{-n}, \dots, x_n + w_n).$$

As $x_k(\cdot)$, we take the limit of the sequence $\{x_k^n(\cdot)\}_{n\geq |k|}$. The system $\{x_k(t),\ k\in\mathbb{Z},\ t\geq 0\}$ is a required one.

By the symbol U, we denote the mapping which puts the set of standard independent Wiener processes $\{w_k,\ k\in\mathbb{Z}\}$ in correspondence to the system $\{x_k(\cdot),\ k\in\mathbb{Z}\}$, i.e.,

$$(x_k(\cdot))_{k\in\mathbb{Z}} = U(x_k + w_k)_{k\in\mathbb{Z}}.$$

The stochastic continuity $U(x_k+w_k)_{k\in\mathbb{Z}}$ in $(\ldots,x_{-n},\ldots,x_n,\ldots)$ in the space $C(\mathbb{R}^+)^{\mathbb{Z}}$ follows from the stochastic continuity of $F_n(x_1+w_1,\ldots,x_n+w_n)$ and condition $2^\star$).

3. Process with heavy diffusing particles

Here, we determine the phase spaces, in which we construct random processes defining the evolution of our system. As was mentioned above, we consider two spaces. The first one is the space of measures, and the second is the space of nondecreasing sequences. Thus, let $\mathcal{H}$ be the set of integer-valued measures μ on a line, for which

$$\lim_{n\to\infty}\frac{\mu([0,n))}{n}=1,\quad \lim_{n\to\infty}\frac{\mu([-n,0))}{n}=1 \tag{1}$$

(by virtue of condition (1), the measure μ is locally finite). By $\mathcal{M}$, we denote the set of nondecreasing sequences $(x_k)_{k\in\mathbb{Z}}$ such that

$$\lim_{k\to\pm\infty}\frac{x_k}{k}=1. \tag{2}$$

We note that conditions (1) and (2) are equivalent in the sense that the measure $\sum_{k\in\mathbb{Z}}\delta_{x_k}$ satisfies condition (1) if and only if $(x_k)_{k\in\mathbb{Z}}$ satisfies condition (2). The following proposition is valid.

Lemma 1. *The measure μ lies in $\mathcal{H}$ if and only if there exists an element $(x_n)_{n\in\mathbb{Z}}$ from $\mathcal{M}$ such that $\mu=\sum_{n\in\mathbb{Z}}\delta_{x_n}$.*

Proof. Let $\mu=\sum_{n\in\mathbb{Z}}\delta_{x_n}\in\mathcal{H}$, and let an enumeration of x_k be chosen such that $x_1\geq 0$ and $x_0<0$. We put each number $m\in\mathbb{N}$ in correspondence to $n_m\in\mathbb{N}$ by the rule

$$n_m-1<x_m\leq n_m. \tag{3}$$

Then we have the inequality

$$\mu([0,n_m-1])<m\leq\mu([0,n_m]).$$

This inequality implies that $\frac{m}{n_m}\to 1$ as $m\to\infty$. Using (3), we see that $\frac{x_m}{m}\to 1$. On the other hand, if $(x_n)_{n\in\mathbb{Z}}\in\mathcal{M}$, we can choose $k\in\mathbb{Z}$ so that $x_{1+k}\geq 0$ and $x_k<0$. We denote $x_n'=x_{n+k}$. It is clear that $(x_n')_{n\in\mathbb{Z}}\in\mathcal{M}$. Taking $\mu([0,n])=m_n$, we have

$$x_{m_n}\leq n<x_{m_n+1}.$$

Hence, since $\frac{x_{m_n}}{m_n}\to 1$, we obtain $\frac{n}{m_n}\to 1$ as $n\to\infty$. □

Condition (1) is needed due to several reasons. First, it follows from Theorem 1 that the system of processes $\{x_k(\cdot),\ k\in\mathbb{Z}\}$, which serves as the mathematical description of our system of particles, exists under some restriction to the initial set of starting particles (condition $2^\star$)). It is easy to see that the existence of limits in (1) ensures the fulfillment of this restriction, if we consider the set $\operatorname{supp}\mu_0$, where $\mu_t=\sum_{i\in\mathbb{Z}}\delta_{x_k(t)}$, as the initial set. Second, (1) is an invariant of the system in $\mathcal{H}$. In other words, the fact that μ_0 satisfies condition (1) implies that μ_t also satisfies (1). The reason is that the particles have no time to strongly deviate from the initial position for a finite time interval, because their motion is similar to the Brownian one. Therefore, our system of particles can be seen as a random process in $\mathcal{H}$. For condition (2), the explanation is analogous.

