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INDEPENDENT INFINITE MARKOV PARTICLE SYSTEMS WITH JUMPS

We investigate independent infinite Markov particle systems (IIMPSs) as measure-valued Markov processes with jumps. We shall give sample path properties and martingale characterizations. In particular, we investigate the Hölder right continuity exponent in the case where each particle participates in the *absorbing α-stable motion on* $(0,\infty)$ *with* $0<\alpha<2$, that is, the time-changed absorbing Brownian motion on $(0,\infty)$ by the increasing $\alpha/2$-stable Lévy processes.

1. Introduction

In the study of infinite Markov particle systems, there are several difficulties, even in independent cases. For instance, "What space of measures is appropriate as a state space of the infinite particle system?", "Is it possible to consider the particle system as a measure-valued diffusion or the measure-valued càdlàg process?", or "Is it possible to characterize the generator as that in the case of finite particle systems?", and so on.

In [3], we considered independent infinite Markov particle systems with immigration on a half-space associated with absorbing Brownian motions. We gave a martingale characterization and investigated sample path properties as a measure-valued diffusion.

In the present paper, we consider more general motion processes with jumps, in particular, *absorbing α-stable motions on* $(0,\infty)$ *with* $0<\alpha<2$. We would like to investigate independent infinite Markov particle systems, which have infinitely many particles near the boundary including points at infinity. In order to control particles near the boundary, we introduce a function $g_0(x)$. Fix a strictly positive C^∞-function $g_0(x)=g_{p,0}(x)$ on $(0,\infty)$, which has the same order as $x\wedge x^{-p}$ for small or large x with $1<p<1+\alpha$ (for other conditions, see Example 3.1 in §3). In this case, the space of counting measures on $(0,\infty)$, $\mathcal{M}_{g_0}$, is defined by

$$\mu\in\mathcal{M}_{g_0} \overset{\text{def}}{\iff} \mu=\sum_n \delta_{x_n} \quad \text{such that} \quad \langle \mu, g_0\rangle = \int g_0(x)\mu(dx)<\infty. \tag{1.1}$$

$\mathcal{M}_{g_0}$ is furnished with the topology

$$\mu_n\to\mu \quad \text{in } \mathcal{M}_{g_0} \overset{\text{def}}{\iff} \sup\langle \mu_n, g_0\rangle<\infty, \langle\mu_n, f\rangle\to\langle\mu, f\rangle \quad \text{for all } f\in C_c, \tag{1.2}$$

where C_c denotes the space of continuous functions with compact supports on $(0,\infty)$. Then it holds that $\langle\mu, g_0\rangle\le\liminf\langle\mu_n, g_0\rangle<\infty$, and thus, $\mu\in\mathcal{M}_{g_0}$. Note that, for each $1\le K<\infty$, we define

$$\begin{cases} \mu\in\mathcal{M}_{g_0,K} \overset{\text{def}}{\iff} \mu\in\mathcal{M}_{g_0}, \langle\mu, g_0\rangle\le K, \\ \mu_n\to\mu \quad \text{in } \mathcal{M}_{g_0,K} \overset{\text{def}}{\iff} \langle\mu_n, f\rangle\to\langle\mu, f\rangle \quad \text{for all } f\in C_c. \end{cases} \tag{1.3}$$

Then $\mathcal{M}_{g_0,K}$ is a Polish space, and $\mu_n\to\mu$ in $\mathcal{M}_{g_0}$ is equivalent to $\mu_n\to\mu$ in $\mathcal{M}_{g,K}$ for some $K\ge1$. Hence , $\mathcal{M}_{g_0}$ is a metrizable separable space (see §2).

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Let $(X_t, \mathbf{P}_\mu)$ be the (indistinguishable) independent infinite Markov particle system (IIMPS) starting from μ, in which each particle participates in the absorbing α-stable motion $(w(t), P_x)$ on $S = (0, \infty)$, i.e., for infinitely many independent motions, $(w_n(t), P_{x_n}) \overset{(\mathrm{d})}{=} (w(t), P_{x_n})$, and we set

$$X_t = \sum_n \delta_{w_n(t)} \quad \text{if } \mu = \textstyle\sum_n \delta_{x_n} \text{ on } S, \quad \text{and} \quad \mathbf{P}_\mu = \prod_n P_{x_n}. \tag{1.4}$$

We shall show that if μ is in $\mathcal{M}_{g_0}$, then $(X_t, \mathbf{P}_\mu)$ is an $\mathcal{M}_{g_0}$-valued Markov process with càdlàg sample paths in $\mathbf{D}([0, \infty) \to \mathcal{M}_{g_0})$ and that $\langle X_t, g_0 \rangle$ is also càdlàg. We shall further investigate the exponent $\lambda > 0$ of the Hölder right continuity of $\langle X_t, g_0 \rangle$ at time zero. Moreover, we shall characterize the generator $\mathcal{L}_0$ of $(X_t, \mathbf{P}_\mu)$ by using the martingale method and also give the semimartingale representation of X_t.

In §2, we consider the IIMPSs in a more general setting. However, in order to investigate IIMPSs as measure-valued processes, we need several assumptions for the transition semigroups of motion processes. We shall give sample path properties, i.e., the exponents of the Hölder (right) continuity, and give the semimartingale representations.

In §3, we give several examples of IIMPSs associated with the well-known motion processes including absorbing stable motions on $(0, \infty)$ and show that they satisfy the conditions given in §2. Other examples are Brownian motions, Brownian motions on $\mathbf{R}^d$, rotation invariant stable Lévy processes (we call *stable motions*) on $\mathbf{R}^d$, or absorbing Brownian motions on $(0, \infty)$.

In §4, we give the proofs of semimartingale representations given in §2.

In §5, we characterize the generators of IIMPSs in the setting given as in §2.

We use the following notation: Let $S \subset \mathbf{R}^d$ be a domain.

- If $x = (x_1, \dots, x_d) \in \mathbf{R}^d$, then $\partial_i^k = \partial^k / (\partial x_i^k)$ and $\partial_i = \partial_i^1$ for each $k = 0, 1, \dots$, $i = 1, \dots, d$. Moreover, $\partial_t = \partial / \partial t$ for time $t \geq 0$.
- $f \in C_c \equiv C_c(S) \overset{\text{def}}{\iff} f$ is a continuous function on S with compact support in S, and $C_c^\infty \equiv C_c^\infty(S) := C_c(S) \cap C^\infty(S)$.
- For each integer $k \geq 0$, $C_b^k := C_b^k(\mathbf{R}^d)|_S$, that is, $f \in C_b^k \overset{\text{def}}{\iff} f$ is the restriction to S of a k-time continuously differentiable function on $\mathbf{R}^d$ with bounded derivatives of order between 0 and k. Moreover, $f \in C_0 \overset{\text{def}}{\iff} f$ is continuous on S, and $f(x) \to 0$ whenever $x \to \partial S$ or $|x| \to \infty$. Furthermore, $C_b := C_b^0$, $C_b^\infty := \bigcap_k C_b^k$, $C_0^k := C_0 \cap C_b^k$ and $C_0^\infty := \bigcap_k C_0^k$.
- For a space D of functions on S, we say that $f \in D^+ \overset{\text{def}}{\iff} f \in D; f \geq 0$.

2. General Settings and Main Results

Let S be a domain of $\mathbf{R}^d$. Let $(w(t), P_x)_{t \geq 0, x \in S}$ be an S-valued Markov process having lifetime $\zeta(w) \in (0, \infty]$ such that $w : [0, \zeta(w)) \to S$ is càdlàg (i.e., right continuous and has left-hand limits). For convenience, we fix an extra point $\Delta \notin S$ and set $w(t) = \Delta$ if $t \geq \zeta(w)$. Moreover, we shall extend functions f on S to $S \cup \{\Delta\}$ by $f(\Delta) = 0$. We denote this path space as $w \in \mathbf{D}([0, \zeta) \to S)$.

Assumption 2.1. Let $(T_t)_{t \geq 0}$ be the transition semigroup of $(w(t), P_x)$, i.e., $T_t f(x) = E_x[f(w(t)) : t < \zeta]$.

(i) (T_t) is a strongly continuous nonnegative contraction semigroup on $(C_0, \|\cdot\|_\infty)$ with generator $(A, \mathcal{D}(A))$, where $\|f\|_\infty = \sup_{x \in S} |f(x)|$.

(ii) $C_c^\infty \subset \mathcal{D}(A)$, and there is a strictly positive function $g_0 \in C_0^\infty$ such that $g_0 \in \mathcal{D}(A)$, and $g_0^{-1} A f \in C_b$ with $g_0^{-1} = 1/g_0$ for every $f \in C_c^\infty \cup \{g_0\}$.

(iii) $\sup_{t \leq T} \|g_0^{-1} T_t g_0\|_\infty < \infty$ for every $T > 0$.

Under this assumption, we introduce a function space $D_{g_0} \subset \mathcal{D}(A)$ as follows:

$$f \in D_{g_0} \overset{\text{def}}{\Longleftrightarrow} f \in \mathcal{D}(A) \quad \text{such that } \|g_0^{-1} f\|_\infty < \infty \text{ and } \|g_0^{-1} A f\|_\infty < \infty.$$

Clearly, $g_0 \in D_{g_0}$, $C_c^\infty \subset D_{g_0}$ and $T_t C_c^\infty \subset D_{g_0}$ for every $t \geq 0$. (Because for $f \in C_c^\infty$, $|A(T_t f)| \leq T_t |Af| \leq C T_t g_0 \leq C' g_0$ with some $C, C' > 0$). Moreover, since C_c^∞ is dense in C_0 and $T_t C_c^\infty \subset D_{g_0}$, D_{g_0} is a core for A (by Prop. 3.3 in Chap. 1 of [2]). However, D_{g_0} may be too large, so we further need the following assumption:

Assumption 2.2. There exist a bounded function $g_1 \in C^\infty$; $g_1 \geq g_0 (> 0)$ and a core $D \subset D_{g_0}$ (we denote $D = D_g$ with $g = (g_0, g_1)$) satisfying the following:

(i) If $f \in D_g$, then $\lim_{t \downarrow 0} \frac{1}{t} \left(T_t(f^2)(x) - f(x)^2 \right)$ exists for each $x \in S$ (we also denote the limit as $Af^2(x) = A(f^2)(x)$, then $\partial_t T_t(f^2)(x) = A T_t(f^2)(x) = T_t A(f^2)(x) \to Af^2(x)$ as $t \downarrow 0$ for each $x \in S$), $Af^2 \in C_b$ and $\|g_1^{-1} A f^2\|_\infty < \infty$.

(ii) For each $T > 0$, $\sup_{t \in [0,T]} \|g_1^{-1} T_t g_1\|_\infty < \infty$.

(iii) For each $0 < s < T$, $\sup_{t \in [s,T]} \|g_0^{-1} T_t g_1\|_\infty < \infty$.

(iv) There exist constants $0 \leq \gamma < 1, \delta > 0$ such that $\sup_{0 \leq t \leq \delta} t^\gamma \|g_0^{-1} T_t g_1\|_\infty < \infty$.

(v) $g_0 \in D_g$.

In §3, we give some examples of semigroups (T_t) satisfying Assumptions 2.1 and 2.2, with an explicit choice of g_0 and g_1.

All through the present paper, we suppose that Assumptions 2.1 and 2.2 are fulfilled and sometime use the notation $\|\cdot\|_{g_0} = \|\cdot / g_0\|_\infty$. Then it holds that, for $f \in D_{g_0}$, $\|f\|_{g_0}, \|Af\|_{g_0} < \infty$ and $|Af^2| \leq C g_1$ with some $C > 0$.

