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SYSTEM OF INTERACTING PARTICLES WITH MARKOVIAN SWITCHING

Most of the published articles on random motions have been devoted to the study of the telegraph process or its generalizations that describe the random motion of a single particle in R^n in a Markov or semi-Markov medium. However, up to our best knowledge, there are no published papers dealing with the interaction of two or more particles which move according to the telegraph processes. In this paper, we construct the system of telegraph processes with interactions, which can be interpreted as a model of ideal gas. In this model, we investigate the free path times of a family of particles, before they are collided with any other particle. We also study the distribution of particles, which is described by telegraph processes with hard collisions and reflecting boundaries, and investigate its limiting properties.

1. Introduction

Let $\{\xi(t), \ t \geq 0\}$ be a Markov process on the phase space $\{0, 1\}$ with generator matrix

$$Q = \lambda \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Definition 1.1. $S(t)$ is the telegraph process if

$$\frac{d}{dt} S(t) = v(-1)^{\xi(t)}, \quad v = const > 0,$$

$$S(0) = y_0.$$

For a set of real numbers $y_1 < y_2 < \cdots < y_n$, we consider a family of independent telegraph processes $S_i(t), \ i = 1, 2, \ldots, n$ with $S_i(0) = y_i$. It is assumed that all the processes have the absolute velocity v and the parameter of the switching process $\lambda > 0$. Moreover, starting from y_i, the process $S_i(t)$ has equal probabilities of initial directions of the motion.

Denote, by $x(y_i, t)$, the position of the particle i at time t, which starts from site y_i. Suppose that the particle $x(y_i, t)$ develops as the telegraph process $S_i(t)$ up to the hard collision with another particle. Under the hard collision of two particles, we mean that, at the time of the collision, the particles change their direction to the opposite, i.e., the particles exchange the telegraph processes that describe their movement. It is easily verified that the positions of the particles $x(y_i, t)$, $i = 1, 2, \ldots, n$ at time t coincide with the order statistics of $S_i(t)$, $i = 1, 2, \ldots, n$ as follows:

$$x(y_1, t) = S_{(1)}(t), x(y_2, t) = S_{(2)}(t), \ldots, x(y_n, t) = S_{(n)}(t). \tag{1}$$

Remark 1.1. It should be noted that each $x(y_i, t)$, $i = 1, 2, \ldots, n$ is not a telegraph process for all $t \geq 0$.

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Remark 1.2. It follows from the description of $x(y_i, t)$ that $x(y_1, t) \le x(y_2, t) \le \cdots \le x(y_n, t)$ for any $t \ge 0$. Such kinds of the model for Wiener processes with coalescence after the collision are called the Arratia flow, and they were studied in [6]-[8].

Various problems such as the number of particle collisions up to time t in the Arratia flow are studied in [9].

Below, the explicit form for the distribution of the meeting instant of two telegraph processes on the line, which started at the same time from different positions on the line, is obtained. We also study the limiting distribution of the meeting instant of two telegraph processes on the line under Kac's condition. It allows us to investigate the system of telegraph processes with interactions, which can be interpreted as a model of ideal gas. In this model, we investigate the free path times of a family of particles before they collide with any other particle. We also study the distribution of particles, which is described by telegraph processes with hard collisions and reflecting boundaries, and investigate its limiting properties.

2. Distribution of the first collision of two telegraph particles

Consider two particles 1 and 2 on a line. Each particle can move in two opposite directions. Starting at $x_i \in R$, $i = 1, 2$, particle i moves with the velocity $v > 0$ in one of two directions during a random time interval that is exponentially distributed with parameter $\lambda > 0$. Then the particle changes its direction and so on. In the sequel, such particle is said to be a telegraph particle as its motion satisfies the telegraph equation [1], [2].

Let $\xi_1(t), \xi_2(t)$ be independent alternating Markov processes with the phase space $\{0, 1\}$ and with the generator matrix Q.

Denote, by $x_i(t)$, the position of particle i at a time moment $t \ge 0$ up to the first collision with another particle. It is easily seen that

$$\frac{d}{dt} x_i(t) = v(-1)^{\xi_i(t)},$$

$$x_i(0) = x_i.$$

We assume that $z = x_2 - x_1 > 0$ and put $\Delta(t) = x_2(t) - x_1(t)$.

Denote $\eta(t) = (\xi_1(t), \xi_2(t))$. Suppose $\eta(0) = (k_1, k_2)$ and define

$$\tau_{(k_1,k_2)}(z) = inf\{t \ge 0 : \ \Delta(t) = 0\}, \quad k_j \in \{0, 1\}.$$

Denote, by $f_{(k_1,k_2)}(t, z)\, dt = P\left(\tau_{(k_1,k_2)}(z) \in dt\right)$, the probability density function (pdf) of $\tau_{(k_1,k_2)}(z)$.