Then we introduce metrics on $\mathcal{H}$ and $\mathcal{M}$. Let us consider the set of mappings from $\mathbb{R}$ onto $[0,1]$,

$$\Phi^+=\{\varphi_k^+,\ k\in\mathbb{N}\},$$

where the functions φ_k^+ are twice continuously differentiable for any k from $\mathbb{N}$, $\varphi_k^+(x) = 0$ for $x \le 0$ or $x \ge k+1$, $\varphi_k^+(x) = 1$ for $x \in [\frac{1}{2}, k]$, and the sequence $\{(\varphi_k^+)''\}_{k\ge 1}$ is uniformly bounded. We denote

$$\Phi^- = \{\varphi_k^- : \ \varphi_k^-(x) = \varphi_k^+(-x), \ x \in \mathbb{R}, \ k \in \mathbb{N}\}.$$

Let $\widetilde{Q} = \{g_k, \ k \in \mathbb{N}\}$ be the set of functions that have the following properties:

1) for any k, the functions g_k are twice continuously differentiable;

2) there exists a constant C such that, for any k and x,

$$|g_k'(x)| < C, \quad |g_k''(x)| < C;$$

3) for any $\mu, \ \nu \in \mathcal{H}$, the relation $\langle g_k, \mu\rangle = \langle g_k, \nu\rangle$ yields the equality of μ and ν.

Denote $Q = \widetilde{Q} \cup \{\varphi_k^+(x - k/2), \ k \in \mathbb{N}\}$. For $\mu, \nu \in \mathcal{H}$, we define

$$\rho_{\mathcal{H}}(\mu,\nu) = d(\mu,\nu) + \sup_{k\ge 1} \frac{|\langle\varphi_k^-, \mu\rangle - \langle\varphi_k^-, \nu\rangle|}{k} + \sup_{k\ge 1} \frac{|\langle\varphi_k^+, \mu\rangle - \langle\varphi_k^+, \nu\rangle|}{k},$$

where

$$d(\mu,\nu) = \sum_{k\ge 1} \frac{1}{2^k} \left[|\langle f_k, \mu\rangle - \langle f_k, \nu\rangle| \wedge 1 \right],$$

and the functions $\varphi_k^+, \ \varphi_k^-$, and f_k belong to $\Phi^+, \ \Phi^-$, and Q, respectively.

We introduce a metric on $\mathcal{M}$ in the following way:

$$\rho_{\mathcal{M}}((x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}}) = \sup_{n\in\mathbb{Z}} \frac{|x_n - y_n|}{1 + |n|}.$$

The following proposition is valid.

Lemma 2. $(\mathcal{H}, \rho_{\mathcal{H}})$ *and* $(\mathcal{M}, \rho_{\mathcal{M}})$ *are complete separable metric spaces.*

In the spaces $\mathcal{H}$ and $\mathcal{M}$, we construct the processes that describe the evolution of our system.

Definition 1. A random process $\{\mu_t, \ t \ge 0\}$ in $\mathcal{H}$ is called a process with heavy diffusing particles, if there exists a system of processes $\{x_k(t), \ k \in \mathbb{Z}, \ t \ge 0\}$, which satisfies conditions $1^\circ) - 5^\circ)$ of Theorem 1 and, for any $t \ge 0$,

$$\mu_t = \sum_{k\in\mathbb{Z}} \delta_{x_k(t)}. \tag{4}$$

For the process $\{X(t), \ t \ge 0\}$ in $\mathcal{M}$, the definition is analogous, if condition (4) is replaced by

$$X(t) = (x_k(t))_{k\in\mathbb{Z}}.$$

By virtue of Theorem 1 and the fact that the diffusing particles which started from the support of the measure $\mu \in \mathcal{H}$ do not deviate strongly from the initial position at an arbitrary time $t > 0$, we can easily prove the following lemma.

Lemma 3. *For an arbitrary measure* $\mu \in \mathcal{H}$ *(*$X \in \mathcal{M}$*), there exists a continuous process with heavy diffusing particles* $\{\mu_t, \ t \ge 0\}$ *(*$\{X(t), \ t \ge 0\}$*), such that* $\mu_0 = \mu$ *(*$X(0) = X$*).*

4. Martingale problem for a finite number of particles

In this section, we consider a process that describes the motion of a finite collection of particles and solve the martingale problem for it. This will allow us to show that the process with heavy diffusing particles is the unique solution of some martingale problem.

We introduce the following notation. Let $\mathfrak{S}^n = \{K = (\alpha_1, \dots, \alpha_p) : \ \alpha_i \subseteq \{1, \dots, n\}, \ p = 1, \dots, n\}$ be the set of partitions of the set $\{1, \dots, n\}$ such that

1) $l < k$ for any $k \in \alpha_i$, $l \in \alpha_{i+1}$, and $i = \{1, \dots, n-1\}$;

2) $\bigcup_{i=1}^p \alpha_i = \{1, \dots, n\}$.