Let $\mathcal{M}_{g_0} = \bigcup_{K \geq 1} \mathcal{M}_{g_0, K}$ be a space of counting measures on S defined as (1.1)-(1.3). Then $\mathcal{M}_{g_0}$ is metrizable and separable. In fact, it is possible to take a countable family of nonnegative functions $\{f_n\}_{n \geq 1} \subset C_c^\infty$ such that $\{\alpha f_n : \alpha \in \mathbf{R}\}_{n \geq 1}$ is dense in $(C_c, \|\cdot\|_\infty)$, and we may assume that $\|f_n\|_{g_0} = 1$. We introduce a metric d on $\mathcal{M}_{g_0}$ such that

$$d(\mu, \nu) = \sum_{n=1}^\infty \frac{1}{2^n} \left(1 \wedge |\langle \mu, f_n \rangle - \langle \nu, f_n \rangle| \right).$$

This induces the same topology as in (1.2). It is easy to see that $(\mathcal{M}_{g_0}, d)$ is not complete. However, for each $K \geq 1$, $(\mathcal{M}_{g_0, K}, d)$ is complete and separable. We also consider another metric ρ such that

$$\rho(\mu, \nu) = (1 \wedge |\langle \mu, g_0 \rangle - \langle \nu, g_0 \rangle|) + d(\mu, \nu).$$

Then $\rho(\mu_n, \mu) \to 0$ is slightly stronger than $\mu_n \to \mu$ in $\mathcal{M}_{g_0}$, and $(\mathcal{M}_{g_0}, \rho)$ is not complete too. However, we can show the Hölder continuities of $\{X_t\}$ under ρ (see Theorem 2.2).

We consider the case where the generator has the form

$$A = A^c + A^d, \tag{2.1}$$

with

$$\begin{aligned} A^c f(x) &= \frac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \partial_{ij}^2 f(x) + \sum_{i=1}^d b^i(x) \partial_i f(x), \\ A^d f(x) &= \int_{S \setminus \{x\}} [f(y) - f(x) - \nabla f(x) \cdot (y - x) I(|y - x| < 1)] \nu(x, dy) \\ &\quad - k(x) f(x) + \sum_{i=1}^d c^i(x) \partial_i f(x) \end{aligned}$$

for $f \in D_g$, where $a^{ij}, b^i \in C_b(S)$, (a^{ij}) is positive definite, $k(x) \geq 0$ denotes the killing rate by jumps, $(c^i(x))$ depends on jumps, and $\nu(x, dy)$ is a Lévy kernel on $S \times (S \setminus \{x\})$ satisfying

$$\sup_{x\in S} \int_{S\setminus\{x\}} (1 \wedge |y-x|^2)\nu(x, dy) < \infty.$$

Let $(X_t, \mathbf{P}_\mu)$ be an independent infinite Markov particle system associated with $(w(t), P_x)$ defined as in (1.4). The generator $\mathcal{L}_0$ of this particle system is given by the following: for $f \in C_c^\infty$ and $\mu \in \mathcal{M}_{g_0}$,

$$\left(\mathcal{L}_0 e^{-\langle \cdot, f\rangle}\right)(\mu) = -\langle \mu, e^f A(1-e^{-f})\rangle e^{-\langle \mu, f\rangle} = -\langle \mu, Af - \Gamma f\rangle e^{-\langle \mu, f\rangle},$$

where $\Gamma f := Af - e^f A(1-e^{-f})$ (a more general formula of $\mathcal{L}_0 F(\mu)$ for functionals $F(\mu)$ is given in §5). In fact, let $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by $\{X_t\}_{t\geq 0}$ and let

$$V_t f(x) = -\log P_x[\exp -f(w(t))] = -\log\left\{1 - T_t(1-e^{-f})(x)\right\}.$$

We have that if $0 \leq s < t$, then

$$\mathbf{E}_\mu\left[e^{-\langle X_t, f\rangle} \middle| \mathcal{F}_s\right] = \exp[-\langle X_s, V_{t-s} f\rangle].$$

It is easy to see that $(V_t)_{t\geq 0}$ is a nonnegative contraction semigroup on C_0. By (ii) of Assumption 2.1, if $f \in C_c^\infty$, then $1 - e^{-f} \in C_c^\infty \subset D_g$. Hence, we have

$$\begin{aligned}
\partial_t V_t f &= \frac{T_t A(1-e^{-f})}{1 - T_t(1-e^{-f})} = \frac{A T_t(1-e^{-f})}{1 - T_t(1-e^{-f})} = e^{V_t f} A(1 - e^{-V_t f}) \\
&\to e^f A(1-e^{-f}) = Af - \Gamma f \quad (t \downarrow 0).
\end{aligned}$$

Note that since $V_t f \leq T_t f$ (by Jensen's inequality), Γ is nonnegative;

$$\Gamma f = Af - \partial_t V_t f|_{t=0+} = \lim_{t\downarrow 0} \frac{1}{t}\left[(T_t f - f) - (V_t f - f)\right] \geq 0.$$

For each $f \in C_c^\infty$, $v_t = V_t f$ is the unique solution to the following equation:

$$\partial_t v_t = e^{v_t} A(1 - e^{-v_t}), \qquad v_0 = f$$

(because $u_t := 1 - e^{-v_t}$ satisfies $\partial_t u_t = A u_t$, $u_0 = 1 - e^{-f}$ and $u_t = T_t(1-e^{-f})$ is the unique solution). Moreover, if $A v_t(x)$ is well-defined for $t > 0$, $x \in S$, then

$$\partial_t v_t = A v_t - \Gamma v_t, \qquad v_0 = f,$$

or, equivalently,

$$v_t = T_t f - \int_0^t T_{t-s} \Gamma v_s ds.$$

If A is given as in (2.1), then $\Gamma = \Gamma^c + \Gamma^d$ with

$$\begin{aligned}
\Gamma^c f(x) &= \frac{1}{2}\sum_{i,j} a^{ij}(x) \partial_i f(x) \partial_j f(x), \\
\Gamma^d f(x) &= \int_{S\setminus\{x\}} \left(e^{-[f(y)-f(x)]} - 1 + [f(y) - f(x)]\right) \nu(x, dy) \\
&\quad + k(x)\left(e^{f(x)} - 1 - f(x)\right).
\end{aligned}$$

First, we mention that, by simple computations,

$$\text{(2.2)} \qquad \begin{cases} \mathbf{E}_\mu[\langle X_t, f\rangle] = \langle \mu, T_t f\rangle, \\ \mathbf{E}_\mu\left[\langle X_t, f\rangle \langle X_t, g\rangle\right] = \langle \mu, T_t f\rangle \langle \mu, T_t g\rangle + \langle \mu, T_t(fg) - (T_t f)(T_t g)\rangle \end{cases}$$

hold for $f \in C_b^+$. Moreover, by using the Markov property and by induction, we have

Proposition 2.1 (Prop.1 in [3]). *For every $0 \le t_1 \le \cdots \le t_n$ and $f_i \in D_g^+, i = 1, 2, \ldots, n$,*

$$\mathbf{E}_\mu \left[\langle X_{t_1}, f_1\rangle \cdots \langle X_{t_n}, f_n\rangle\right] \le \prod_{i=1}^n \langle \mu, T_{t_i} f_i\rangle + C_1^{(n)} \sum_{i=1}^n \prod_{j\neq i} \langle \mu, T_{t_j} f_j\rangle$$

$$+C_2^{(n)} \sum_{i_1=1}^n \sum_{i_2\neq i_1} \prod_{j\neq i_1, i_2} \langle \mu, T_{t_j} f_j\rangle + \cdots + C_{n-1}^{(n)} \sum_{j=1}^n \langle \mu, T_{t_j} f_j\rangle + C_n^{(n)},$$

where $C_k^{(n)}, k = 1, \cdots, n$ are positive constants depending on $(n, \{\|f_i\|_\infty\}_{i\le n})$.

Hence, for $\mu \in \mathcal{M}_{g_0}$, if $t > 0$, then $\langle X_t, g_1\rangle$ is in $L^k(\mathbf{P}_\mu)$ for every $k \ge 1$ by (iii) of Assumption 2.2. Furthermore, using Jensen's inequality and Fubini's theorem, one can show that, for every $t \ge 0$, $j, k \ge 1$,

$$\langle X_t, g_0\rangle, \int_0^t \langle X_s, g_0\rangle^j \langle X_s, g_1\rangle ds \quad \text{are in } L^k(\mathbf{P}_\mu) \tag{2.3}$$

by (iii), (iv) of Assumption 2.2 (note that $g_0 \le g_1$).

We next introduce a nonnegative operator Q as $Qf = Af^2 - 2fAf$ for $f \in D_g$, which is well-defined by (i) of Assumption 2.2 and plays an important role to investigate the Hölder (right) continuity exponents. The nonnegativity follows from $(T_t f^2 - f^2) - 2f(T_t f - f) \ge (T_t f)^2 - 2f T_t f + f^2 = (T_t f - f)^2 \ge 0$. Moreover, the assumption yields $Qf \le C g_1$ for $f \in D_g$ with some $C > 0$.

Remark 2.1. (i) If we further assume that, for each $x \in S$, $P_x(\zeta \ge t) = o(t)$ as $t \downarrow 0$, i.e., $\lim_{t\downarrow 0} \frac{1}{t}(T_t 1 - 1)(x) = 0$, then we have, for $f \in C_c^\infty$,

$$e^{-2\|f\|_\infty} Qf \le 2\Gamma f \le e^{2\|f\|_\infty} Qf. \tag{2.4}$$

Indeed,

$$\begin{aligned} Qf(x) &= \lim_{t\downarrow 0} \frac{1}{t} \left(T_t f^2 - 2f T_t f + f^2\right)(x) \\ &= \lim_{t\downarrow 0} \frac{1}{t} \left[T_t (f - f(x))^2(x) + f(x)^2 (1 - T_t 1)(x)\right] \end{aligned}$$

and

$$\begin{aligned} \Gamma f(x) &= Af(x) - e^{f(x)} A(1 - e^{-f})(x) \\ &= \lim_{t\downarrow 0} \frac{1}{t} \left[(T_t f - f)(x) - e^{f(x)} \left\{T_t(1 - e^{-f}) - (1 - e^{-f(x)})\right\}(x)\right] \\ &= \lim_{t\downarrow 0} \frac{1}{t} \left[T_t \left(e^{-[f - f(x)]} - 1 + [f - f(x)]\right)(x) + (e^f - 1 - f)(x)(1 - T_t 1)(x)\right]. \end{aligned}$$

Hence, by using $e^x - 1 - x = x^2 e^{\theta x}/2$ with some $\theta \in (0, 1)$, we have (2.4).

(ii) If A is given as in (2.1), then $Q = Q^c + Q^d$ with $Q^c = 2\Gamma^c$, i.e., $Q^c f(x) = \sum_{i,j=1}^d a^{ij}(x) \partial_i f(x) \partial_j f(x)$ and

$$Q^d f(x) = \int_{S\setminus\{x\}} [f(y) - f(x)]^2 \nu(x, dy) + k(x) f(x)^2.$$

(iii) In case of $g_0 \neq g_1$, $\mathcal{L}_0(\exp -\langle \cdot, g_0\rangle)(\mu)$ may not be well-defined for all $\mu \in \mathcal{M}_{g_0}$, because $\langle \mu, \Gamma g_0\rangle$ may be infinite for some $\mu \in \mathcal{M}_{g_0}$.