Lemma 2.1. *For $t \ge \frac{z}{2v}$,*

$$f_{(0,1)}(t, z) = e^{-2\lambda t}\delta(z - 2vt) + \frac{z\lambda}{2v^2} e^{-2\lambda t} \frac{I_1\left(\frac{\lambda}{v}\sqrt{4v^2t^2 - z^2}\right)}{\sqrt{4v^2t^2 - z^2}}, \tag{2}$$

$$f_{(0,0)}(t, z) = f_{(1,1)}(t, z) = \frac{z\lambda}{2v^2} e^{-2\lambda t} \int_{z/2v}^{t} \frac{I_1\left(\frac{\lambda}{v}\sqrt{4v^2u^2 - z^2}\right)}{\sqrt{4v^2u^2 - z^2}} \frac{I_1(2\lambda(t - u))}{(t - u)} du, \tag{3}$$

and

(4)
$$f_{(1,0)}(t, z) = \frac{z\lambda}{2v^2} e^{-2\lambda t} \int_{z/2v}^{t} \frac{I_1\left(\frac{\lambda}{v}\sqrt{4v^2u^2 - z^2}\right)}{\sqrt{4v^2u^2 - z^2}} \int_0^{t-u} \frac{I_1(2\lambda(t - u - v))}{(t - u - v)} \frac{I_1(2\lambda v)}{v} dv du.$$

Proof. Let us consider the Laplace transforms of $\tau_{(k_1,k_2)}(z)$, $k_i \in \{0,1\}$.

$$\varphi_{(k_1,k_2)}(s,z) = E\left[e^{-s\tau_{(k_1,k_2)}(z)}\right], \quad s>0.$$

By using the renewal theory, we can obtain the following system of integral equations for these Laplace transforms:

$$\begin{aligned}
\varphi_{(0,1)}(s,z) =& e^{-\frac{s+2\lambda}{2v}z} + \frac{\lambda}{2v}\int_0^z e^{-\frac{s+2\lambda}{2v}u}\varphi_{(1,1)}(s,z-u)\,du \\
&+\frac{\lambda}{2v}\int_0^z e^{-\frac{s+2\lambda}{2v}u}\varphi_{(0,0)}(s,z-u)\,du \\
=& e^{-\frac{s+2\lambda}{2v}z} + \frac{\lambda}{2v}e^{-\frac{s+2\lambda}{2v}z}\int_0^z e^{\frac{s+2\lambda}{2v}u}\left(\varphi_{(1,1)}(s,u)+\varphi_{(0,0)}(s,u)\right)du
\end{aligned}$$

$$\begin{aligned}
\varphi_{(0,0)}(s,z) =& \lambda\int_0^\infty e^{-(s+2\lambda)u}\varphi_{(0,1)}(s,z)\,du \ + \lambda\int_0^\infty e^{-(s+2\lambda)u}\varphi_{(1,0)}(s,z)\,du \\
=& \frac{\lambda}{s+2\lambda}\left(\varphi_{(0,1)}(s,z)+\varphi_{(1,0)}(s,z)\right),
\end{aligned}$$

$$\begin{aligned}
\varphi_{(1,1)}(s,z) =& \lambda\int_0^\infty e^{-(s+2\lambda)u}\varphi_{(0,1)}(s,z)\,du \ + \lambda\int_0^\infty e^{-(s+2\lambda)u}\varphi_{(1,0)}(s,z)\,du \\
=& \frac{\lambda}{s+2\lambda}\left(\varphi_{(0,1)}(s,z)+\varphi_{(1,0)}(s,z)\right),
\end{aligned}$$

$$\begin{aligned}
\varphi_{(1,0)}(s,z) =& \frac{\lambda}{2v}\int_0^\infty e^{-\frac{s+2\lambda}{2v}u}\left(\varphi_{(0,0)}(s,z+u)+\varphi_{(1,1)}(s,z+u)\right)du \\
=& \frac{\lambda}{2v}e^{\frac{s+2\lambda}{2v}z}\int_z^\infty e^{-\frac{s+2\lambda}{2v}u}\left(\varphi_{(0,0)}(s,u)+\varphi_{(1,1)}(s,u)\right)du.
\end{aligned}$$

It is easily seen that

$$\varphi_{(0,0)}(s,z) = \varphi_{(1,1)}(s,z). \tag{5}$$

Taking into account that

$$\varphi_{(0,0)}(s,z)+\varphi_{(1,1)}(s,z) = \frac{2\lambda}{s+2\lambda}\left(\varphi_{(0,1)}(s,z)+\varphi_{(1,0)}(s,z)\right), \tag{6}$$

we have

$$\begin{aligned}
\frac{\partial}{\partial z}\varphi_{(0,1)}(s,z) &= -\frac{s+2\lambda}{2v}\varphi_{(0,1)}(s,z) + \frac{\lambda^2}{v(s+2\lambda)}\left(\varphi_{(0,1)}(s,z)+\varphi_{(1,0)}(s,z)\right), \\
\frac{\partial}{\partial z}\varphi_{(1,0)}(s,z) &= \frac{s+2\lambda}{2v}\varphi_{(1,0)}(s,z) - \frac{\lambda^2}{v(s+2\lambda)}\left(\varphi_{(0,1)}(s,z)+\varphi_{(1,0)}(s,z)\right).
\end{aligned}$$

It is well known [3] that $\varphi_{(1,0)}(s,z)$ and $\varphi_{(0,1)}(s,z)$ satisfy the equation

$$det\begin{pmatrix} \frac{\partial}{\partial z}+\frac{(s+2\lambda)}{2v}-\frac{\lambda^2}{v(s+2\lambda)} & -\frac{\lambda^2}{v(s+2\lambda)} \\ \frac{\lambda^2}{v(s+2\lambda)} & \frac{\partial}{\partial z}-\frac{(s+2\lambda)}{2v}+\frac{\lambda^2}{v(s+2\lambda)} \end{pmatrix} f(z) = 0.$$

By calculating the determinant, we get

$$\frac{\partial^2}{\partial z^2}f(z) - \frac{s^2+4\lambda s}{4v^2}f(z) = 0.$$

Solving this equation, we have

$$f(z) = C_1 e^{\sqrt{s^2+4\lambda s}\ \frac{z}{2v}} + C_2 e^{-\sqrt{s^2+4\lambda s}\ \frac{z}{2v}}.$$