By $|K|$, we denote the number p, and $K(i)$ is an element in $\alpha \in K$, for which $i \in \alpha$. Let

$K \in \mathfrak{S}^n$. We denote

$$S_K^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_i = x_{i+1} \Leftrightarrow K(i) = K(i+1), \ i = 1, \ldots, n-1\},$$

$$E^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_i \leq x_{i+1}, \ i = 1, \ldots, n-1\},$$

and

$$\mathfrak{S}_K^- = \{(\beta_1, \ldots, \beta_q) : \ q = |K| - 1, \ \forall\, i \ \exists\, j \ \beta_i \supseteq K(j)\}.$$

Let $\widehat{C}(\mathbb{R}^n)$ be a class of continuous functions on $\mathbb{R}^n$ which are symmetric relative to all permutations of coordinates and become zero at infinity. We take $f \in \widehat{C}(\mathbb{R}^n)$ and $K \in \mathfrak{S}^n$. We define

$$f_K(y_1, \ldots, y_{|K|}) = f\big|_{S_K^n}(x_1, \ldots, x_n),$$

where $x \in S_K^n$, $(x_1, \ldots, x_n) = (y_1, \ldots, y_1, \ldots, y_{|K|}, \ldots, y_{|K|})$. Let $\mathcal{D}_{\mathbb{R}}^{(n)}$ be a collection of functions f from $\widehat{C}(\mathbb{R}^n)$ such that

1) for any $K \in \mathfrak{S}^n$, the function f_K is twice continuously differentiable on $E^{|K|}$;

2) for any $K = (\alpha_1, \ldots, \alpha_p) \in \mathfrak{S}^n$, any derivative of the function f_K, whose order is at most two, can be extended to a continuous function on $S_K^n \cup \left(\bigcup_{P \in \mathfrak{S}_K^-} S_P^n\right)$. Moreover, for any $P^i = (\alpha_1, \ldots, \alpha_i \cup \alpha_{i+1}, \ldots, \alpha_p) \in \mathfrak{S}_K^-$, the relation

$$\Delta_K f_K\big|_{y_i = y_{i+1}} = \Delta_{P^i} f_{P^i}, \tag{5}$$

where $\Delta_K f_K(y_1, \ldots, y_p) = \sum_{j=1}^p \frac{1}{\#\alpha_j} \frac{\partial^2}{\partial y_j^2} f_K(y_1, \ldots, y_p)$, is valid.

On $\mathcal{D}_{\mathbb{R}}^{(n)}$, we consider the operator

$$\mathfrak{G}_{\mathbb{R}}^{(n)} f(x_1, \ldots, x_n) = \frac{1}{2} \Delta_n f(x_1, \ldots, x_n),$$

where Δ_n is the n-dimensional Laplace operator.

Remark 1. For any function $f \in \mathcal{D}_{\mathbb{R}}^{(n)}$ and any $K \in \mathfrak{S}^n$, the equality

$$\Delta_n f\big|_{S_K^n} = \Delta_K f_K$$

holds. Here, the contraction is performed gradually, by descending from one face to another until we reach S_K^n.

Then we consider the system $\{x_k(\cdot), \ k = 1, \ldots, n\}$, which satisfies conditions 1°) – 5°) of Theorem 1, and denote $X = (x_1(\cdot), \ldots, x_n(\cdot))$. The following lemma is valid.

Lemma 4. *$\{X(t), \ t \geq 0\}$ is the unique solution $(\mathfrak{G}_{\mathbb{R}}^{(n)}, \mathcal{D}_{\mathbb{R}}^{(n)})$-martingale problem.*

Proof. First, we show that $\{X(t), \ t \geq 0\}$ is a solution of the $(\mathfrak{G}_{\mathbb{R}}^{(n)}, \mathcal{D}_{\mathbb{R}}^{(n)})$-martingale problem. By taking $f \in \mathcal{D}_{\mathbb{R}}^{(n)}$ and applying the Itô formula to $f(X(t))$, we obtain

$$f(X(t)) - f(X(0)) - \frac{1}{2} \sum_{k,l=1}^n \int_0^t \frac{\partial^2 f}{\partial x_k \partial x_l}(X(s)) d\langle x_k(\cdot), x_l(\cdot)\rangle_s = \text{martingale}. \tag{6}$$

We now calculate

$$\frac{1}{2} \int_0^t \sum_{k,l=1}^n \frac{\partial^2 f}{\partial x_k \partial x_l}(X(s)) d\langle x_k(\cdot), x_l(\cdot)\rangle_s =$$

$$= \frac{1}{2} \int_0^t \sum_{k,l=1}^n \frac{\partial^2 f}{\partial x_k \partial x_l}(X(s)) \frac{1}{m_k(s)} \mathbb{I}_{\{x_k(s) = x_l(s)\}} ds =$$

$$= \frac{1}{2} \int_0^t \sum_{K \in \mathfrak{S}^n} \left[\sum_{k,l=1}^n \frac{\partial^2 f}{\partial x_k \partial x_l}(X(s)) \frac{1}{m_k(s)} \mathbb{I}_{\{x_k(s) = x_l(s)\}} \right] \mathbb{I}_{S_K^n}(X(s)) ds.$$

Since the derivatives of the function f_K for any $K \in \mathfrak{S}^n$ have a continuous extension onto $S_K^n \cup \left(\bigcup_{P \in \mathfrak{S}_K^-} S_P^n\right)$, we can rewrite (6) in the form

$$f(X(t)) - f(X(0)) - \frac{1}{2}\int_0^t \sum_{K \in \mathfrak{S}^n} \Delta_K f_K(X(s)) \mathbb{I}_{S_K^n}(X(s)) ds =$$

$$= f(X(t)) - f(X(0)) - \frac{1}{2}\int_0^t \Delta_n f(X(s)) ds = \text{martingale}.$$