By (2.2), we have, for $f \in C_c^\infty$,

$$\mathcal{L}_0 \langle \cdot, f\rangle(\mu) = \langle \mu, Af\rangle, \quad \mathcal{L}_0 \langle \cdot, f\rangle^2(\mu) = 2\langle \mu, f\rangle\langle \mu, Af\rangle + \langle \mu, Qf\rangle.$$

Moreover, we can see the following:

Proposition 2.2. *Let $\mu \in \mathcal{M}_{g_0}$. For $f \in D_g$,*

$$\begin{aligned} M_t(f) &= \langle X_t, f\rangle - \langle X_0, f\rangle - \int_0^t \mathcal{L}_0\langle \cdot, f\rangle(X_s)ds, \\ N_t(f) &= \langle X_t, f\rangle^2 - \langle X_0, f\rangle^2 - \int_0^t \mathcal{L}_0\langle \cdot, f\rangle^2(X_s)ds \end{aligned}$$

are $\mathbf{P}_\mu$-martingales and $M_t(f)^2 - \int_0^t \langle X_s, Qf\rangle ds$ is also a $\mathbf{P}_\mu$-martingale. In particular,

$$\mathbf{E}_\mu[M_t(f)^2] = \mathbf{E}_\mu\left[\int_0^t \langle X_s, Qf\rangle ds\right] = \int_0^t \langle \mu, T_s Qf\rangle ds.$$

Proof. Let $f \in D_g$. If the particle system is finite, it is easy to check that $M_t(f)$, $N_t(f)$ are martingales. Moreover, by a simple computation, we have

$$\begin{aligned} &M_t(f)^2 - \int_0^t \langle X_s, Qf\rangle ds \\ &= N_t(f) - 2\langle X_0, f\rangle M_t(f) + \left(\int_0^t \langle X_s, Af\rangle ds\right)^2 \\ &\quad -2\langle X_t, f\rangle \int_0^t \langle X_s, Af\rangle ds + 2\int_0^t \langle X_s, f\rangle\langle X_s, Af\rangle ds \\ &= N_t(f) - 2\langle X_0, f\rangle M_t(f) - 2\int_0^t [M_t(f) - M_s(f)]\langle X_s, Af\rangle ds. \end{aligned}$$

This is a martingale. Hence, the above results are valid for finite particle systems. Let $X_t^{(n)} = \sum_{k\le n} \delta_{w_k(t)}$ with $X_0^{(n)} = \mu^{(n)}$. For $f \in D_g$, recall $|f|, |Af| \le Cg_0, |Qf| \le Cg_1$ with some $C > 0$. By (2.3) for each fixed $t \ge 0$, under $\mathbf{P}_\mu$, $\langle X_t^{(n)}, g_0\rangle \uparrow \langle X_t, g_0\rangle$ and $\int_0^t \langle X_s^{(n)}, g_0\rangle^i \langle X_s^{(n)}, g_1\rangle^j ds \uparrow \int_0^t \langle X_s, g_1\rangle^i \langle X_s, g_1\rangle^j ds$ a.s. and in L^k for every $k \ge 1$ as $n \to \infty$ ($i = 0, 1, 2, j = 0, 1$). Therefore, the results are valid for the infinite case. $\square$

Theorem 2.1. *Let $\mu \in \mathcal{M}_{g_0}$.*

(i) *If the motion process is a continuous Markov process in $\mathbf{C}([0,\zeta) \to S)$, then $\{X_t\}$ is in $\mathbf{C}([0,\infty) \to \mathcal{M}_{g_0})$ and $\{\langle X_t, g_0\rangle\}$ is in $\mathbf{C}([0,\infty) \to \mathbf{R})$, $\mathbf{P}_\mu$-a.s.*

(ii) *If the motion process is a discontinuous Markov process in $\mathbf{D}([0,\zeta) \to S)$, then $\{X_t\}$ is in $\mathbf{D}([0,\infty) \to \mathcal{M}_{g_0})$ and $\{\langle X_t, g_0\rangle\}$ is in $\mathbf{D}([0,\infty) \to \mathbf{R})$, $\mathbf{P}_\mu$-a.s.*

Proof. Fix $T > 0$. Let $X^{(n)} = \{X_t^{(n)}\}_{t\le T}$ be the n-particle system such that $X_0^{(n)} = \mu^{(n)}$ given as in the previous proof. We denote the corresponding martingale part by $\{M_t^{(n)}(f)\}$. Recall $Qg_0 \le Cg_1, |Ag_0| \le Cg_0$ with some $C > 0$. By Assumption 2.2 (iv),

$$\begin{aligned} \int_0^T \langle \mu, T_t Q g_0\rangle dt &\le C\int_0^T \langle \mu, T_t g_1\rangle dt \\ &\le C\sup_{t\le T} t^\gamma \|g_0^{-1}T_t g_1\|_\infty \int_0^T t^{-\gamma} dt\ \langle \mu, g_0\rangle \\ &\le C_T\langle \mu, g_0\rangle, \end{aligned}$$

where C_T are positive constants. We note that $\mu^{(m)} \leq \mu^{(n)}$ if $m < n$. By Proposition 2.2 and Doob's maximal inequality, we have, for $m < n$,

$$\begin{aligned}\mathbf{E}_\mu \left[\sup_{t\leq T}\left|M_t^{(n)}(g_0) - M_t^{(m)}(g_0)\right|^2\right] &\leq 4\int_0^T \langle \mu^{(n)} - \mu^{(m)}, T_tQg_0\rangle dt \\ &\leq 4C_T\langle \mu^{(n)} - \mu^{(m)}, g_0\rangle \\ &\to 0\end{aligned}$$

as $n > m \to \infty$. Moreover, $\|g_0^{-1}T_tg_0\|_\infty < \infty$ by Assumption 2.1. If $m < n$, then

$$\begin{aligned}\mathbf{E}_\mu\left[\int_0^T \langle X_t^{(n)} - X_t^{(m)}, |Ag_0|\rangle dt\right] &\leq C'\int_0^T \langle \mu^{(n)} - \mu^{(m)}, T_tg_0\rangle dt \\ &\leq C'_T \sup_{t\leq T}\|g_0^{-1}T_tg_0\|_\infty\langle \mu^{(n)} - \mu^{(m)}, g_0\rangle \\ &\to 0 \quad (n > m \to \infty),\end{aligned}$$

where C', C'_T are positive constants. By Proposition 2.2, this yields

$$\mathbf{E}_\mu\left[\sup_{t\leq T}|\langle X_t^{(n)} - X_t^{(m)}, g_0\rangle|\right] \to 0 \quad (n > m \to \infty).$$

Thus, there is a subsequence $\{X^{(n_k)}\}_{k\geq 1}$ such that

$$\sup_{t\leq T}|\langle X_t^{(n_k)} - X_t^{(n_j)}, g_0\rangle| \to 0 \quad \text{as } j,k \to \infty,\ \mathbf{P}_\mu\text{-a.s.} \tag{2.5}$$

Moreover, (2.5) is also valid for $f \in C_c^+$ instead of g_0. Hence, for $\mathbf{P}_\mu$-a.a. ω, there is a positive number $K = K(\omega) \geq 1$ such that $\{X^{(n_k)}(\omega)\}$ is a Cauchy sequence in $\mathbf{C}([0,T] \to (\mathcal{M}_{g_0,K}, d))$ (or in $\mathbf{D} = \mathbf{D}([0,T] \to (\mathcal{M}_{g_0,K}, d))$). Since $(\mathcal{M}_{g_0,K}, d)$ is complete, the limit $\widetilde{X} = \{\widetilde{X}_t\}_{t\leq T}$ exists in $\mathbf{C} = \mathbf{C}([0,T] \to \mathcal{M}_{g_0})$ (or $\mathbf{D} = \mathbf{D}([0,T] \to \mathcal{M}_{g_0})$), which is a version of $\{X_t\}_{t\leq T}$. Hence, $\mathbf{P}_\mu(\widetilde{X}_r = X_r$ for all $r \in \mathbf{Q}^+) = 1$, and, by Fatou's lemma, we have, for $f \in C_c^+ \cup \{g_0\}$,

$$\langle \widetilde{X}_t, f\rangle = \lim_{r(\in\mathbf{Q}^+)\downarrow t}\langle X_r, f\rangle \geq \langle X_t, f\rangle \quad \text{for every } t \geq 0,\ \mathbf{P}_\mu\text{-a.s.},$$

Therefore,

$$\sup_{t\leq T}|\langle \widetilde{X}_t - X_t, f\rangle| \leq \sup_{t\leq T}|\langle \widetilde{X}_t - X_t^{(n_k)}, f\rangle| \to 0 \quad (k \to \infty), \qquad \mathbf{P}_\mu\text{-a.s.},$$

that is, $\widetilde{X}_t = X_t$ for all $t \geq 0$, $\mathbf{P}_\mu$-a.s. □

We now investigate the exponents of the Hölder (right) continuity of $\langle X_t, g_0\rangle$. First, we consider the continuous case.

Theorem 2.2 (Hölder continuity). *Let $(w(t), P_x)$ be a continuous Markov process in $\mathbf{C}([0,\zeta) \to S)$ with a transition semigroup (T_t) satisfying Assumptions 2.1 and 2.2. Let $\mu \in \mathcal{M}_{g_0}$. The following holds with $\mathbf{P}_\mu$-probability one:*

- (i) *$\{\langle X_t, g_0\rangle\}$ is locally $(1/2-\varepsilon)$-Hölder continuous at $t > 0$ and $((1-\gamma)/2-\varepsilon)$-Hölder right continuous at $t = 0$ for sufficiently small $\varepsilon > 0$, where the constant $0 \leq \gamma < 1$ is as in Assumption 2.2* (iv).
- (ii) *Let $\langle \mu, g_1\rangle < \infty$, in particular, for $g_1 \geq g_0$. If it is possible to take $g_1(x) = g_0(x)$, then $\{\langle X_t, g_0\rangle\}$ is locally $(1/2-\varepsilon)$-Hölder continuous at $t \geq 0$ for sufficiently small $\varepsilon > 0$.*

Moreover, the same results hold for $\{X_t\}$ under the metric ρ.

Proof. The proof is similar to that of Theorem 2 in [3]. First, we show the local $((1-\gamma)/2-\varepsilon)$-Hölder continuity of $\{\langle X_t, g_0\rangle\}$. By Kolmogorov's continuity theorem, it is enough to show that, for each $T>0$ and for large $k\in\mathbf{N}; k(1-\gamma)>1$, there are constants $C_{T,k}>0$ such that

$$\mathbf{E}_\mu\left[|\langle X_t, g_0\rangle - \langle X_s, g_0\rangle|^{2k}\right] \le C_{T,k}(t-s)^{k(1-\gamma)}$$

for all $0\le s<t\le T$. First, we note that

$$\int_s^t \langle \mu, T_u g_0\rangle du \le \sup_{u\in[0,T]} \|g_0^{-1}T_u g_0\|_\infty \langle \mu, g_0\rangle (t-s).$$

Hence, by Jensen's inequality and using Proposition 2.1 and Assumption 2.1 (iii), we have that, for each $T>0$ and for each $k\in\mathbf{N}$, there are constants $C_{T,k}^{(0)}, C_{T,k}^{(1)}>0$ such that

$$\mathbf{E}_\mu\left[\left(\int_s^t \langle X_u, g_0\rangle du\right)^k\right] \le C_{T,k}^{(0)}\left(\int_s^t \langle \mu, T_u g_0\rangle du\right)^k \le C_{T,k}^{(1)}(t-s)^k$$

for all $0\le s<t\le T$. As $|Ag_0|\le Cg_0$ by Assumption 2.1 (ii), we further obtain

$$\mathbf{E}_\mu\left[\left(\int_s^t \langle X_u, |Ag_0|\rangle du\right)^{2k}\right] \le C_{T,k}^{(2)}(t-s)^{2k}.$$