The constants obtained from the system of integral equations yield

$$\varphi_{(0,1)}(s,z) = e^{-\frac{z}{2v}\sqrt{s^2+4\lambda s}} \tag{7}$$

and

$$\varphi_{(1,0)}(s,z) = \frac{s+2\lambda-\sqrt{s^2+4\lambda s}}{s+2\lambda+\sqrt{s^2+4\lambda s}} e^{-\frac{z}{2v}\sqrt{s^2+4\lambda s}}. \tag{8}$$

Taking Eqs.(5) and (6) into account, we have

$$\varphi_{(0,0)}(s,z) = \varphi_{(1,1)}(s,z) = \frac{s+2\lambda-\sqrt{s^2+4\lambda s}}{2\lambda^2} e^{-\frac{z}{2v}\sqrt{s^2+4\lambda s}}. \tag{9}$$

The inverse Laplace transformation of $\varphi_{(0,1)}(s,z)$ yields the following pdf ([4], p. 239, formula 88):

$$\begin{aligned} f_{(0,1)}(t,z) =& \mathcal{L}^{-1}\left(e^{-\frac{z}{2v}\sqrt{s^2+4\lambda s}}, t\right) \\ =& e^{-2\lambda t}\delta(z-2vt) + 2z\lambda e^{-2\lambda t}\frac{I_1\left(\frac{\lambda}{v}\sqrt{4v^2t^2-z^2}\right)}{\sqrt{4v^2t^2-z^2}}, \quad t \geq \frac{z}{2v}. \end{aligned}$$

Hence, Eq.(2) is proved, and

$$P\left(\tau_{(0,1)}(z) \in dt\right) = e^{-2\lambda t}\delta(z-2vt)\,dt + 2z\lambda e^{-2\lambda t}\frac{I_1\left(\frac{\lambda}{v}\sqrt{4v^2t^2-z^2}\right)}{\sqrt{4v^2t^2-z^2}}dt.$$

It is easily verified that

$$\exp\left\{-\frac{z}{2v}\sqrt{s^2+4\lambda s}\right\} = \exp\left\{-\frac{z}{2v}s + \int_0^\infty \left(1-e^{-sy}\right)\frac{\lambda}{v}\frac{e^{-2\lambda y}I_1(2\lambda y)}{y}dy\right\}. \tag{10}$$

Then the following inverse Laplace transform comes from [12], p.237, no.49:

$$\mathcal{L}^{-1}\left(1+\frac{s-\sqrt{s^2+4\lambda s}}{2\lambda}, t\right) = e^{-2\lambda t}\frac{I_1(2\lambda t)}{t}.$$

It is easily seen that the following condition holds:

$$\int_0^\infty (1\wedge t)e^{-2\lambda t}\frac{I_1(2\lambda t)}{t}dt < +\infty. \tag{11}$$

It is well known that the distribution, whose Laplace transform can be represented as the right-hand side of Eq.(10) under condition (11), belongs to the infinitely divisible distribution [13]. Therefore, the pdf $f_{(0,1)}(t,z)$ is the infinitely divisible density function. Using [12], p. 237, no. 49, we get

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{s+2\lambda-\sqrt{s^2+4\lambda s}}{s+2\lambda+\sqrt{s^2+4\lambda s}}, t\right) =& \frac{1}{4\lambda^2}\,\mathcal{L}^{-1}\left(\left(s+2\lambda-\sqrt{s^2+4\lambda s}\right)^2, t\right) \\ =& e^{-2\lambda t}\int_0^t \frac{I_1(2\lambda(t-v))}{(t-v)}\frac{I_1(2\lambda v)}{v}dv. \end{aligned}$$

By calculating

$$\mathcal{L}^{-1}\left(\frac{s+2\lambda-\sqrt{s^2+4\lambda s}}{s+2\lambda+\sqrt{s^2+4\lambda s}}e^{-\frac{z}{2v}\sqrt{s^2+4\lambda s}}\right),$$

we obtain Eq.(4).

It is easily seen that $f_{(0,1)}(t,z)$ is a heavy tail probability density function w.r.t. t. Indeed, by using the asymptotic expansion for $I_1(t)$ [5], we have

$$\lim_{t\to+\infty}\sqrt{2\pi t}I_1\left(t\right)\,e^{-t}=1. \tag{12}$$

Therefore,

$$E\left[\tau_{(0,1)}\left(z\right)\right]^{\alpha}\geq 2z\lambda\int_{\frac{z}{2v}}^{\infty}t^{\alpha}e^{-2\lambda t}\frac{I_1\left(\frac{\lambda}{v}\sqrt{4v^2t^2-z^2}\right)}{\sqrt{4v^2t^2-z^2}}dt=+\infty,\quad \text{for } \alpha\geq\frac{1}{2}.$$

It is easily verified that $E\left[\tau_{(0,1)}\left(z\right)\right]^{\alpha}<\infty$ for $0\leq\alpha<\frac{1}{2}$.

For $\tau_{(0,1)}(z)$ at time $t=0$, particles move in opposite directions to meet each other, whereas, for $\tau_{(0,1)}(z)$ at time $t=1$, particles move in opposite directions far away from each other.

Hence, $E\tau_{(0,1)}\left(z\right)\leq\ E\tau_{(1,0)}\left(z\right)$ and $f_{(1,0)}\left(t,z\right)$ is also a heavy tail density function w.r.t. t.