Let us verify the uniqueness of the solution. For $g \in \widehat{C}(\mathbb{R}^n) \cap C^{2,\alpha}(\mathbb{R}^n)$ and $\lambda > 0$, we consider the equation

$$\lambda f - \Delta_n f = g, \tag{7}$$

whose solution is sought in the class $\mathcal{D}_{\mathbb{R}}^{(n)}$. Since $f \in \mathcal{D}_{\mathbb{R}}^{(n)}$ is a continuous symmetric function satisfying condition (5), relation (7) is equivalent to the collection of equations of the elliptic type

$$\lambda h_K(y) - \Delta_K h_K(y) = g_K(y), \quad y \in E^p, \tag{8}$$

$$h_K(y_1, \ldots, y_p)\big|_{y_i = y_{i+1}} = h_{P^i}(y_1, \ldots, y_i, y_{i+2}, \ldots, y_p), \tag{9}$$

where $K = (\alpha_1, \ldots, \alpha_p) \in \mathfrak{S}^n$, $P^i = (\alpha_1, \ldots, \alpha_i \cup \alpha_{i+1}, \ldots, \alpha_p) \in \mathfrak{S}_K^-$, $i = 1, \ldots, p-1$, and the functions f and h are connected by the equality $h = f\big|_{E^n}$. Since (8) is an elliptic equation in E^p with the continuous boundary conditions (9), problem (8)–(9) has the unique solution (see Theorem 6.13 [8]). The possibility of a continuous extension of the derivatives of the functions f_K up to the second order inclusively follows from Lemma 6.18 [8] on the regularity of the solution of an equation of the elliptic type near the boundary. Hence, $R(\lambda - \Delta)$ is dense in $\widehat{C}(\mathbb{R}^n)$. Using the fact that $\mathfrak{G}_{\mathbb{R}}^{(n)}$ satisfies the maximum principle on the set $\mathcal{D}_{\mathbb{R}}^{(n)}$ which is, in turn, dense in $\widehat{C}(\mathbb{R}^n)$, due to Theorem 4.4.1 [7], we obtain the uniqueness of the solution. □

We note that the phase space of a process with heavy diffusing particles is the space of locally finite integer-valued measures on $\mathbb{R}$. In order to solve the martingale problem for it, we will use the previous lemma and will find the generator of the process

$$\mu_t^n = \sum_{k=1}^n \delta_{x_k(t)} \tag{10}$$

in the space $\mathcal{H}_n = \{\mu \in \mathcal{H} : \ \mu(\mathbb{R}) = n\}$ with a metric of weak convergence .

Since we deal with the process, whose values are measures, it is convenient to define the generator on polynomials which depend on measures, i.e., on functions of the form

$$F_{\varphi,m} = \langle \varphi, \mu^{\otimes m} \rangle = \int \varphi(x_1, \ldots, x_m) \mu(dx_1) \ldots \mu(dx_m)$$

(see, e.g., [6], [4], [3], [7]). The role of derivatives will be played by derivatives in the sense of Dawson [3], which are calculated by the rule

$$\frac{\delta F_{\varphi,m}(\mu)}{\delta \mu(x)} = \sum_{j=1}^m \int \cdots \int_{\mathbb{R}^{m-1}} \varphi(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_m) \prod_{i \neq j} \mu(dx_i),$$

$$\frac{\delta^2 F_{\varphi,m}(\mu)}{\delta \mu(x) \delta \mu(y)} =$$

$$= \sum_{j=1}^m \sum_{k=1}^m \int \cdots \int_{\mathbb{R}^{m-2}} \varphi(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_m) \prod_{i \neq j,k} \mu(dx_i).$$

We now write the generator for the process $\{\mu_t, \ t \geq 0\}$. It is composed from two parts. The first and second parts are responsible, respectively, for the diffusion and for the sticking and the summation of masses. Hence, we have

$$\mathfrak{G}F_{\varphi,m}(\mu)=\frac{1}{2}\iint_{\mathbb{R}^2}\frac{\partial^2}{\partial x\partial y}\frac{\delta^2F_{\varphi,m}(\mu)}{\delta\mu(x)\delta\mu(y)}\delta_x(dy)\mu(dx)+\frac{1}{2}\int_{\mathbb{R}}\frac{d^2}{dx^2}\frac{\delta F_{\varphi,m}(\mu)}{\delta\mu(x)}\mu^*(dx), \tag{11}$$

where $\mu^*=\sum_{y\in\operatorname{supp}\mu}\delta_y$.