Moreover, by (iii) and (iv) of Assumption 2.2, it holds that, for any $0\le s<t\le T$,

$$\int_s^t \langle \mu, T_u g_1\rangle du \le \sup_{u\le T} u^\gamma \|g_0^{-1}T_u g_1\|_\infty \langle \mu, g_0\rangle \int_s^t u^{-\gamma}du \le C_T(t-s)^{1-\gamma}$$

with some constant C_T. Recall Proposition 2.2. Since, for each fixed $s\ge 0$, $\{N_t^s(g_0)\}_{t\ge s}$; $N_t^s(g_0):=M_t(g_0)-M_s(g_0)$ is a continuous martingale with quadratic variation

$$[N_t^s(g_0)] = [M(g_0)]_t - [M(g_0)]_s = \int_s^t \langle X_u, Qg_0\rangle du,$$

we have, by using the Burkholder–Davis–Gundy inequality and by $Qg_0\le Cg_1$,

$$\mathbf{E}_\mu\left[(M_t(g_0)-M_s(g_0))^{2k}\right] \le C_{T,k}^{(3)}\mathbf{E}_\mu\left[\left(\int_s^t \langle X_u, Qg_0\rangle du\right)^k\right] \le C_{T,k}^{(4)}(t-s)^{k(1-\gamma)},$$

where the constants $C_{T,k}^{(i)}$, $i=2,3,4$, depend only on (T,k). Thus, the $((1-\gamma)/2-\varepsilon)$-Hölder continuity of $\{\langle X_t, g_0\rangle\}$ in $0\le t\le T$ follows. Furthermore, if $\langle \mu, g_1\rangle<\infty$, then, by $\langle \mu, T_u g_1\rangle \le \langle \mu, g_1\rangle \|g_1^{-1}T_u g_1\|_\infty$ and (ii) of Assumption 2.2, we have,

$$\int_s^t \langle \mu, T_u g_1\rangle du \le \sup_{u\in[0,T]} \|g_1^{-1}T_u g_1\|_\infty \langle \mu, g_1\rangle (t-s).$$

Thus, $\{\langle X_t, g_0\rangle\}$ is locally $(1/2-\varepsilon)$-Hölder continuous at $t\ge 0$, $\mathbf{P}_\mu$-a.s. For general $\mu\in\mathcal{M}_{g_0}$, if $t>0$, then, by (iii) of Assumption 2.2,

$$\mathbf{E}_\mu[\langle X_t, g_1\rangle] = \langle \mu, T_t g_1\rangle \le \langle \mu, g_0\rangle \|g_0^{-1}T_t g_1\|_\infty < \infty.$$

Thus, $\langle X_t, g_1\rangle<\infty$, $\mathbf{P}_\mu$-a.s. Therefore, the locally $(1/2-\varepsilon)$-Hölder continuity of $\{\langle X_t, g_0\rangle\}$ at $t>0$ and the $((1-\gamma)/2-\varepsilon)$-Hölder right continuity at $t=0$ follow. Finally, in the definition of the metric ρ for $n\ge 1$, we can take $\{f_n\}\subset (C_c^\infty)^+$ such that

$$\|f_n\|_{g_0} + \|Af_n\|_{g_0} + \|Qf_n\|_{g_0} \le 1.$$

Hence we can get the same inequalities for $\rho(X_t, X_s)^{2k}$ instead of $|\langle X_t, g_0\rangle - \langle X_s, g_0\rangle|^{2k}$. □

For more specific cases (e.g., the Brownian motion or absorbing Brownian motion), it is possible to discuss the non-Hölder continuities as in [3]. The Hölder continuity exponent is determined by the order of t in $\mathbf{E}_\mu[\langle X_t, g_1\rangle] = \langle \mu, T_t g_1\rangle$.

Next, we consider the discontinuous case. In this case, we can only discuss the Hölder right continuity exponents at a fixed time.

Theorem 2.3 (Hölder right continuity at $t = 0$)**.** *Let $(w(t), P_x)$ be a discontinuous Markov process in $\mathbf{D}([0, \zeta(w)) \to S)$ with a transition semigroup (T_t) satisfying Assumptions 2.1 and 2.2. Let $\mu \in \mathcal{M}_{g_0}$. The following holds with $\mathbf{P}_\mu$-probability one.*

(i) *$\{\langle X_t, g_0\rangle\}$ is $((1-\gamma)/2 - \varepsilon)$-Hölder right continuous at $t = 0$ for sufficiently small $\varepsilon > 0$, where the constant $0 \le \gamma < 1$ is in* (iv) *of Assumption 2.2.*

(ii) *If $\langle \mu, g_1\rangle < \infty$, in particular, if $g_1(x) = g_0(x)$, then $\{\langle X_t, g_0\rangle\}$ is $(1/2 - \varepsilon)$-Hölder right continuous at $t = 0$ for sufficiently small $\varepsilon > 0$.*

If $t > 0$, then $\langle X_t, g_1\rangle < \infty$, $\mathbf{P}_\mu$-a.s. Hence, the following is immediately obtained.

Corollary 2.1. *Let $\mu \in \mathcal{M}_{g_0}$. For each fixed $t_0 > 0$, with $\mathbf{P}_\mu$-probability one, it holds that $\{\langle X_t, g_0\rangle\}$ is $(1/2-\varepsilon)$-Hölder right continuous at $t = t_0$ for sufficiently small $\varepsilon > 0$.*

By using the following proposition, the above theorem can be shown similarly to the continuous case. However, we only use the square moment, i.e., $p = 2$.

Proposition 2.3. *Let $p > 1$. On a probability space $(\Omega, \mathcal{F}, P)$, let $\{M_t\}$ be a right-continuous $L^p(P)$-martingale starting from $M_0 = 0$. For small $T > 0$, if there exist constants $\beta > 0$ and $C_{p,T} > 0$ such that*

$$E[|M_t|^p] \le C_{p,T} t^{p\beta} \qquad \textit{for all } 0 \le t \le T,$$

then

$$\limsup_{t\downarrow 0} \frac{|M_t|}{t^\beta \log 1/t} = 0, \qquad P\textit{-a.s.}$$

Proof. For each integer n, set $t_n = 1/2^n$ and let

$$Z_n = \sup_{t_{n+1}\le t < t_n} \frac{|M_t|}{t^\beta \log 1/t}.$$

By Doob's maximal inequality, we have

$$\begin{aligned} E[|Z_n|^p] &\le \frac{E[\sup_{t_{n+1}\le t<t_n} |M_t|^p]}{t_{n+1}^{p\beta}(\log 1/t_n)^p} \\ &\le \left(\frac{p}{p-1}\right)^p \frac{E|M_{t_n}|^p}{(t_n/2)^{p\beta}(\log 2^n)^p} \\ &\le \left(\frac{p2^\beta}{(p-1)\log 2}\right)^p \frac{C_{p,T}}{n^p}. \end{aligned}$$

Hence, $E\left[\sum_n |Z_n|^p\right] = \sum_n E[|Z_n|^p] < \infty$. This yields

$$P\left(\limsup_{t\downarrow 0} \frac{|M_t|}{t^\beta \log 1/t} = 0\right) = P\left(\lim_{n\to\infty} Z_n = 0\right) = 1.$$

□

We give a semimartingale representation of $(X_t, \mathbf{P}_\mu)$. The following result can be shown like Theorem 6.1.3 in [1]. The proof is given in §4.

Theorem 2.4 (Semimartingale Representation of $(X_t, \mathbf{P}_\mu)$). *Under Assumptions 2.1 and 2.2, we suppose that the generator A of the motion process is given as in* (2.1). *If $\mu \in \mathcal{M}_{g_0}$, then $(X_t, \mathbf{P}_\mu)$ has the following semimartingale representation: For $f \in D_g$,*

$$\langle X_t, f\rangle = \langle X_0, f\rangle + \int_0^t \langle X_s, Af\rangle ds + M_t^c(f) + M_t^d(f),$$

where

$$M_t^c(f) \quad \text{is a continuous } L^2\text{-martingale}$$

with quadratic variation $[M^c(f)]_t = \int_0^t \langle X_s, Q^c f\rangle ds = 2\int_0^t \langle X_s, \Gamma^c f\rangle ds$ *and*

$$M_t^d(f) = \int_0^t \int_{\mathcal{M}_{g_0}^{\pm}} \langle \mu, f\rangle \widetilde{N}(ds, d\mu) \quad \text{is a purely discontinuous } L^2\text{-martingale}$$

with $\widetilde{N} = N - \widehat{N}$ is the martingale measure such that, for $\Delta X_u = X_u - X_{u-}$,

$$\begin{aligned} N(ds, d\mu) &= \sum_{u;\Delta X_u \neq 0} \delta_{(u,\Delta X_u)}(ds, d\mu) : \quad \text{the jump measure of } \{X_t\}, \\ \widehat{N}(ds, d\mu) &= ds \int X_s(dx) \left(\int \nu(x, dy)\delta_{(\delta_y - \delta_x)} + k(x)\delta_{-\delta_x} \right)(d\mu) : \\ &\qquad\qquad \text{the compensator of } N, \end{aligned}$$

where $\mathcal{M}_{g_0}^{\pm}$ is the family of signed-measures of $\mu^+ - \mu^-; \mu^+, \mu^- \in \mathcal{M}_{g_0}$.

3. Examples of Motion Processes and the Hölder (Right) Continuity Exponents

In this section, we shall investigate the exponents of the Hölder right continuity of sample paths of independent Markov particle systems associated with motion processes given in the following examples:

Example 3.1. (i) *Brownian motion and stable motion on* $\mathbf{R}^d$: Let $p > d$ and $g_p(x) := (1 + |x|^2)^{-p/2}$. We define function spaces C_p, C_p^2 by $f \in C_p \equiv C_p(\mathbf{R}^d) \overset{\text{def}}{\iff} f \in C(\mathbf{R}^d)$ and $\|f/g_p\| < \infty$, $f \in C_p^2 \overset{\text{def}}{\iff} f \in C_b^2(\mathbf{R}^d); |f|, |\partial_i f|, |\partial_{ij}^2 f| \le C g_p$ for all $i, j = 1, \cdots, d$ with some constant $C = C(f)$. In this case, we can take $g_p(x)$ as $g_0(x)$ and $D_g = C_p^2 \equiv C_p^2(\mathbf{R}^d)$ (see the following proof of Theorem 3.1).

(a) If $(w(t), P_x)$ is the Brownian motion on $\mathbf{R}^d$, then $p > d$ and

$$Af = \frac{1}{2}\triangle f = \frac{1}{2}\sum_{i=1}^d \partial_i^2 f, \qquad Qf = |\nabla f|^2 = \sum_{i=1}^d (\partial_i f)^2.$$

(b) If $(w(t), P_x) = (w^\alpha(t), P_x^\alpha)$ is the α-stable motion on $\mathbf{R}^d$ $(0 < \alpha < 2)$, that is, a rotation-invariant α-stable Lévy process on $\mathbf{R}^d$, then $d < p < d + \alpha$ and

$$\begin{aligned} Af(x) &= A^\alpha f(x) = -(-\triangle)^{\alpha/2} f(x) \\ &= c\int_{\mathbf{R}^d \setminus \{0\}} [f(x+y) - f(x) - \nabla f(x)\cdot y I(|y| < 1)] \frac{dy}{|y|^{d+\alpha}} \\ &= c\int_{\mathbf{R}^d \setminus \{x\}} [f(y) - f(x) - \nabla f(x)\cdot (y-x) I(|y-x| < 1)] \frac{dy}{|y-x|^{d+\alpha}}, \end{aligned}$$

$$\begin{aligned} Qf(x) &= Q^\alpha f(x) = c\int_{\mathbf{R}^d \setminus \{0\}} |f(x+y) - f(x)|^2 \frac{dy}{|y|^{d+\alpha}} \\ &= c\int_{\mathbf{R}^d \setminus \{x\}} |f(y) - f(x)|^2 \frac{dy}{|y-x|^{d+\alpha}}, \end{aligned}$$

where $c > 0$ is a suitable constant.