Let us consider the following so-called Kac's condition (or the hydrodynamic limit): denote, by $\lambda=\varepsilon^{-2}$, $v=c\varepsilon^{-1}$, as $\varepsilon>0$, i.e., $v\to+\infty$, and $\lambda\to+\infty$, such that $\frac{v^2}{\lambda}\to c^2$. It was proved in [1] that, under Kac's condition, the telegraph process $x\left(t\right)$ weakly converges to the Wiener process $w\left(t\right)\sim N\left(0,c^2t\right)$.

Denote $f\left(t,z\right)=\frac{cz\,\exp\left(-\frac{c^2z^2}{4t}\right)}{2\sqrt{\pi}t^{3/2}}$. It is well known that $f(t,z)$ is the pdf of a collision instant of two particles moving according to Wiener paths $w(t)$, where $z>0$ is the distance between starting points of the particles.

Lemma 2.2. *For each $k_1,k_2\in\{0,1\}$, $f_{(k_1,k_2)}(t,z)$ weakly converges to $f(t,z)$ under Kac's condition.*

Proof. It follows from Eqs.(7)-(9) that

$$\lim_{\varepsilon\to 0}\varphi_{(k_1,k_2)}\left(s,z\right)\ =e^{-zc\sqrt{s}}.$$

Passing to the inverse Laplace transform, we have

$$f\left(t,z\right)=\mathcal{L}^{-1}\left(e^{-zc\sqrt{s}}\right)=\frac{cz\,\exp\left(-\frac{c^2z^2}{4t}\right)}{2\sqrt{\pi}t^{3/2}}.$$

Therefore, under Kac's conditions, not only the telegraph process weakly converges to the Wiener process, but the first meeting instant of two telegraph processes weakly converges to the first meeting instant of the corresponding two Wiener processes.

Remark 2.1. It should be noted that, instead of two telegraph processes $x(y_1,t)$, $x(y_2,t)$ on the line, we can consider the bivariate process $\overrightarrow{x}(t)=(x(y_1,t),x(y_2,t))$ on the plane. The process $\overrightarrow{x}(t)$ is driven by the switching process $\eta(t)$. Denote $l=\{(x,y):x=y;x,y\in\mathbb{R}\}$. For this case,

$$\tau_{(k_1,k_2)}\left(z\right)=inf\left\{t\geq 0:\ \overrightarrow{x}(t)\in l\right\}$$

.

3. Estimation of the number of particle collisions

Denote, by $N_{(0,1)}\left(t,z\right)$, the number of collisions of particles $x_i\left(t\right),\ i=1,2$, during the time $(0,t)$, $t>0$, assuming $\eta\left(0\right)=\left(0,1\right)$.

Consider the renewal function $H_{(0,1)}\left(t,z\right)=EN_{(0,1)}\left(t,z\right)$. By using the Laplace transform for the general renewal function [16], it follows from Eqs. (7)-(8) that the Laplace transform $\hat{H}_{(0,1)}\left(s,z\right)=\mathcal{L}\left(H_{(0,1)}\left(t,z\right),s\right)$ of $H_{(0,1)}\left(t,z\right)$ w.r.t. t has the form

$$\hat{H}_{(0,1)}(s,z) = \frac{e^{-\frac{z}{2v}\sqrt{s^2+4\lambda s}}}{s}\sum_{k=0}^{\infty}\left(\frac{s+2\lambda-\sqrt{s^2+4\lambda s}}{s+2\lambda+\sqrt{s^2+4\lambda s}}\right)^k$$
$$= e^{-\frac{z}{2v}\sqrt{s^2+4\lambda s}}\left(\frac{s+2\lambda+\sqrt{s^2+4\lambda s}}{2s\sqrt{s^2+4\lambda s}}\right).$$

It is easily verified that

$$\mathcal{L}^{-1}\left(\frac{s+2\lambda+\sqrt{s^2+4\lambda s}}{2s\sqrt{s^2+4\lambda s}}\right) = \frac{1}{2} + \left(\left(\frac{1}{2}+\lambda t\right)I_0(2\lambda t) + \lambda t I_1(2\lambda t)\right)e^{-2\lambda t}.$$

Therefore,

$$(13)\quad H_{(0,1)}(t) = \int_{\frac{z}{2v}}^{t} e^{-2\lambda u}\left(\delta(z-2vu) + 2z\lambda\frac{I_1\left(\frac{\lambda}{v}\sqrt{4v^2u^2-z^2}\right)}{\sqrt{4v^2u^2-z^2}}\right)$$
$$\times\left(\frac{1}{2} + e^{-2\lambda(t-u)}\left(\left(\frac{1}{2}+\lambda(t-u)\right)I_0(2\lambda(t-u)) + \lambda(t-u)I_1(2\lambda(t-u))\right)\right)du$$
$$= \frac{e^{-\frac{\lambda z}{v}}}{2} + z\lambda\int_{\frac{z}{2v}}^{t} e^{-2\lambda u}\frac{I_1\left(\frac{\lambda}{v}\sqrt{4v^2u^2-z^2}\right)}{\sqrt{4v^2u^2-z^2}}du$$
$$+ e^{-2\lambda t}\left(\left(\frac{1}{2}+\lambda\left(t-\frac{z}{2v}\right)\right)I_0\left(2\lambda\left(t-\frac{z}{2v}\right)\right) + \lambda\left(t-\frac{z}{2v}\right)I_1\left(2\lambda\left(t-\frac{z}{2v}\right)\right)\right)$$
$$+ e^{-2\lambda t}z\lambda\int_{\frac{z}{2v}}^{t}\frac{I_1\left(\frac{\lambda}{v}\sqrt{4v^2u^2-z^2}\right)}{\sqrt{4v^2u^2-z^2}}$$
$$\times\left(1 + \left((1+2\lambda(t-u))I_0(2\lambda(t-u)) + \lambda(t-u)I_1(2\lambda(t-u))\right)\right)du.$$