We now define the domain $\mathcal{D}_{\mathcal{H}_n}$ for the as-introduced operator. Consider the class of functions

$$\Phi_n=\{\varphi:\ \exists\widetilde{\varphi}\in\mathcal{D}^{(n)}_{\mathbb{R}}\ \exists K\in\mathfrak{S}^n\ \widetilde{\varphi}\big|_{S^n_K}=\varphi\}. \tag{12}$$

Let

$$\mathcal{D}^{(n)}=\mathrm{sp}\left\{F_{\varphi,m}:\ \varphi\in\Phi_n,\ m\le n\right\}.$$

Lemma 5. *The process $\{\mu^n_t,\ t\ge 0\}$, that is given by formula* (10) *is the unique solution of the $\left(\mathfrak{G},\mathcal{D}^{(n)}\right)$ - martingale problem.*

Proof. We now verify that the process μ^n_t is a solution of the $\left(\mathfrak{G},\mathcal{D}^{(n)}\right)$-martingale problem. To this end, we take the function $F_{\varphi,m}\in\mathcal{D}^{(n)}$ and apply the Itô formula to $F_{\varphi,m}(\mu^n_t)$. We obtain

$$F_{\varphi,m}(\mu^n_t)=\sum_{k_1,\ldots,k_m}\varphi(x_{k_1}(t),\ldots,x_{k_m}(t))=\sum_{k_1,\ldots,k_m}\varphi(x_{k_1}(0),\ldots,x_{k_m}(0))+$$
$$+\frac{1}{2}\sum_{k_1,\ldots,k_m}\sum_{i,j=1}^m\int_0^t\varphi''_{i,j}(x_{k_1}(s),\ldots,x_{k_m}(s))d\langle x_{k_i}(\cdot),x_{k_j}(\cdot)\rangle_s+$$
$$+\sum_{k_1,\ldots,k_m}\sum_{i=1}^m\int_0^t\varphi'_i(x_{k_1}(s),\ldots,x_{k_m}(s))dx_{k_i}(s).$$

We calculated

$$\sum_{k_1,\ldots,k_m}\sum_{i,j=1}^m\int_0^t\varphi''_{i,j}(x_{k_1}(s),\ldots,x_{k_m}(s))d\langle x_{k_i}(\cdot),x_{k_j}(\cdot)\rangle_s=$$
$$\sum_{k_1,\ldots,k_m}\sum_{i,j=1}^m\int_0^t\frac{\varphi''_{i,j}(x_{k_1}(s),\ldots,x_{k_m}(s))}{m_{k_i}(s)}\mathbb{I}_{\{\tau_{k_i,k_j}<s\}}ds=$$
$$=\sum_{i=1}^m\int_0^t\sum_{k_1,\ldots,k_m}\frac{\varphi''_{i,i}(x_{k_1}(s),\ldots,x_{k_m}(s))}{m_{k_i}(s)}ds+$$
$$+\sum_{i\ne j}\int_0^t\sum_{\{k_1,\ldots,k_m\}\setminus\{k_j\}}\sum_{k_j}\frac{\varphi''_{i,j}(x_{k_1}(s),\ldots,x_{k_m}(s))}{m_{k_i}(s)}\mathbb{I}_{\{\tau_{k_i,k_j}<s\}}ds=$$
$$=\int_{\mathbb{R}}\frac{d^2}{dx^2}\frac{\delta F_{\varphi,m}(\mu)}{\delta\mu(x)}(\mu^n_t)^*(dx)+\int_0^t\sum_{i\ne j}\sum_{\{k_1,\ldots,k_m\}\setminus\{k_j\}}\varphi''_{i,j}(\ldots,x_{k_i}(s),\ldots,x_{k_i}(s),\ldots)ds=$$
$$=\int_{\mathbb{R}}\frac{d^2}{dx^2}\frac{\delta F_{\varphi,m}(\mu)}{\delta\mu(x)}(\mu^n_t)^*(dx)+\iint_{\mathbb{R}^2}\frac{\partial^2}{\partial x\partial y}\frac{\delta^2F_{\varphi,m}(\mu)}{\delta\mu(x)\delta\mu(y)}\delta_x(dy)\mu^n_t(dx).$$

The uniqueness of the solution follows from the fact that there exist the constants $\{C_k,\ k=1,\ldots,m\}$ for any $f\in\mathcal{D}^{(n)}_{\mathbb{R}}$ and the functions $\{F_{\varphi_k,k},\ k=1,\ldots,m\}\subset\mathcal{D}^{(n)}$ such that

$$f(x)=\sum_{k=1}^m C_kF_{\varphi_k,k}(\mu)\quad\text{and}\quad\mathfrak{G}^{(n)}_{\mathbb{R}}f(x)=\sum_{k=1}^m C_k\mathfrak{G}F_{\varphi_k,k}(\mu),$$

where $\mu=\sum_{k=1}^n\delta_{x_k}$. □

Let $\mathcal{D}^{(n)}_0=\mathrm{sp}\{F_{\varphi,m}\in\mathcal{D}^{(n)}:\ \ \varphi-\text{has a compact support}\}$. Lemma 5 and Theorem 4.6.2 [7] yield the following proposition.