(ii) *Absorbing Brownian motion and absorbing stable motion on* $(0, \infty)$: Let $h_0(v)$ be a C^∞-function on $(0, \infty)$ such that $0 < h_0 \le 1$ on $(0, \infty)$, $h_0(v) = v$ for $v \in (0, 1/2]$ and $h_0(v) = 1$ for $v \ge 1$. Let $p > 1$ and set

$$g_{p,0}(x) := g_p(x)h_0(x) \quad \text{for } x \in (0, \infty) \text{ with } g_p(x) = (1 + x^2)^{-p/2}.$$

We further define $f \in C_{p,0} \overset{\text{def}}{\Longleftrightarrow} f \in C(\mathbf{R})|_{(0,\infty)}$ and $\|f/g_{p,0}\| < \infty$. Moreover, for $k \ge 2$, we set $f \in C^k_{p,0} \overset{\text{def}}{\Longleftrightarrow} f \in C^k_b(\mathbf{R})|_{(0,\infty)}; |f|, |f''| \le Cg_{p,0}$ and $|f'| \le Cg_p$ with some constant $C = C(f)$. In this case, $g_0(x)$ is given as $g_{p,0}(x)$.

(a) If $(w(t), P_x) = (w^0(t), P^0_x)$ is the absorbing Brownian motion on $(0, \infty)$, then $p > 1$. Moreover, we can take $D_g = C^2_{p,0}$, and A, Q are the same as in the case of the Brownian motion on $\mathbf{R}$.

(b) If $(w(t), P_x) = (w^{-,\alpha}(t), P^{-,\alpha}_x)$ is the absorbing α-stable motion on $(0, \infty)$ $(0 < \alpha < 2)$, i.e., the time-changed absorbing Brownian motion on $(0, \infty)$ by the increasing $\alpha/2$-stable Lévy process $y^{\alpha/2}(t)$ on $[0, \infty)$ starting from 0; $w^{-,\alpha}(t) = w^0(y^{\alpha/2}(t))$, where $\{w^0(t)\}, \{y^{\alpha/2}(t)\}$ are independent, then $1 < p < 1 + \alpha$, and we can take $D_g = C^3_{p,0}$ (see the following proof of Theorem 3.1). Moreover, for a function f on $(0, \infty)$, let $\overline{f}$ be an extension of f on $\mathbf{R}$ defined as

$$\overline{f}(z) = \begin{cases} f(z) & (z > 0), \\ f(0+) & (z = 0), \\ -f(-z) & (z < 0). \end{cases}$$

The generator $A = A^{-,\alpha}$ is given as

$$A^{-,\alpha} f(x) = A^\alpha \overline{f}(x).$$

We can also write that if $0 < \alpha < 1$, then

$$\begin{aligned} A^{-,\alpha} f(x) &= c \int_{\mathbf{R}\setminus\{x\}} \left[\overline{f}(y) - \overline{f}(x)\right] \frac{dy}{|y - x|^{1+\alpha}} \\ &= c \int_0^\infty [f(y) - f(x)] K(x, y) dy - 2cf(x) \int_0^\infty \frac{dy}{(y + x)^{1+\alpha}} \\ &= c \int_0^\infty [f(y) - f(x)] K(x, y) dy - \frac{2c}{\alpha} x^{-\alpha} f(x) \end{aligned}$$

and that if $1 \le \alpha < 2$, then (in §4 of [4], we have some misprints)

$$\begin{aligned} A^{-,\alpha} f(x) &= c \int_{\mathbf{R}\setminus\{x\}} \left[\overline{f}(y) - \overline{f}(x) - (\overline{f})'(x)(y - x) I(|y - x| < 1)\right] \frac{dy}{|y - x|^{1+\alpha}} \\ &= c \int_0^\infty \left[f(y) - f(x) - f'(x)(y - x) I(|y - x| < 1)\right] K(x, y) dy \\ &\quad + c \int_0^\infty [-2f(x) + f'(x)(y + x) I(y + x < 1) \\ &\quad - f'(x)(y - x) I(|y - x| < 1))] \frac{dy}{(y + x)^{1+\alpha}} \\ &= c \int_0^\infty \left[f(y) - f(x) - f'(x)(y - x) I(|y - x| < 1)\right] K(x, y) dy \\ &\quad - \frac{2c}{\alpha} x^{-\alpha} f(x) + f'(x) c(x), \end{aligned}$$

where

$$K(x, y) = \frac{I(y \ne x)}{|y - x|^{1+\alpha}} - \frac{1}{(y + x)^{1+\alpha}}$$

and

$$c(x) = c\int_0^\infty [(y+x)I(y+x<1) - (y-x)I(|y-x|<1))]\,\frac{dy}{(y+x)^{1+\alpha}}.$$

(Note that if $0<x<1$, then

$$c(x) = \begin{cases} \dfrac{2}{\alpha}\left(x^{1-\alpha} - x(2x+1)^{-\alpha}\right) - \dfrac{1}{\alpha-1}\left(1-(2x+1)^{1-\alpha}\right) \sim \dfrac{2}{\alpha}x^{1-\alpha} & (1<\alpha<2), \\ 2\left(1-\dfrac{x}{2x+1}\right) - \log(2x+1) \sim 2 & (\alpha=1). \end{cases}$$

as $x\downarrow 0$. We can also show that $c(x)$ is positive.) Moreover, we have, for $0<\alpha<2$,

$$Qf(x) = Q^{-,\alpha}f(x) = c\int_0^\infty |f(y)-f(x)|^2K(x,y)dy + \frac{2c}{\alpha}x^{-\alpha}f(x)^2.$$

With the above motion processes, we have the following results:

Theorem 3.1. *Let g_0 be given in the following each case. Let $\mu\in\mathcal{M}_{g_0}$, and let $\varepsilon>0$ denote any small number.*

(i) *Continuous case; Brownian motion on $\mathbf{R}^d$ or absorbing Brownian motion on $(0,\infty)$. Under $\mathbf{P}_\mu$, the following holds with probability one.*
  - *(a) Let the motion process be the Brownian motion on $\mathbf{R}^d$ and $g_0=g_p$ with $p>d$. Then $\{\langle X_t, g_p\rangle\}$ is locally $(1/2-\varepsilon)$-Hölder continuous in $t\ge 0$.*
  - *(b) Let $d=1$. If the motion process is the absorbing Brownian motion on $(0,\infty)$ and $g_0=g_{p,0}$ with $p>1$, then $\{\langle X_t, g_{p,0}\rangle\}$ is locally $(1/2-\varepsilon)$-Hölder continuous at $t>0$ and $(1/4-\varepsilon)$-Hölder right continuous at $t=0$. Moreover, if $\langle\mu, g_p\rangle<\infty$, then $\{\langle X_t, g_p\rangle\}$ is locally $(1/2-\varepsilon)$-Hölder continuous at $t\ge 0$.*

(ii) *Discontinuous case; the stable motion on $\mathbf{R}^d$ or absorbing stable motion on $(0,\infty)$. Under $\mathbf{P}_\mu$, the following holds with probability one.*
  - *(a) Let the motion process be the α-stable motion on $\mathbf{R}^d$ with $0<\alpha<2$ and $g_0=g_p$ with $d<p<d+\alpha$. Then $\{\langle X_t, g_p\rangle\}$ is $(1/2-\varepsilon)$-Hölder right continuous at $t=0$.*
  - *(b) Let $d=1$, let the motion process be the absorbing α-stable motion on $(0,\infty)$ with $0<\alpha<2$, and let $g_0=g_{p,0}$ with $1<p<1+\alpha$. $\{\langle X_t, g_{p,0}\rangle\}$ is $(1/(2(\alpha\vee 1))-\varepsilon)$-Hölder right continuous at $t=0$. Moreover, in the case of $1<\alpha<2$, if $\langle\mu, g_1\rangle<\infty$ with $g_1(x)=g_p(x)h_0(x)^{2-\alpha}$, then $\{\langle X_t, g_{p,0}\rangle\}$ is $(1/2-\varepsilon)$-Hölder right continuous at $t=0$.*

Corollary 3.1. *In the above discontinuous case, if $t_0>0$, then, under $\mathbf{P}_\mu$, $\{\langle X_t, g_0\rangle\}$ is $(1/2-\varepsilon)$-Hölder right continuous at $t=t_0$ for sufficiently small $\varepsilon>0$, where $\mu\in\mathcal{M}_{g_0}$.*

Proof of Theorem 3.1.

It suffices to check that the conditions in Assumptions 2.1 and 2.2 are fulfilled with suitable $g_1\in C^\infty$ and $0\le\gamma<1$;

(i) (a) $g_1(x)=g_p(x)$, $\gamma=0$. (b) $g_1(x)=g_p(x)$, $\gamma=1/2$.

(ii) (a) $g_1(x)=g_p(x)$, $\gamma=0$. (b) Let h_0 be given as in (ii) of Example 3.1. Let $h_1\in C^\infty; 0<h_1\le 1$, $h_1(v)=v\log(1/v)$ for $v\in(0,1/e]$ and $h_1(v)=1$ for $v\ge 1$. If $0<\alpha<1$, then $g_1(x)=g_{p,0}(x)$, $\gamma=0$. If $\alpha=1$, then $g_1(x)=g_p(x)h_1(x)$, $\gamma=\delta$ for any small $0<\delta<1$. If $1<\alpha<2$, then $g_1(x)=g_p(x)h_0(x)^{2-\alpha}$, $\gamma=1-1/\alpha$.

It is well known that the following in Prop. 2.3 of [5] holds: Let (T_t) (resp., (T_t^α)) be a transition semigroup of the Brownian motion on $\mathbf{R}^d$ (resp., of the α-stable motion on $\mathbf{R}^d$). For $f\in C_0(\mathbf{R}^d)$, if there exists a constant $L\in\mathbf{R}$ such that $\lim_{|x|\to\infty}|x|^pf(x)=L$, then

$$\lim_{|x|\to\infty} \sup_{t\geq 0} |x|^p T_t f(x) = L \qquad \text{if } p > d$$
$$\lim_{|x|\to\infty} \sup_{t\geq 0} |x|^p T_t^\alpha f(x) = L \qquad \text{if } d < p < d+\alpha.$$

(i) *Continuous case.*

(a) By $C_p^2 \subset \mathcal{D}(A)$ and $AC_p^2 \subset C_p$, and by the above result, Assumption 1 follows with $D_{g_0} = C_p^2$. Furthermore, the fact that C_p^2 is stable under multiplication (i.e., if $f \in C_p^2$, then $f^2 \in C_p^2$) yields Assumption 2 with $g_1 = g_0 = g_p$, $D_g = C_p^2$ and $\gamma = 0$.

(b) It is essentially proved in [3].

(ii) *Discontinuous case.* Let $0 < \alpha < 2$ and $d < p < d + \alpha$.