It follows from Eq.(13) that, by putting $\lambda = \varepsilon^{-2}$, $v = c\varepsilon^{-1}$, we have

$$H_{(0,1)}(t,z) = O\left(\varepsilon^{-1}\right) = O\left(\sqrt{\lambda}\right) = O(v) \quad \text{as } \varepsilon \to 0.$$

For y, y^* such as $y < y^*$ and a fixed $T > 0$, denote $\widetilde{\tau} = inf\{T; t : x(y,t) - x(y^*,t) = 0\}$. Almost in the same way, we can show that, for all $k_1, k_2 \in \{0,1\}$,

$$H_{(k_1,k_2)}(t,z) = O\left(\varepsilon^{-1}\right) = O\left(\sqrt{\lambda}\right) = O(v) \quad \text{as } \varepsilon \to 0.$$

Lemma 3.1. *There exist $C > 0$ such that, for any two points y, y^* $(y < y^*)$,*

$$E\widetilde{\tau} \le C(y^* - y).$$

Proof.

$$E\tau_{(0,1)} = \int_{\frac{z}{2v}}^{T} te^{-2\lambda t}\left[\delta(z-2vt) + 2z\lambda\frac{I_1\left(\frac{\lambda}{v}\sqrt{4v^2t^2-z^2}\right)}{\sqrt{4v^2t^2-z^2}}\right]dt$$
$$\le \frac{z}{2v} + 2z\lambda\int_{\frac{z}{2v}}^{T} t\frac{I_1\left(\frac{\lambda}{v}\sqrt{4v^2t^2-z^2}\right)}{\sqrt{4v^2t^2-z^2}}dt$$
$$= \frac{z}{2v} + \frac{z}{2v}\left(I_0\left(\frac{\lambda}{v}\sqrt{4T^2v^2-z^2}\right) - 1\right) \le Cz,$$

where $C = \frac{I_0(2T\lambda)}{2v}$.

Now

$$\begin{aligned} E\tau_{(1,0)} =& \int_{z/2v}^{T} t f_{(1,0)}\left(t,z\right)dt \\ =& 2z\lambda \int_{z/2v}^{T} te^{-2\lambda t} \int_{z/2v}^{t} \frac{I_1\left(\frac{\lambda}{v}\sqrt{4v^2u^2-z^2}\right)}{\sqrt{4v^2u^2-z^2}} \\ & \times \int_{0}^{t-u} \frac{I_1\left(2\lambda\left(t-u-r\right)\right)}{\left(t-u-r\right)} \frac{I_1\left(2\lambda r\right)}{r} drdudt < Cz, \end{aligned}$$

where $C = \frac{\lambda}{v}\int_0^T te^{-2\lambda t}\int_0^t \frac{I_1(2\lambda u)}{u}\int_0^{t-u}\frac{I_1(2\lambda(t-u-r))}{(t-u-r)}\frac{I_1(2\lambda r)}{r}drdudt.$

4. Free path times of a family of particles

Since we consider the model of an ideal gas, it is natural to assume that the number of particles is very large. As an example, we consider a model with an infinite number of particles, and study the free path of the particles before collisions.

Consider the segment $[0,S]\subset\mathbb{R}$ and an increasing sequence of different points $\{y_n;\ n\geq 1\}$ from this segment. As above, we consider a family of independent telegraph processes $S_k(t)$, $k\geq 1$ and trajectories $x(y_k,t)$ of particles, which satisfy Eq. (1).

Introduce the following random times:

$$\tau_1 = T > 0,$$

$$\tau_k = inf\left\{T; t : \left(x\left(y_k,t\right) - x\left(y_{k-1},t\right)\right) = 0\right\}, \quad k\geq 2.$$

The random variable τ_k is the duration of the free motion of the particle with number k up to the collision with a particle starting with a smaller number or till T (finite) if none collision occurs.

Lemma 4.1. *Suppose* $\{y_n;\ n\geq 1\}\subset [0,S]$, $0<S<+\infty$ *is a sequence of different points. Then*

$$\sum_{k=1}^{\infty}\tau_k < +\infty \quad \text{a.s.}$$

Proof. Consider the following random times

$$\widetilde{\tau}_1 = T,$$

$$\widetilde{\tau}_k = inf\left\{T; t : \left(S_k\left(t\right) - S_{k-1}\left(t\right)\right) = 0\right\}, \quad k\geq 2.$$

It is easily seen that $\tau_k\leq\widetilde{\tau}_k$ for all $k\geq 1$.

Hence, if we show that $\sum_{k=1}^{\infty}\widetilde{\tau}_k<+\infty$ a.s., we prove the lemma. Since $\widetilde{\tau}_k\geq 0$, it is sufficient to prove that

$$\sum_{k=1}^{\infty}E\widetilde{\tau}_k < +\infty.$$

Consider the set of numbers $y_1<y_2<\cdots<y_n$. It follows from Lemma 2.1 that there exists $C>0$ such that, for any $k\geq 2$,

$$E\widetilde{\tau}_k\leq C\left(y_k-y_{k-1}\right).$$

Hence, we have

$$\lim_{n\to\infty}\sum_{k=2}^{n}E\widetilde{\tau}_k \ \leq C\lim_{n\to\infty}\sum_{k=2}^{n}\left(y_k-y_{k-1}\right) \ = C\sum_{k=2}^{\infty}\left(y_k-y_{k-1}\right)\leq CS. \tag{14}$$

Therefore, it follows from Eq.(14) that $\sum_{k=1}^{\infty}\widetilde{\tau}_k$ converges almost surely. This concludes the proof.