Lemma 6. *The process $\{\mu_t^n,\ t \ge 0\}$, given by formula* (10) *is the unique solution of the $\left(\mathfrak{G}, \mathcal{D}_0^{(n)}\right)$-martingale problem.*

5. Martingale problem for the process with heavy diffusing particles

In this section, we will show that the process with heavy diffusing particles is the unique solution of the martingale problem with a generator of the form (11) and will find its domain. As was mentioned above, we are not able to derive the continuous processes in $\mathbb{R}$ which would define the trajectories of particles from a continuous $\mathcal{H}$-valued process. But this is of importance for the proof of the uniqueness of the solution of the martingale problem. Therefore, we are forced to seek solutions among measure-valued processes of the form $\sum_{k\in\mathbb{Z}} \delta_{x_k(t)} \in \mathcal{H}$, where $x_k(\cdot)$ are continuous processes. In this connection, the definition of the martingale problem for a process with heavy diffusing particles will be different from that commonly accepted in the literature (see, e.g., [3]–[11]).

We begin from the domain of the generator. We take

$$\mathcal{D} = \mathrm{sp}\{F \in \mathcal{D}_0^{(n)} :\ n \in \mathbb{N}\}.$$

Definition 2. A strictly Markov continuous process $\{(x_n(t))_{n\in\mathbb{Z}},\ t \ge 0\}$ in $\mathcal{M}$ is called a solution of the $(\mathfrak{G}, \mathcal{D})$-martingale problem, if

1) the measure-valued process $\mu_t = \sum_{k\in\mathbb{Z}} \delta_{x_k(t)}$ in $\mathcal{H}$ satisfies the following condition:
$$F(\mu_t) - F(\mu_0) - \int_0^t \mathfrak{G}(F(\mu_s))ds$$
is a martingale for an arbitrary function $F \in \mathcal{D}$;
2) for each k and any function $f \in C^2([0, +\infty))$ which is bounded together with its derivatives and satisfies the condition $f''(0) = 0$, the difference
$$f(x_{k+1}(t) - x_k(t)) - \frac{1}{2}\int_0^t f''(x_{k+1}(s) - x_k(s)) \left[\frac{1}{\sqrt{\nu_{k+1}(s)}} + \frac{1}{\sqrt{\nu_k(s)}}\right] ds$$
is a martingale, where

$$\nu_k(t) = \#\{i :\ x_i(t) = x_k(t)\}. \tag{13}$$

Remark 2. Condition 2) of definition 2 guarantees that the processes $x_k(\cdot)$ and $x_l(\cdot)$ after the coincidence do not come apart, i.e.,

$$(x_k(t) - x_l(t))\mathbb{I}_{\{t>\tau_{k,l}\}} = 0.$$

We now formulate the theorem which is our main result.

Theorem 2. *A process with heavy diffusing particles is the unique solution of the $(\mathfrak{G}, \mathcal{D})$-martingale problem.*

Proof. We verify that the process with heavy diffusing particles is a solution of the $(\mathfrak{G}, \mathcal{D})$ - martingale problem. For this purpose, we show firstly that it is a strictly Markov process in $\mathcal{M}$. Let U be the mapping that is constructed in Section 1. We take the collection of standard Wiener processes $\{w_k,\ k \in \mathbb{Z}\}$ and consider

$$X(x, s, t) = (x_k(x, s, t))_{k\in\mathbb{Z}} = U(x_k + w_k(t) - w_k(s))_{k\in\mathbb{Z}} \quad t \ge s$$

for any $x \in \mathcal{M}$ and $s \ge 0$. We note that, for fixed $s \ge 0$ and $x \in \mathcal{M}$, the process $X(x, s, t + s)$ is a process with heavy diffusing particles. In view of the fact that the particles do not strongly deviate from their initial positions for a finite time interval, it is easy to verify that, at fixed s and t, $X(x, s, s + t)$ is stochastically continuous in x in the space $\mathcal{M}$. Since the space $\mathcal{M}$ is complete and separable, the stochastic continuity of the process $X(x, s, t)$ in x yields its measurability (see [12]).

We take $x \in \mathcal{M}$ and consider $X(\cdot) = X(x, 0, \cdot)$. We can verify the Markov property of the process $X(\cdot)$ in the standard way, by using the following relation

$$X(X(x, s, r), r, t) = X(x, s, t), \quad s \le r \le t.$$

Let now τ be a finite Markovian moment relative to the filtration

$$(\mathcal{F}_t^w)_{t\ge 0} = (\sigma(w_k(s),\ s \le t,\ k \in \mathbb{Z}))_{t\ge 0},$$

and $f(x) = \widetilde{f}(x_{-n}, \dots, x_n)$, where $\widetilde{f} \in C_b(\mathbb{R}^{2n+1})$. If τ is a discrete Markovian moment, then, according to Proposition 3.1.3 [7], we have

$$E[f(X(\tau + t))|\mathcal{F}_\tau^w] = E[f(X(t))|X(\tau)]. \tag{14}$$

In another case, we will approximate τ by a nonincreasing sequence of discrete Markovian moments $\{\tau_n\}_{n\ge 1}$ and, by using (14) for τ_n, the continuity of the filtration $(\mathcal{F}_t^w)_{t\ge 0}$ on the right, and the stochastic continuity of $\pi_n \circ X(x, 0, t)$ in x, we will prove (14) for τ. This is sufficient for the strict Markov property of the process $X(\cdot)$ in $\mathcal{M}$ to be valid.