(a) It is easy to see that if $f \in C_c^\infty$, then $\partial_i T_t f = T_t(\partial_i f)$. Therefore, we have $T_t^\alpha C_c^\infty \subset C_p^2$ for every $t \geq 0$. Moreover, it is well known that $C_p^2 \subset C_0^2 \subset \mathcal{D}(A^\alpha)$. Thus, $D_g = C_p^2$ is a core. It suffices only to show that $A^\alpha C_p^2 \subset C_p$. $A^\alpha f \in C(\mathbf{R}^d)$ is clear by Lebesgue's convergence theorem. We prove that, for $f \in C_p^2$, $\|g_p^{-1} A^\alpha f\|_\infty < \infty$, i.e., $|A^\alpha f(x)| \leq C|x|^{-p}$ for sufficiently large $|x|$. For simplicity of the notation, we omit the superscript "α" as $A^\alpha = A$. In the following, we fix $x; |x| \geq 2$ and use the same symbol C as any finite x-independent constant. Let $f^{(2)} = (\partial_{ij}^2 f)$ and $|f^{(2)}(x)| = \max_{i,j} |\partial_{ij}^2 f(x)|$. Note that $|f^{(2)}(x)| \leq C|x|^{-p}$. Therefore, if $|y| \leq 1$, then

$$|f(x+y) - f(x) - \nabla f(x)\cdot y| = |f^{(2)}(x+\theta y)||y|^2/2 \leq C|x+\theta y|^{-p}|y|^2 \leq C|x|^{-p},$$

where $\theta = \theta(x,y) \in (0,1)$ (note that $|x+\theta y| \geq |x| - |y| \geq |x| - 1 \geq |x|/2$). Thus,

$$\begin{aligned}
|Af(x)| &= c\int_{\mathbf{R}^d\setminus\{0\}} [|f(x+y) - f(x) - \nabla f(x)\cdot y I(|y|<1)|] \frac{dy}{|y|^{d+\alpha}} \\
&\leq c\int_{|y|<1} |f(x+y) - f(x) - \nabla f(x)\cdot y| \frac{dy}{|y|^{d+\alpha}} \\
&\quad + c\int_{|z-x|\geq 1} |f(z)| \frac{dz}{|z-x|^{d+\alpha}} + c|f(x)| \int_{|y|\geq 1} \frac{dy}{|y|^{d+\alpha}} \\
&\leq C|x|^{-p} \int_{|y|<1} |y|^{-d-\alpha+2} dy + c\int_{|z-x|\geq 1} |f(z)| \frac{dz}{|z-x|^{d+\alpha}} + C|x|^{-p} \\
&\leq C|x|^{-p} + c\int_{|z-x|\geq 1} |f(z)| \frac{dz}{|z-x|^{d+\alpha}}.
\end{aligned}$$

For the second term, we divide the integral area into $\{|z-x| \geq 1\} = \{|z-x| > \delta|x|\} \cup \{1 \leq |z-x| \leq \delta|x|\}$ with $0 < \delta < 1/2$ and denote each integral by $I_1(x)$, $I_2(x)$, respectively. We have

$$I_1(x) \leq |x|^{-d-\alpha}\delta^{-d-\alpha} \int_{\mathbf{R}^d} |f(z)| dz = C|x|^{-d-\alpha} \leq C|x|^{-p}$$

and

$$I_2(x) = |x|^{-p} \int_{1\leq|z-x|\leq\delta|x|} \left|\frac{x}{z}\right|^p (|z|^p|f(z)|) \frac{dz}{|z-x|^{d+\alpha}} \leq C|x|^{-p} \int_{|y|\geq 1} \frac{dy}{|y|^{d+\alpha}} = C|x|^{-p},$$

because $|z| \geq (1-\delta)|x|$ if $|z-x| \leq \delta|x|$. Therefore, if $f \in C_p^2$, then $|A^\alpha f(x)| \leq C|x|^{-p}$ for $|x| \geq 2$ with some constant $C = C(f)$.

(b) Let $(T_t^{-,\alpha})$ be a transition semigroup of the absorbing α-stable motion $(w^{-,\alpha}(t), P_x^{-,\alpha})$ on $(0,\infty)$. Note that, in this case, $D_g = C_{p,0}^3$ is not stable under multiplication. For simplicity of the notation, we omit the superscript "α" as $T_t^{-,\alpha} = T_t^-$, $A^{-,\alpha} = A^-$, $T_t^\alpha = T_t$, $A^\alpha = A$, and so on. We will show the following:

(B1) $C_{p,0}^3 \subset \mathcal{D}(A^-)$, $T_t^- C_c^\infty \subset C_{p,0}^3$ for every $t \ge 0$, $A^- C_{p,0}^3 \subset C_{p,0}$ and $\sup_{t\ge 0, 0<x\le 1} |x^{-1} T_t^- g_{p,0}(x)| < \infty$ (this yields Assumption 2.1, and $C_{p,0}^3$ is a core).

(B2) For every $f \in C_{p,0}^3$, $\partial_t T_t^- f^2(x) = A^- T_t^- f^2(x) = T_t^- A^- f^2(x)$ $(x > 0)$, $A^- f^2 \in C_b$ and $\|g_1^{-1} Q^- f\|_\infty < \infty$ (this yields (i) of Assumption 2.2).

(B3) For each $0 < \beta \le 1$, $\sup_{t\ge 0} T_t^-(y^\beta)(x) \le 2(1+\beta)x^\beta$ for all $x > 0$ (this yields (ii) of Assumption 2.2).

(B4) For each $0 < \beta \le 1$, $\sup_{0<x\le 1} x^{-1} T_t^{-,\alpha}(y^\beta)(x) \le C_\beta\, t^{-(1-\beta)/\alpha}$ with a constant $C_\beta > 0$ depending only on β (this yields (iii) and (iv) of Assumption 2.2).

Note that we take $\gamma = (1-\beta)/\alpha$ in Assumption 2.2. More exactly, if $0 < \alpha < 1$, then we take $\beta = 1$, i.e., $\gamma = 0$. If $\alpha = 1$, then $\beta = 1 - \delta$ for any small $0 < \delta < 1$, i.e., $\gamma = \delta$. If $1 < \alpha < 2$, then $\beta = 2 - \alpha$, i.e., $\gamma = 1 - 1/\alpha$.

Let $p^\alpha(x)$ be the density of the α-stable motion on $\mathbf{R}$ starting from 0. It is well known that $p_t^\alpha(x)$ satisfies the relations $p_t^\alpha(x) = t^{-1/\alpha} p_1^\alpha(t^{-1/\alpha} x)$ (the scaling property) and $p_1^\alpha(x) \le C(1 \wedge |x|^{-1-\alpha})$. The transition density $p_t^-(x,y) \equiv p_t^{-,\alpha}(x,y)$ of the absorbing α-stable motion on $(0,\infty)$ is given as

$$p_t^-(x,y) = p_t^\alpha(y-x) - p_t^\alpha(y+x) = -\int_{-x}^x \partial_v p_t^\alpha(y+v)dv.$$

Hence, by using the integration by parts, we have, for $0 < \beta \le 1$,

$$\begin{aligned} T_t^-(y^\beta)(x) &= \int_0^\infty y^\beta p_t^-(x,y)dy = \left(\int_0^x + \int_x^\infty\right) y^\beta p_t^-(x,y)dy \\ &\le x^\beta \int_0^x p_t^-(x,y)dy - \int_x^\infty dy \int_{-x}^x y^\beta \partial_v p_t^\alpha(y+v)dv \\ &\le x^\beta + \int_{-x}^x dv \left(x^\beta p_t^\alpha(x+v) + \beta \int_x^\infty y^{\beta-1} p_t^\alpha(y+v)dy \right) \\ &\le 2x^\beta + \beta x^{\beta-1} \int_{-x}^x dv \int_x^\infty p_t^\alpha(y+v)dy \\ &\le 2(1+\beta)x^\beta. \end{aligned}$$

Thus, we have (B3) and, hence, the last claim of (B1). Moreover,

$$\begin{aligned} T_t^-(y^\beta)(x) &= -\int_{-x}^x dv \int_0^\infty y^\beta \partial_v p_t^\alpha(y+v)dy \\ &= \beta \int_{-x}^x dv \int_0^\infty y^{\beta-1} p_t^\alpha(y+v)dy \\ &= \beta \int_{-x}^x dv \int_0^\infty y^{\beta-1} t^{-1/\alpha} p_1^\alpha(t^{-1/\alpha}(y+v))dy \\ &= \beta t^{(\beta-1)/\alpha} \int_{-x}^x dv \int_0^\infty z^{\beta-1} p_1^\alpha(z + t^{-1/\alpha} v)dz \\ &\le C t^{-(1-\beta)/\alpha} x. \end{aligned}$$

It is easy to see that

$$\int_0^\infty z^{\beta-1} p_1^\alpha(z+u)dz \quad \text{is bounded in } u \in \mathbf{R}.$$

Therefore, we have (B4).

If $f \in C_{p,0}^3$, then $T_t^- f = T_t \overline{f}$ and $\overline{f} \in C_p^2(\mathbf{R}^d) \subset \mathcal{D}(A)$. Hence, we have $A^- f = A\overline{f}$ and $C_{p,0}^3 \subset \mathcal{D}(A^-)$. Moreover, $(T_t^- f)^{(k)} = T_t(\overline{f}^{(k)})$ for $f \in C_c^\infty$ $(k \ge 0)$ yields $T_t^- C_c^\infty \subset C_{p,0}^3$

by the last claim of (B1). In order to show that $A^- C^3_{p,0} \subset C_{p,0}$, it suffices to prove that

$$A^- f(x) = O(x) \quad \text{as } x \downarrow 0 \text{ for } f \in C^3_{p,0}. \tag{3.1}$$

Let $0 < x \le 1$.

$$\begin{aligned} A^- f(x) &= c \int_{\mathbf{R}\setminus\{0\}} \left[\overline{f}(y+x) - \overline{f}(x) - \nabla \overline{f}(x) y I(|y|<1)\right] \frac{dy}{|y|^{1+\alpha}} \\ &= c \int_{-x}^{x} [f(y+x) - f(x) - f'(x)y] \frac{dy}{|y|^{1+\alpha}} \\ &\quad + c \int_x^\infty [f(y+x) - f(y-x) - 2f(x)] \frac{dy}{y^{1+\alpha}}. \end{aligned} \tag{3.2}$$

We denote the first term on the right-hand side as $J_1(x)$. For the second term, we divide the integral into $\displaystyle\int_1^\infty + \int_x^1 dy$ and denote the corresponding terms as $J_2(x)$ and $J_3(x)$, respectively. In the following, we use the same symbol C as any finite constant, independent of $0 < x \le 1$. It is easy to see that, by $|f''(x)| \le Cx$,

$$\begin{aligned} |J_1(x)| &\le c \int_0^1 dt(1-t) \int_{-x}^{x} |f''(x+ty)| y^2 \frac{dy}{|y|^{1+\alpha}} \\ &\le C \int_{-x}^{x} (x+|y|) y^2 \frac{dy}{|y|^{1+\alpha}} \\ &= Cx^{3-\alpha} \quad (= o(x) \quad \text{as } x \downarrow 0) \end{aligned}$$

and, by $f(y+x) - f(y-x) = 2f'(y - x + \theta(y+x))x$ for some $\theta \in (0,1)$,

$$\begin{aligned} |J_2(x)| &\le c \int_1^\infty [|f(y+x) - f(y-x)| + 2|f(x)|] \frac{dy}{|y|^{1+\alpha}} \\ &\le Cx\|f'\|_\infty \\ &\le Cx. \end{aligned}$$

By using Taylor's formula for f at $0+$ and $f(0+) = 0$, we see that

$$\begin{aligned} & f(y+x) - f(y-x) - 2f(x) \\ &= \int_0^1 dt(1-t) \left[f''(t(y+x))(y+x)^2 - f''(t(y-x))(y-x)^2 - 2f''(tx)x^2\right] \\ &= \int_0^1 dt(1-t) \Big[\{f''(t(y+x)) - f''(t(y-x))\}(y^2+x^2) \\ &\qquad\qquad + 2\{f''(t(y+x)) + f''(t(y-x))\} xy - 2f''(tx)x^2\Big]. \\ &= \int_0^1 dt(1-t) \Big[\{f''(t(y+x)) - f''(t(y-x))\}(y^2+x^2) \\ &\qquad\qquad + 2\{f''(t(y+x)) + f''(t(y-x))\} xy - 2f''(tx)x^2\Big]. \\ &= \int_0^1 dt(1-t) \Big[2f^{(3)}(t(y-x+2\theta x))x(y^2+x^2) \\ &\qquad\qquad + 2\{f''(t(y+x)) + f''(t(y-x))\} xy - 2f''(tx)x^2\Big] \end{aligned}$$

with some $\theta \in (0,1)$. Thus, by $|f''(x)| \le Cx$,

$$
\begin{aligned}
|J_3(x)| &\le c\int_x^1 \left[2\|f^{(3)}\|_\infty x(y^2+x^2) + C(y+x)xy + 2Cx^3\right]\frac{dy}{y^{1+\alpha}} \\
&\le Cx\int_x^1 (y^2+xy+x^2)\frac{dy}{y^{1+\alpha}} \\
&\le 3Cx\int_0^1 y^2\frac{dy}{y^{1+\alpha}} = 3Cx\int_0^1 y^{1-\alpha}dy = \frac{3C}{2-\alpha}x.
\end{aligned}
$$