Note that Lemma 4.1 for Wiener particles was proved in [8].

Let us denote, by $N_{(k_1,k_2,\dots,k_n)}(t, y_1, y_2, \dots, y_n)$, $k_i \in \{0,1\}$, $y_1 < y_2 < \cdots < y_n$, the number of collisions of particles $x(y_i,t)$, $i = 1,2,\dots,n$ during time $(0,t)$, $t > 0$ assuming $\eta(0) = (k_1, k_2, \dots, k_n)$.
Then it is easily seen that

$$\begin{aligned} H_{(k_1,k_2,\dots,k_n)}(t, y_1, y_2, \dots, y_n) &= EN_{(k_1,k_2,\dots,k_n)}(t, y_1, y_2, \dots, y_n) \\ &= \sum_{i=1}^{n-1} H_{(k_i,k_{i+1})}(t, y_{i+1} - y_i), \end{aligned}$$

where $H_{(k_i,k_{i+1})}(t, y_{i+1} - y_i)$ can be calculated similarly to Eq. (13).

5. Random motion with reflecting boundaries

Consider a set of real numbers $\{y_i; i = 1,\dots,n\} \subset (0,b)$, where $b > 0$ and $y_1 < y_2 < \cdots < y_n$. Let $S_1(t), S_2(t), \dots, S_n(t)$ be independent telegraph processes. It is assumed that all processes have absolute velocity v and parameter of switching process λ and, starting from y_i, the process $S_i(t)$ has equal probabilities of initial directions of the motion. We suppose that 0 and b are two reflecting boundaries such that if a process reaches boundary 0 or b, then it changes the velocity direction to the opposite one. Consider the family of particles with trajectories $x(y_1,t), x(y_2,t), \dots, x(y_n,t)$, where every $x(y_i,t)$ coincides respectively with processes $S_i(t)$ before particle i has first hard collision with another particle or equivalently to the first intersection of the process $S_i(t)$ with another process. After the first hard collision of the particle $x(y_i,t)$ with another particle, say $x(y_j,t)$, they will switch the telegraph processes that describe their trajectories so, $S_i(t)$ will coincide with the trajectory of $x(y_j,t)$ and so on.

It is easily seen that the trajectories of the particles $x(y_k,t)$, $k = 1,2,\dots,$ coincides with the order statistics of $S_i(t)$, $i = 1,2,\dots,$ as follows:

$$x(y_1,t) = S_{(1)}(t), x(y_2,t) = S_{(2)}(t), \dots, x(y_n,t) = S_{(n)}(t). \tag{15}$$

Let us introduce the distribution functions $F_{y_r}(x) = P\{x(y_r,t) < x\}$. Denote, by $M_k^{(l)}$, $l = 1,2,\dots,C_n^k$, different k-element subsets of the set $M = \{1,2,\dots,n\}$. It follows from Eqs. (15) that

$$F_{y_r}(x) = P\{x(y_r,t) < x\} = \sum_{k=r}^{n} \sum_{l=1}^{C_n^k} \prod_{i \in M_k^{(l)}} P(S_i(t) < x) \prod_{j \in M \setminus M_k^{(l)}} P(S_j(t) \geq x).$$

For some particular cases, we have

$$F_{y_1}(x) = P(x(y_1,t) < x) = 1 - \prod_{i=1}^{n} P(S_i(t) \geq x),$$

$$\begin{aligned} F_{y_{n-1}}(x) = P(x(y_{n-1},t) < x) = \sum_{k=1}^{n} \prod_{i=1, i\neq k}^{n} & P(S_i(t) < x) P(S_k(t) \geq x) \\ & + \prod_{i=1}^{n} P(S_i(t) < x), \end{aligned}$$

$$F_{y_n}(x) = P(x(y_n,t) < x) = \prod_{i=1}^{n} P(S_i(t) < x).$$

Let us study the limiting distribution of $S_k(t)$, $k = 1, \ldots, n$ as $t \to +\infty$. Denote, by $N(t)$, the number of Poisson events that have occurred in the interval $(0, t)$, and let s_j, $j \geq 0$, be instants, at which Poisson events occur, and $s_0 = 0$. We assume that the instants s_j denote the times of a change of the direction of $S_k(t)$.

Lemma 5.1. *Suppose that $f(x)$ is an integrable function on $[0, b]$. Then*

$$P\left(\lim_{T\to+\infty} \frac{1}{T}\int_0^T f(S_k(t))\,dt = \frac{1}{b}\int_0^b f(x)\,dx\right) = 1.$$

Proof In the sequel, we will use the well-known strong law of large numbers for a Poisson process

$$P\left(\lim_{T\to+\infty} \frac{N(T)}{T} = \lambda\right) = 1. \tag{16}$$

Since, during the time $s_{j+1} - s_j$, the particle covers the distance of $(s_{j+1} - s_j)v$, the number $\left[\frac{(s_{j+1}-s_j)v}{2b}\right]$ is equal to the double number of passages of the segment $[0, b]$ by the particle.