We now verify condition 1) of definition 2. We take $\mu_t = \sum_{k\in\mathbb{Z}} \delta_{x_k(t)}$. Analogously to the proof of Lemma 5, the function $F_{\varphi_k,k} \in \mathcal{D}$ satisfies the relation

$$F_{\varphi,m}(\mu_t) - F_{\varphi,m}(\mu_0) - \frac{1}{2}\int_0^t F_{\varphi,m}(\mu_s)ds = \alpha(t),$$

where $\alpha(t) = \sum_{k_1,\dots,k_m} \sum_{i=1}^m \int_0^t \varphi_i'(x_{k_1}(s), \dots, x_{k_m}(s))dx_{k_i}(s)$. We now show that $\alpha(t)$ is a martingale. For convenience, we assume that $m = 1$. Hence,

$$\alpha(t) = \sum_{k\in\mathbb{Z}} \int_0^t \varphi'(x_k(s))dx_k(s).$$

We denote

$$\alpha_n(t) = \sum_{k=-n}^{n} \int_0^t \varphi'(x_k(s))dx_k(s).$$

$\{\alpha_n\}_{n\ge 1}$ is a sequence in the space of continuous square-integrable martingale with the metric

$$\rho(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\sqrt{E\int_0^n (\alpha(t) - \beta(t))^2 dt} \wedge 1\right).$$

Consider

$$E\int_0^T (\alpha_n(t) - \alpha_{n+p}(t))^2 dt = E\int_0^T \left(\sum_{|k|=n+1}^{n+p} \int_0^t \varphi'(x_k(s))dx_k(s)\right)^2 dt =$$

$$= \int_0^T \sum_{|l|,|k|=n+1}^{n+p} \left(E\int_0^t \varphi'(x_k(s))\varphi'(x_l(s))d\langle x_k(\cdot), x_l(\cdot)\rangle_s\right) dt =$$

$$= E\int_0^T \int_0^t \sum_{|l|,|k|=n+1}^{n+p} \frac{\varphi'(x_k(s))\varphi'(x_l(s))}{\sqrt{m_k(s)m_l(s)}} \mathbb{I}_{\{s\ge\tau_{k,l}\}} dsdt =$$

$$= E\int_0^T \int_0^t \sum_{|k|=n+1}^{n+p} \frac{\varphi'(x_k(s))^2}{m_k(s)} m_k(s)dsdt = E\int_0^T \int_0^t \sum_{|k|=n+1}^{n+p} \varphi'(x_k(s))^2 dsdt.$$

Since the function φ' has a compact support, and $\{\mu_t,\ t \ge 0\} \in \mathbb{H}$, we have $\rho(\alpha_n, \alpha) \to 0$. This implies that α is a continuous square-integrable martingale. Condition 2) of the definition can be verified analogously.

We prove the uniqueness in the following way. Let

$$\mu_t = \sum_{k\in\mathbb{Z}} \delta_{x_k(t)}$$

is a solution of the martingale problem for $(\mathfrak{G}, \mathcal{D})$. Consider the process ν_n given by equality (13). We note that $x_k(t)$ converges monotonically to infinity for any $t \in [0, T]$. Like the proof of the Dini theorem about a monotonically increasing sequence of continuous functions on an interval, we can analogously verify that the process $\nu_n(t)$ is bounded with probability 1 on $[0, T]$. Remark 2 implies that it does not decrease for any $n \in \mathbb{Z}$.

We now separate a finite collection G of atoms of a measure μ_0 which are positioned in succession at $x_{n_1}(0), \ldots, x_{n_2}(0)$. We take $\tau'_k,\ k > n_2$, to be the times of the "sticking" of $x_{n_2}(t)$ and $x_k(t)$, and let $\tau''_k,\ k < n_1$, be the times of the "sticking" of $x_{n_1}(t)$ and $x_k(t)$. Since $\nu_n(t)$ is a nondecreasing bounded process, $\{\tau'_k\}$ and $\{\tau''_k\}$ are nondecreasing sequences which converge to infinity with probability 1. We define $\{\sigma_k,\ k \geq 1\}$ as the union of $\{\tau'_k\}$ and $\{\tau''_k\}$ sorted in the ascending order. For a fixed $T > 0$, we denote $\sigma'_k = \sigma_k \wedge T,\ k \geq 1$. Let

$$\delta = [x_{n_1}(0) - x_{n_1-1}(0)] \wedge [x_{n_2+1}(0) - x_{n_2}(0)], \quad \widetilde{\delta} = \frac{\delta}{3} \text{ and}$$

$$\theta_{1,1} = \inf\{t:\ \mu_t([x_{n_1-1}(0) + \widetilde{\delta}, x_{n_1}(0) - \widetilde{\delta}]) > 0\} \wedge \\ \wedge \inf\{t:\ \mu_t([x_{n_2}(0) + \widetilde{\delta}, x_{n_2+1}(0) - \widetilde{\delta}]) > 0\} \wedge T.$$