Therefore, we have (3.1). Next, we prove (B2). Fix $f \in C^3_{p,0}$ and denote $h \equiv h_f = f^2$. Note that if $f'(0+) \neq 0$, then $h''(0+) \neq 0$, i.e., $h \notin C^2_{p,0}$ ($\overline{h} \notin C^2(\mathbf{R})$). However, there exists a sequence $\{h_n\}_{n\ge 1} \subset C^3_{p,0}$ such that $h_n, h_n' \to h, h'$ uniform, respectively, and $h_n'' \to h''$ locally uniform and uniformly bounded as $n \to \infty$. Note that it holds that $\overline{h}_n, \overline{h}_n' \to \overline{h}, \overline{h}'$ as $n \to \infty$ uniformly on $\mathbf{R}$. By the formula of $A^- h_n$ (see (3.2)), it is possible to extend A^- for h, that is, there is a function $g \in C$ such that $A^- h_n \to g$ locally uniform and uniformly bounded, thus $g = A^- h$. Moreover, $T_t^- h_n = T_t\overline{h}_n \to T_t\overline{h} = T_t^- h$, $(T_t^- h_n)' = T_t(\overline{h}_n)' \to T_t(\overline{h})' = (T_t^- h)'$ uniformly as $n \to \infty$, respectively. Hence, it holds that $A^- T_t^- h_n \to A^- T_t^- h$ and $T_t^- A^- h_n \to T_t^- A^- h$. Thus, we have the first claim. To show $Q^- f(x) \le Cg_1(x)$ for all $x > 0$, it is enough to consider it for $0 < x \le 1$ (because even if $f^2 \notin C^2_{p,0}$, but $f \in C^2_{p,0}$, then it holds that $|A^- f^2(x)| \le C(1 \wedge x^{-p})$ for all $x > 0$ as in the case of A^α). Let $0 < x \le 1$. $Q^- = Q^{-,\alpha}$ is also expressed as

$$
\begin{aligned}
Q^- f(x) &= c\int_{-x}^x [f(y+x) - f(x)]^2\frac{dy}{|y|^{1+\alpha}} \\
&+ \ c\int_x^\infty \left[\{f(y+x) - f(y-x)\}\{f(y+x) + f(y-x) - 2f(x)\} + 2f(x)^2\right]\frac{dy}{y^{1+\alpha}}.
\end{aligned}
$$

The first term on the right-hand side is bounded by

$$
c\|f'\|_\infty \int_{-x}^x |y|^{1-\alpha}dy \le Cx^{2-\alpha}.
$$

For the second term, we divide the integral into $\displaystyle\int_1^\infty + \int_x^1 dy$. Since $|f(y+x) - f(y-x)| \le 2x\|f'\|_\infty$ and $f(x)^2 \le C_1x^2$, the first integral is bounded by Cx. Furthermore, by using $|f(y+x) + f(y-x) - 2f(x)| \le C_2(y+x)$, we have that the second integral is bounded by

$$
\int_x^1 \left[2x\|f'\|_\infty C_2(y+x) + 2C_1x^2\right]\frac{dy}{y^{1+\alpha}} \le \begin{cases} C(x \vee x^{2-\alpha}) & (\alpha \neq 1), \\ Cx\log 1/x & (\alpha = 1). \end{cases}
$$

Thus, we get $Q^- f(x) \le Cg_1(x)$ for $0 < x \le 1$; so, for all $x > 0$. Therefore, (B2) is proved. □

4. Proofs of Theorem 2.4

We always assume that Assumptions 2.1 and 2.2 are satisfied, and A is given as in (2.1).

Proof of Theorem 2.4.

In this independent case, the semimartingale representation will be almost evident. However, we shall show the representation only by using the properties of Theorem 4.1 described as below. The following proposition is needed to prove the uniqueness of the solution to the martingale problem in §5.

Lemma 4.1. *For each $f \in C_c^\infty$ and $T > 0$, $\displaystyle\sup_{t\in[0,T]} \|g_0^{-1}\partial_t V_t f\|_\infty < \infty$.*

Proof. Since $\|V_t f\|_\infty \le \|f\|_\infty$ and $|A(1-e^{-f})| \le C g_0$ by (ii) of Assumption 2.1, we have

$$|\partial_t V_t f| = |e^{V_t f} T_t A(1-e^{-f})| \le C e^{\|f\|_\infty} T_t g_0.$$

Hence, the claim follows. □

Theorem 4.1. *For* $f \in C_c^\infty$,

$$e^{-\langle X_t, f\rangle} - e^{-\langle X_0, f\rangle} - \int_0^t \mathcal{L}_0 e^{-\langle \cdot, f\rangle}(X_s) ds$$

is a $\mathbf{P}_\mu$*-martingale. Moreover,*

$$H_t(f) = \exp\left[-\langle X_t, f\rangle + \int_0^t \langle X_s, Af - \Gamma f\rangle ds\right]$$

is also a $\mathbf{P}_\mu$*-martingale.*

Proof. By the above lemma, we see that if $s < t$, then

$$\begin{aligned}
\partial_t \mathbf{E}_\mu \left[e^{-\langle X_t, f\rangle} \middle| \mathcal{F}_s \right] &= \partial_t e^{-\langle X_s, V_{t-s} f\rangle} \\
&= \partial_{u=0+} e^{-\langle X_s, V_{t-s+u} f\rangle} \\
&= \partial_{u=0+} \mathbf{E}_\mu \left[e^{-\langle X_t, V_u f\rangle} \middle| \mathcal{F}_s \right] \\
&= \mathbf{E}_\mu \left[\partial_{u=0+} e^{-\langle X_t, V_u f\rangle} \middle| \mathcal{F}_s \right] \\
&= \mathbf{E}_\mu \left[\mathcal{L}_0 e^{-\langle \cdot, f\rangle}(X_t) \middle| \mathcal{F}_s \right],
\end{aligned}$$

where $\partial_{u=0+}$ denotes the right differential operator at $u = 0$. Hence, the first claim follows. The second claim follows from Cor. 3.3 of Chap. 2 in [2]. □

We proceed to the proof of Theorem 2.4. Let, for $f \in C_c^\infty$,

$$G_t(f) = \exp\left[-\int_0^t \langle X_s, Af - \Gamma f\rangle ds\right]$$

be a continuous process of bounded variation. Since $H_t(f)$ is a martingale,

$$Z_t(f) = \exp[-\langle X_t, f\rangle] = H_t(f) G_t(f)$$

is a semimartingale, more exactly, a special semimartingale, i.e., a bounded variation part is (locally) integrable. In fact, by Prop. 3.2 of Chap. 2 in [2], we have

$$\begin{aligned}
dZ_t(f) &= H_t(f) dG_t(f) + G_t(f) dH_t(f) \\
&= -\langle X_t, Af - \Gamma f\rangle Z_t(f) dt + d(\text{martingale}).
\end{aligned} \tag{4.1}$$

On the other hand, $\langle X_t, f\rangle$ is also a special semimartingale. Hence, by (1.10) of Chap. 4 in [6], $\langle X_t, f\rangle$ has the following expression:

$$\langle X_t, f\rangle = \langle X_0, f\rangle + C_t(f) + M_t^c(f) + \widetilde{N}_t(f) + N_t(f),$$

where $C_t(f)$ is a continuous process of locally bounded variation, $M_t^c(f)$ is a continuous L^2-martingale with quadratic variation $[M^c(f)]_t$, and

$$\begin{aligned}
\widetilde{N}_t(f) &= \int_0^t \int_{\mathcal{M}^\pm} \langle \mu, f\rangle I(\|\mu\| < 1) \widetilde{N}(ds, d\mu), \\
N_t(f) &= \int_0^t \int_{\mathcal{M}^\pm} \langle \mu, f\rangle I(\|\mu\| \ge 1) N(ds, d\mu)
\end{aligned}$$

with the jump measure N of $\{X_t\}$, its compensator $\widehat{N}$, and $\widetilde{N} = N - \widehat{N}$. By using Itô's formula, we have

$$
\begin{aligned}
\text{(4.2)} \qquad dZ_t(f) &= Z_{t-}(f)\Bigg\{-dC_t(f) + \frac{1}{2}d[M^c(f)]_t \\
&\quad + \int_{\mathcal{M}^\pm} \left[e^{-\langle \mu, f\rangle} - 1 + \langle \mu, f\rangle\right] I(\|\mu\| < 1)\widehat{N}(dt, d\mu) \\
&\quad + \int_{\mathcal{M}^\pm} \left[e^{-\langle \mu, f\rangle} - 1\right] I(\|\mu\| \geq 1)N(dt, d\mu)\Bigg\} + d(\text{martingale}) \\
&= Z_{t-}(f)\Bigg\{-\left(dC_t(f) + \int_{\{\|\mu\|\geq 1\}} \langle \mu, f\rangle \widehat{N}(dt, d\mu)\right) + \frac{1}{2}d[M^c(f)]_t \\
&\quad + \int_{\mathcal{M}^\pm} \left[e^{-\langle \mu, f\rangle} - 1 + \langle \mu, f\rangle\right] \widehat{N}(dt, d\mu)\Bigg\} + d(\text{martingale}).
\end{aligned}
$$

If we set

$$
B_t(f) = C_t(f) + \int_0^t \int_{\{\|\mu\|\geq 1\}} \langle \mu, f\rangle \widehat{N}(ds, d\mu),
$$

then by expressions (4.1), (4.2) and by the uniqueness of the special semimartingale with predictable locally bounded part (see Theorem 2.1.1 in [6]), we have

$$
\begin{aligned}
&-dB_t(f) + \frac{1}{2}d[M^c(f)]_t + \int \left[e^{-\langle \mu, f\rangle} - 1 + \langle \mu, f\rangle\right] \widehat{N}(dt, d\mu) \\
&\quad = \quad -\langle X_t, Af - \Gamma f\rangle dt \\
&\quad = \quad \left[-\langle X_t, Af\rangle + \langle X_t, \Gamma^c f\rangle + \langle X_t, \Gamma^d f\rangle\right] dt
\end{aligned}
$$

Hence, it is easy to see that

$$
\begin{aligned}
B_t(f) &= \int_0^t \langle X_s, Af\rangle ds, \\
[M^c(f)]_t &= 2\int_0^t \langle X_s, \Gamma^c f\rangle ds = \int_0^t \langle X_s, Q^c f\rangle ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \int \left[e^{-\langle \mu, f\rangle} - 1 + \langle \mu, f\rangle\right] \widehat{N}(ds, d\mu) \\
&\quad = \int_0^t \langle X_s, \Gamma^d f\rangle ds \\
&\quad = \int_0^t ds \int X_s(dx)\Bigg\{\int \left(e^{-[f(y)-f(x)]} - 1 + [f(y) - f(x)]\right) \nu(x, dy) \\
&\qquad + k(x)\left(e^{f(x)} - 1 - f(x)\right)\Bigg\}.
\end{aligned}
$$

Therefore, we have

$$
\widehat{N}(ds, d\mu) = ds \int X_s(dx) \left(\int \nu(x, dy)\delta_{(\delta_y - \delta_x)} + k(x)\delta_{-\delta_x}\right)(d\mu).
$$

Finally, it is possible to extend $f \in C_c^\infty$ to $f \in D_g$. The proof of Theorem 2.4 is completed.

□

5. Martingale Problem for $\mathcal{L}_0$

The following assumption is needed to prove the well-posedness of the martingale problem.

Assumption 5.1. For each $f \in (C_c^\infty)^+$, $t > 0$, $AV_t f = -A\log(1 - T_t(1 - e^{-f}))$ is well-defined, and $AV_t f$ is continuous in t under the norm $\|\cdot/g_1\|_\infty$, i.e.,

$$\|(AV_t f - AV_{t_0} f)/g_1\|_\infty \to 0 \quad (t \to t_0).$$

In the following, we suppose that the generator A of the motion process has the form of (2.1).