Hence,

$$\begin{aligned}
\lim_{T\to+\infty} \frac{1}{T}\int_0^T f(S_k(t))\,dt &= \lim_{T\to+\infty} \frac{1}{T}\sum_{i=0}^{N(T)} \int_{s_j}^{s_{j+1}} f(S_k(t))\,dt \\
&= \lim_{T\to+\infty} \frac{1}{T}\sum_{i=0}^{N(T)} \left(\left[\frac{(s_{j+1}-s_j)v}{2b}\right]\frac{2}{v}\int_0^b f(x)\,dx + r_i\right) a.s., \qquad (17)
\end{aligned}$$

where $r_i = \int_{u_i}^{u_i+\vartheta_i} f(S_k(t))dt$, u_i, ϑ_i are independent random variables, and u_i is uniformly distributed on $[0, 2b]$, and ϑ_i has the following pdf:

$$g(t) = \frac{\lambda}{v}e^{-\frac{\lambda t}{v}}\left(1 - e^{-\frac{2\lambda b}{v}}\right)^{-1} I_{\{0\leq t\leq 2b\}}.$$

Therefore,

$$\begin{aligned}
Er_i = E\int_{u_i}^{u_i+\vartheta_i} f(S_k(t))dt &= \frac{\lambda}{2bv\left(1-e^{-\frac{2\lambda b}{v}}\right)}\int_0^{2b} dx \int_0^{2b} dp \int_x^{x+p} dt f(S_k(t)) e^{-\frac{\lambda p}{v}} \\
&= -\frac{1}{2b}\int_0^{2b} dx \int_x^{x+2b} dt f(S_k(t)) \frac{e^{-\frac{2\lambda b}{v}}}{\left(1-e^{-\frac{2\lambda b}{v}}\right)} \\
&+ \frac{1}{2b\left(1-e^{-\frac{2\lambda b}{v}}\right)}\int_0^{2b} dx \int_0^{2b} dp f\left(S_k(x+p)\right) e^{-\frac{\lambda p}{v}} \qquad (18) \\
&= -\frac{2e^{-\frac{2\lambda b}{v}}}{\left(1-e^{-\frac{2\lambda b}{v}}\right)}\int_0^b f(x)dx + \frac{1}{\lambda b}\int_0^b f(x)dx.
\end{aligned}$$

The strong law of large numbers for $\{r_i, i \geq 1\}$ yields

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} r_i = Er_i \quad a.s.$$

Since $\theta_j = s_{j+1} - s_j$, $j = 1, 2, \ldots$, are independent exponentially distributed random variables, we have the strong law of large numbers takes the form

$$\begin{aligned}\lim_{N\to+\infty}\frac{1}{N}\sum_{j=1}^{N}\left[\frac{(s_j-s_{j-1})v}{2b}\right] &= E\left[\frac{(s_j-s_{j-1})v}{2b}\right]\\ &= \sum_{n=1}^{\infty}n\left(e^{-\frac{2n\lambda b}{v}}-e^{-\frac{2(n+1)\lambda b}{v}}\right)=\frac{e^{-\frac{2\lambda b}{v}}}{1-e^{-\frac{2\lambda b}{v}}}.\end{aligned}\tag{19}$$

Combining Eqs. (16)-(19), we get

$$\begin{aligned}\lim_{T\to+\infty}\frac{1}{T}\int_0^T f\left(S_k\left(t\right)\right)dt=\lim_{T\to+\infty}\frac{N\left(t\right)}{T}\frac{1}{N\left(T\right)}\sum_{j=1}^{N(T)}\left(\int_{s_{j-1}}^{s_j}f(S_k(t))\right)\\ =\frac{1}{b}\int_0^b f(x)dx \quad a.s.\end{aligned}$$

This concludes the proof.

Therefore, the limiting distribution of $S_k\left(t\right)$ as $t\to+\infty$ for all $k=1,\ldots,n$ is the uniform distribution on $[0,b]$.

Lemma 5.2. *Suppose that the initial distribution of a telegraph particle with reflecting boundaries* 0 *and* b *is uniform on* $[0,b]$. *Then it remains uniform for all* $t>0$.

Proof Denote, by $p\left(t,x\mid y_k\right)$, the probability density of the process position $S_k\left(t\right)$ at time t. It was shown in [15] that, for $x\in[0,b]$,

$$p\left(t,x\mid y_k\right)=\frac{1}{b}+\frac{2}{b}e^{-\lambda t}\sum_{n=1}^{\infty}\left\{\left[\cosh\left(\theta_n t\right)\;+\frac{\lambda}{\theta_n}\sinh\theta_n t\;\right]\cos\left(\frac{\pi n y_k}{b}\right)\,\cos\left(\frac{\pi n x}{b}\right)\;\right\},$$

where

$$\theta_n=\left(\lambda^2-\frac{\pi^2v^2}{b^2}n^2\right)^{1/2}.$$

It is easily seen that, for any $t>0$ and $x\in[0,b]$,

$$p\left(t,x\right)=\frac{1}{b}\int_0^b p\left(t,x\mid y_k\right)dy_k=\frac{1}{b}.$$

Now let us consider the system of processes $\overline{S}_k\left(t\right)$ with the limiting distribution of the respective processes $S_k\left(t\right)$, $k=1,2,\ldots,n$. According to Lemmas 5.1 and 5.2, the processes $\overline{S}_k\left(t\right)$, $k=1,2,\ldots n$, are independent and have the uniform distribution on $[0,b]$ for each $t\geq 0$.