Consider

$$\mu_t^{1,1} = \sum_{k \in n_1, \ldots, n_2} \delta_{x_k(t)}.$$

We take $F_{\varphi,m} \in \mathcal{D}_0^{(n)}$, where $n = \#G$, and verify that

$$F_{\varphi,m}(\mu^{1,1}_{t\wedge\theta_{1,1}}) - \int_0^{t\wedge\theta_{1,1}} \mathfrak{G}(F_{\varphi,m}(\mu_s^{1,1}))ds$$

is a martingale. Consider a function $\psi \in \mathcal{D}$ such that $\psi(x_1, \ldots, x_n) = \varphi(x_1, \ldots, x_n)$ for

$$x_i \in \left[x_{n_1}(0) - \widetilde{\delta}, x_{n_2}(0) + \widetilde{\delta}\right]$$

and $\psi(x_1, \ldots, x_n) = 0$ for

$$x_i \notin \left[x_{n_1-1}(0) + \widetilde{\delta}, x_{n_2+1}(0) - \widetilde{\delta}\right],$$

where $i = 1, \ldots, n$. We have that

$$F_{\varphi,m}(\mu^{1,1}_{t\wedge\theta_{1,1}}) - \int_0^{t\wedge\theta_{1,1}} \mathfrak{G}(F_{\varphi,m}(\mu_s^{1,1}))ds = F_{\psi,m}(\mu_{t\wedge\theta_{1,1}}) - \int_0^{t\wedge\theta_{1,1}} \mathfrak{G}(F_{\psi,m}(\mu_s))ds$$

is a martingale.

Let now $\{w_k,\ k \in \mathbb{Z}\}$ be some system of independent standard Wiener processes which is independent of $\{x_k(\cdot),\ k \in \mathbb{Z}\}$. In the standard way, we transfer the processes $w_k,\ k \in \mathbb{Z}$ and $x_k(\cdot),\ k \in \mathbb{Z}$ into a single probability space and take

$$(y_k(\cdot))_{k=1,\ldots,n} = F_n((x_k(\theta_{1,1}) + w_k(\cdot))_{k=1,\ldots,n}).$$

Then the random process

$$\widehat{\mu}_t = \mu_t^{1,1} \mathbb{I}_{\{t<\theta_{1,1}\}} + \sum_{k=1}^n \delta_{y_k(t-\theta_{1,1})} \mathbb{I}_{\{t\geq\theta_{1,1}\}}.$$

is a solution of the $\left(\mathfrak{G}, \mathcal{D}_0^{(n)}\right)$-problem of martingales. This result and Lemma 6 imply that the family $\{x_{n_1}(\cdot), \ldots, x_{n_2}(\cdot)\}$ satisfies the conditions of Theorem 1, if we replace t by $t \wedge \theta_{1,1}$ in 1°)–5°). At the time $\theta_{1,1}$, the particles, which have started from G, change their position $\mathcal{F}_{\theta_{1,1}}$ in a measurable manner. Now, these particles form the set $G_{1,1}$. We now use the strict Markov property of the input process $\{(x_n(t))_{n\in\mathbb{Z}},\ t \geq 0\}$. The particles, which have started from $G_{1,1}$, also behave themselves in the corresponding manner until the time $\theta_{1,2}$ which is constructed analogously to $\theta_{1,1}$, etc. Since $x_k(\cdot)$ are continuous processes, and, until the time σ'_1, the trajectories $x_{n_1-1}(\cdot)$, $x_{n_1}(\cdot)$ and $x_{n_2}(\cdot)$, $x_{n_2+1}(\cdot)$ do not intersect, we have $\theta_{1,k} \to \sigma'_1$ as $k \to \infty$. Therefore, the system $\{x_{n_1}(\cdot), \dots, x_{n_2}(\cdot)\}$ satisfies conditions 1°)–5°) of Theorem 1 until the time σ'_1. Then, by virtue of the strict Markov property, we start our reasoning again after the time σ'_1. We obtain again that the system $\{x_{n_1}(\cdot), \dots, x_{n_2}(\cdot)\}$ satisfies conditions 1°)–5°) until the time σ'_2, etc. In this case, only a finite number of times σ'_k are different from T with probability 1. The presented consideration implies that the trajectories of particles, which have started from G, are ordered local continuous martingales with required characteristics. Since G is arbitrary, the trajectories of all particles, which started from atoms of the measure μ_0, possess the same properties. By virtue of Theorem 1, the distribution for such a system is unique. Theorem 2 is proved. □

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Yuriy Fedkovych Chernivtsi National University, 2, Kotsyubyns'kyi Str., Chernivtsi 58012, Ukraine

E-mail address: `vitalik@imath.kiev.ua`