For $\eta \in \mathcal{M}_{g_0}$, let $F(\eta) = \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle) \in \mathcal{D}_0 \overset{\text{def}}{\Longleftrightarrow} \Phi(x) \in C^\infty(\mathbf{R}^n)$ be a polynomial growth function with polynomial growth derivatives of all orders and $f_i \in D_g$, $i = 1, \dots, n$. For this $F(\eta)$, the generator $\mathcal{L}_0$ of X_t will be extended to the following form:

$$\begin{aligned}
\mathcal{L}_0 F(\eta) \;=\;& \sum_{i=1}^n \partial_i \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle)\langle \eta, A f_i\rangle \\
&+ \frac{1}{2}\sum_{i,j=1}^n \partial_{ij}^2 \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle)\langle \eta, Q^c(f_i, f_j)\rangle \\
&+ \int_S \Bigg\{ \int_{S\setminus\{x\}} \nu(x, dy)\Bigg[\Phi(\langle \eta, f_1\rangle + f_1(y) - f_1(x), \dots, \langle \eta, f_n\rangle + f_n(y) - f_n(x)) \\
&\qquad -\Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle) - \sum_{i=1}^n \partial_i \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle)(f_i(y) - f_i(x))\Bigg] \\
&\qquad +k(x)\Bigg[\Phi(\langle \eta, f_1\rangle - f_1(x), \dots, \langle \eta, f_n\rangle - f_n(x)) - \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle) \\
&\qquad +\sum_{i=1}^n \partial_i \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle) f_i(x)\Bigg]\Bigg\}\eta(dx),
\end{aligned}$$

where

$$Q^c(f, g)(x) = \sum_{i,j} a^{ij}(x)\partial_i f(x)\partial_i g(x).$$

Theorem 5.1 (Martingale Problem for $(\mathcal{L}_0, \mathcal{D}_0, \mu)$)**.** *Under Assumptions 2.1, 2.2, and 5.1, we suppose that the generator A is given as in* (2.1). *Let $\mu \in \mathcal{M}_{g_0}$.*

(i) $\mathbf{P}_\mu(X_0 = \mu) = 1$ *and for* $F(\eta) = \Phi(\langle \eta, f_1\rangle, \dots, \langle \eta, f_n\rangle) \in \mathcal{D}_0$,

$$M_t^F = F(X_t) - F(X_0) - \int_0^t \mathcal{L}_0 F(X_s)ds \quad \textit{is a } \mathbf{P}_\mu\textit{-martingale},$$

(ii) *If there is a probability measure $\mathbf{Q}_\mu$ on $\mathbf{D} = \mathbf{D}([0,\infty) \to \mathcal{M}_{g_0})$ such that the canonical process $\widetilde{X}_t(\omega) = \omega(t)$ $(\omega \in \mathbf{D})$ satisfies the same conditions as $(X_t, \mathbf{P}_\mu)$ in* (i) *and*

$$\int_0^t \langle \widetilde{X}_s, g_1\rangle ds < \infty \quad \mathbf{Q}_\mu\textit{-a.s. for all } t \ge 0,$$

then $\mathbf{Q}_\mu = \mathbf{P}_\mu \circ X^{-1}$ on $\mathbf{D}$, that is, the martingale problem for $(\mathcal{L}_0, \mathcal{D}_0, \mu)$ on $\mathbf{D}$ is well-posed.

Proof. (i) is easily obtained, so we prove (ii). We always fix any $f \in (C_c^\infty)^+$ and $T > 0$. To prove the uniqueness of the martingale problem, it is enough to show that $\exp(-\langle \widetilde{X}_t, V_{T-t} f\rangle)$, $0 \le t \le T$, is a $\mathbf{Q}_\mu$-martingale (because this implies the uniqueness in the sense of finite dimensional distributions, and the separability of $\mathcal{M}_{g_0}$ yields the uniqueness in the sense of distributions on $\mathbf{D}$). By Lemma 4.1, we have

$$\partial_t V_t f \quad \text{is continuous in } t \text{ under the norm } \|\cdot\|_{g_0} = \|\cdot/g_0\|_\infty.$$

Moreover, by Assumption 5.1, we see that

$$\Gamma V_t f \in C_b \quad \text{is continuous in } t \text{ under the norm } \|\cdot/g_1\|_\infty$$

and $v_t = v_t^T = V_{T-t}f$ $(0 \le t \le T)$ is the unique solution to the equation;

$$(\partial_t + A - \Gamma)v_t = 0 \quad \text{and} \quad v_T = f.$$

Let $\Phi(v) = e^{-v}$. It is not difficult to check that $(\widetilde{X}_t, \mathbf{Q}_\mu)$ has the same semimartingale representation as $(X_t, \mathbf{P}_\mu)$ in Theorem 2.4. Hence, by using the above results and Itô's formula, we can show that the following quantity is a $\mathbf{Q}_\mu$-martingale:

$$\begin{aligned}
&\Phi(\langle \widetilde{X}_t, v_t\rangle) - \Phi(\langle \widetilde{X}_0, v_0\rangle) - \int_0^t \Phi'(\langle \widetilde{X}_s, v_s\rangle)\langle \widetilde{X}_s, \partial_s v_s + Av_s\rangle ds \\
&\qquad - \int_0^t \Phi''(\langle \widetilde{X}_s, v_s\rangle)\langle \widetilde{X}_s, \Gamma^c v_s\rangle ds \\
&\qquad - \int_0^t \int_{\mathcal{M}_\pm} \Big[\Phi(\langle \widetilde{X}_s + \eta, v_s\rangle) - \Phi(\langle \widetilde{X}_s, v_s\rangle) - \Phi'(\langle \widetilde{X}_s, v_s\rangle)\langle \eta, v_s\rangle\Big] \widehat{N}(ds d\eta) \\
= {} & \exp[-\langle \widetilde{X}_t, v_t\rangle] - \exp[-\langle \widetilde{X}_0, v_0\rangle] + \int_0^t \langle \widetilde{X}_s, \partial_s v_s + Av_s\rangle \exp[-\langle \widetilde{X}_s, v_s\rangle] ds \\
&\qquad - \int_0^t \langle \widetilde{X}_s, \Gamma^c v_s\rangle \exp[-\langle \widetilde{X}_s, v_s\rangle] ds - \int_0^t \langle \widetilde{X}_s, \Gamma^d v_s\rangle \exp[-\langle \widetilde{X}_s, v_s\rangle] ds \\
= {} & \exp[-\langle \widetilde{X}_t, v_t\rangle] - \exp[-\langle \widetilde{X}_0, v_0\rangle] + \int_0^t \langle \widetilde{X}_s, (\partial_s + A - \Gamma)v_s\rangle \exp[-\langle \widetilde{X}_s, v_s\rangle] ds \\
= {} & \exp[-\langle \widetilde{X}_t, V_{T-t}f\rangle] - \exp[-\langle \widetilde{X}_0, V_T f\rangle].
\end{aligned}$$

Therefore, we have the desired result. □

Corollary 5.1 (Martingale Problem for Examples)**.** *Let $\mu \in \mathcal{M}_{g_0}$. The martingale problems for $(\mathcal{L}_0, \mathcal{D}_0, \mu)$ with the motion processes of Example 3.1 are well-posed.*

Proof. It is enough to show that the conditions of Assumption 5.1 are fulfilled. Fix $f \in (C_c^\infty)^+$, and let $h = 1 - e^{-f}$ (then $h \in C_c^\infty$). It is shown in §3 that $T_t C_c^\infty \subset D_g$ and $\sup_t \|T_t g_0\|_{g_0} < \infty$. We shall show that $V_t f := -\log(1 - T_t h) \in D_g$, that is, $AV_t f$ is well-defined, and $AV_t f$ is continuous in t under the norm $\|\cdot\|_{g_0} = \|\cdot/g_0\|_\infty$ $(\ge \|\cdot/g_1\|_\infty)$ in each example. In the following, we use notations (1) T_t, (2) T_t^0, (3) T_t^α, (4) $T_t^{-,\alpha}$ for T_t, and similarly for V_t, A.

(1) Brownian motion on $\mathbf{R}^d$ ($g_0 = g_1 = g_p$ with $p > d$, and $D_g = C_p^2$).

For simplicity, we consider the case of $d = 1$. Since $(T_t h)' = T_t(h')$, we have

$$\begin{aligned}
(V_t f)' &= e^{V_t f}(T_t h)' = e^{V_t f} T_t(h') \\
(V_t f)'' &= e^{2V_t f}(T_t(h'))^2 + e^{V_t f} T_t(h'').
\end{aligned} \tag{5.1}$$

Hence, $V_t f \in C_p^2$, and it is easy to see that $AV_t f = (V_t f)''/2$ is continuous under $\|\cdot\|_{g_p}$.

(2) Absorbing Brownian motion on $(0, \infty)$ ($g_0 = g_{p,0}$ with $p > 1$, and $D_g = C_{p,0}^2$).

In this case, $T_t^0 h = T_t \overline{h}$, and the above yields the desired result.

(3) α-stable motion on $\mathbf{R}^d$ ($g_0 = g_1 = g_p$ with $d < p < d + \alpha$, and $D_g = C_p^2$).

As in (1), for simplicity, we consider the case $d = 1$ (the case $d \ge 2$ is essentially the same). We have (5.1) and, thus, $V_t^\alpha f \in C_p^2$. By $h \in C_c^\infty$, it is easy to see that

$(A^\alpha h)' = A^\alpha(h')$. For $\partial_t V_t^\alpha f = e^{V_t^\alpha f} T_t^\alpha A^\alpha h$, this result yields

$$\begin{aligned}(\partial_t V_t^\alpha f)' &= e^{V_t^\alpha f}\left((V_t^\alpha f)' T_t^\alpha A^\alpha h + T_t^\alpha A^\alpha(h')\right)\\ (\partial_t V_t^\alpha f)'' &= e^{V_t^\alpha f}\Big\{(V_t^\alpha f)'\left((V_t^\alpha f)' T_t^\alpha A^\alpha h + T_t^\alpha A^\alpha(h')\right)\\ &\quad + (V_t^\alpha f)'' T_t^\alpha A^\alpha h + (V_t^\alpha f)' T_t^\alpha A^\alpha(h') + T_t^\alpha A^\alpha(h'')\Big\}.\end{aligned}$$

Hence, by $|A^\alpha h|, |A^\alpha(h')|, |A^\alpha(h'')| \le C g_p$ with some constant $C > 0$, we have $\partial_t V_t^\alpha f \in C_p^2$. Furthermore, since $\sup_t \|T_t g_p\|_{g_p} < \infty$, we can show that $\sup_t \|A^\alpha(\partial_t V_t^\alpha f)\|_{g_p} < \infty$ by the same way as in the proof of $A^\alpha C_p^2 \subset C_p$ in §3. Thus, in this case, it easily follows that

$$\|A^\alpha(V_t^\alpha f - V_{t_0}^\alpha f)\|_{g_p} = \|\int_{t_0}^t A^\alpha(\partial_s V_s^\alpha f) ds\|_{g_p} \le |t - t_0| \sup_s \|A^\alpha(\partial_s V_s^\alpha f)\|_{g_p}.$$

Therefore, we have the continuity of $A^\alpha V_t^\alpha f$.

(4) Absorbing α-stable motion on $(0,\infty)$ ($g_0 = g_{p,0}$ with $1 < p < 1+\alpha$, and $D_g = C_{p,0}^3$).

Note that $T_t^{-,\alpha} h = T_t^\alpha \overline{h}$, $A^{-,\alpha} h = A^\alpha \overline{h}$ and $V_t^{-,\alpha} f = V_t^\alpha \overline{f}$. By computing $(V_t^{-,\alpha} f)'''$, we see that $V_t^{-,\alpha} f \in C_{p,0}^3$. Moreover, the continuity of $A^{-,\alpha} V_t^{-,\alpha} f$ follows in the same way as above. □

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