In this case, we denote, by $x_k\left(t\right)$, $k=1,2,\ldots,n$, the positions of particles at time $t\geq 0$. It is easy to see that, for every $t\geq 0$, the processes $x_k\left(t\right)$ are the order statistics of $S_k\left(t\right)$, $k=1,2,\ldots,n$, namely

$$x_1\left(t\right)=\overline{S}_{(1)}\left(t\right),x_2\left(t\right)=\overline{S}_{(2)}\left(t\right),\ldots,x_n\left(t\right)=\overline{S}_{(n)}\left(t\right).$$

Consider the function

$$p\left(x\right)=P(S_k(t)<x)=\begin{cases}\frac{x}{b}, & x\in[0,b],\\ 0, & x\notin[0,b].\end{cases}$$

It is easily verified that the distributions $\pi_k\left(\cdot\right)$ of the positions of particles $x_k\left(t\right)$, $k\in\{1,2,\ldots,n\}$ are as follows:

$$\pi_k(x) = P(x_k(t) < x) = I_{p(x)}(k, n-k+1),$$

where

$$I_{p(x)}(k, n-k+1) = \frac{\int_0^{p(x)} t^{k-1}(1-t)^{n-k}dt}{\int_0^1 t^{k-1}(1-t)^{n-k}dt}.$$

Let us study the number of collisions $C_{(1,2,\dots,n)}(0,t)$ of particles $x_k(t)$, $k = 1,2,\dots,n$ during the time interval $(0,t)$. It is easy to see that $C_{(1,2,\dots,n)}(0,t)$ is a number of intersections of $\overline{S}_k(t)$, $k = 1,2,\dots,n$, for each $t > 0$.

Denote, by $I_{(k,l)}(0,t)$, $k \neq l$, the number of intersection of the processes $\overline{S}_k(t)$ and $\overline{S}_l(t)$ during the time interval $(0,t)$. Then it is easily verified that

$$C_{(1,2,\dots,n)}(0,t) = \sum_{1\le k<l\le n} I_{(k,l)}(0,t).$$

Therefore, let us analyze the distribution of $I_{(k,l)}(0,t)$, $k \neq l$.
Since $\overline{S}_k(t)$ and $\overline{S}_l(t)$ have the uniform distribution on $[0,b]$, the probability of their intersections $I_{(k,l)}(t, t+\triangle t)$ during $(t, t+\triangle t)$ satisfies the following inequalities for $x \in (a,b)$:

$$\begin{aligned}\frac{1}{4}P\left(\left|\overline{S}_k(t) - \overline{S}_l(t)\right| \le 2\triangle tv\right)e^{-2\lambda\triangle t} &\le P\left(N_{(k,l)}(t,t+\triangle t) \ge 1\right)\\ &\le P\left(\left|\overline{S}_k(t) - \overline{S}_l(t)\right| \le 2\triangle tv\right).\end{aligned} \tag{20}$$

By using

$$P\left(\left|\overline{S}_k(t) - \overline{S}_l(t)\right| \le 2\triangle tv\right) = O(\triangle t)$$

and

$$\frac{1}{4}P\left(\left|\overline{S}_k(t) - \overline{S}_l(t)\right| \le 2\triangle tv\right)e^{-2\lambda\triangle t} = O(\triangle t),$$

we get

$$P\left(I_{(k,l)}(t,t+\triangle t) \ge 1\right) = O(\triangle t). \tag{21}$$

It is easily verified that, for $n \ge 2$,

$$\begin{aligned}P\left(I_{(k,l)}(t,t+\triangle t) = n\right) &\le P\left(\left|\overline{S}_k(t) - \overline{S}_l(t)\right| \le 2\triangle tv\right)\left(1-e^{-\lambda\triangle t}\right)^{2(n-1)}\\ +P\left(\{\overline{S}_k(t), \overline{S}_l(t) \in [0, 2\triangle tv]\} \cup \{\overline{S}_k(t), \overline{S}_l(t) \in [b-2\triangle tv, b]\}\right)&\left(1-e^{-\lambda\triangle t}\right)^{(n-1)}\\ &\le \frac{4\triangle tv}{b}(\lambda\triangle t)^{2(n-1)} + 2\frac{(2\triangle tv)^2}{b^2}(\lambda\triangle t)^{(n-1)}.\end{aligned} \tag{22}$$

In view of Eqs. (21) and (22), we conclude that

$$P\left(I_{(k,l)}(t,t+\triangle t) = 1\right) = O(\triangle t).$$

Therefore, for $x \in (a,b)$,

$$
\begin{aligned}
O(\triangle t) &= \frac{1}{4} P\left(\left|\overline{S}_k(t) - \overline{S}_l(t)\right| \le 2\triangle t v\right) e^{-2\lambda \triangle t} \\
&\le EI_{(k,l)}(t, t+\triangle t) \le P\left(\left|\overline{S}_k(t) - \overline{S}_l(t)\right| \le 2\triangle t v\right) \\
&+ \frac{4\triangle t v}{b} \sum_{n\ge 1} n\left((\lambda\triangle t)^{2(n-1)} + \frac{2\triangle t v}{b}(\lambda \triangle t)^{(n-1)}\right) \\
&= \frac{4\triangle t v}{b} + o(\triangle t). \qquad (23)
\end{aligned}
$$

It is easily seen the additive property of $EI_{(k,l)}(t_1, t_2)$: for any $s \in (t_1, t_2)$,

$$EI_{(k,l)}(t_1, t_2) = EI_{(k,l)}(t_1, s) + EI_{(k,l)}(s, t_2).$$

Hence, there exists a constant $c > 0$ such that

$$EI_{(k,l)}(0, t) = ct.$$

This implies that

$$EC_{(1,2,\ldots,n)}(0, t) = \frac{n(n-1)}{2} ct.$$

Relations (20) and (23) yield the following estimation for the factor c:

$$\frac{v}{b} \le c \le \frac{4v}{b}.$$

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