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ON LATTICE OSCILLATOR-TYPE GIBBS SYSTEMS WITH SUPERSTABLE MANY-BODY POTENTIALS

The grand canonical correlation functions of lattice oscillator-type Gibbs systems with a general one-body phase measure space and many-body superstable interaction potentials are found in the thermodynamic limit at low activities as a solution of the ordered lattice Kirkwood–Salzburg equation. For special choices of the measure space, they describe the equilibrium states of lattice classical and quantum linear oscillator systems and the states of stochastic gradient lattice systems of interacting oscillators with Gibbs initial states.

In this paper, we will find the thermodynamic limit of the finite-volume grand canonical correlation functions of oscillator-type unbounded spin lattice Gibbs systems with the inverse temperature $\beta \geq 0$, the one-body phase measure space $(\Omega, e^{-\beta u}P^0)$, an interaction potential energy $U(\omega_X), X \subset \mathbb{Z}^d$, and an external potential $u(\omega)$, where $\omega_X = (\omega_x \in \Omega, x \in X)$. They are given by

$$\rho^{\Lambda}(\omega_X) = \Xi_{\Lambda}^{-1}\chi_{\Lambda}(X) \sum_{Y \subseteq \Lambda \backslash X} z^{|X|+|Y|} \int \exp\{-\beta U(\omega_{X \cup Y})\} P(d\omega_Y),$$

where $|Y|$ is a number of sites in Y, χ_{Λ} is the characteristic function of a hyper-cube Λ, $z \in \mathbb{C}$ is the activity, the grand partition function Ξ_{Λ} coincides with the sum in the numerator for $X = \emptyset$, and

$$P(d\omega_Y) = \prod_{y \in Y} P(d\omega_y), \quad P(d\omega_y) = e^{-\beta u(\omega_y)} P^0(d\omega_y).$$

The thermodynamic limit demands that Λ be enlarged to $\mathbb{Z}^d$. The interaction potential energy is given by

$$U(\omega_X) = \sum_{Y \subseteq X} u_Y(\omega_Y), \quad u_y(\omega_y) = 0,$$

where $u_Y(\omega_Y) = u_{Y+a}(\omega_Y), |Y| = k, a \in \mathbb{Z}^d$ is the k-body translation invariant potential. The potential energy is an unbounded function, and $P^0(\Omega) = \infty$ for oscillator-type or abstract unbounded spin systems. The space Ω can be considered as a metric space (σ-algebra is associated with Borel sets), which is a discrete union of finite balls, and the measure P^0 is finite on them. Classical lattice oscillator systems are described by $\omega = q \in \mathbb{R} = \Omega, P^0(dq) = dq$. In the quantum case, Ω is the space of continuous paths, and $P^0(d\omega) = dqP^{\beta}_{q,q}(dw), q \in \mathbb{R}$, where $P^{\beta}_{q,q}(dw)$ is the conditional Wiener measure. Stochastic lattice gradient systems of interacting oscillators can also be represented as generalized Gibbs oscillator-type lattice systems with $P^0(d\omega) = dqP_q(dw), q \in \mathbb{R}$, where $P_q(dw)$ is the Wiener measure [1].

The results of this paper allows one to find weak solutions (without the introduction of additional Wiener paths used in [1]) of the diffusion hierarchy for correlation functions

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of the infinite component stochastic gradient differential equation

$$\dot{q}_x(t) = -\partial u^0(q_x) - \sum_{|Y|\geq 2} \partial_x u^0_Y(q_Y) + \beta^{-\frac{1}{2}} \dot{w}_x(t), \qquad x \in \mathbb{Z}^d,$$

where the dot means the time derivative, $q_x \in \mathbb{R}$, $u^0_Y(q_Y)$ is a $|k|$-body translation invariant short-range superstable interaction potential, u^0 is an external potential, $\dot{w}_x(t)$ are the independent processes of white noise, $\partial_x = \frac{\partial}{\partial q_x}$, and the summation is performed over finite subsets of $\mathbb{Z}^d$. The initial states for such systems have to be Gibbsian with a special structure generated by many-body interaction potentials considered in this paper and more general than in [1], where we considered a pair interaction of stochastic oscillators: $u^0_Y(q_Y) = 0, |Y| > 2$.

We assume that the superstability condition ([2],[3]) holds for the positive and negative parts of the many-body potentials. In other words, there exists a non-negative function v on Ω such that

(1)
$$|u^{\pm}_Y(\omega_Y)| \leq J_Y \sum_{y\in Y} v(\omega_y), \quad N_0 = \int e^{\beta(\gamma v^{1+\zeta}(\omega) - u(\omega))} P^0(d\omega) < \infty, \quad \zeta \geq 0, \gamma, \beta > 0,$$

where $u_Y(\omega_Y) = u^+_Y(\omega_Y) - u^-_Y(\omega_Y), u^{\pm} \geq 0$, $||J|| = \max_x \sum_{x\in Y} J_Y < \infty$, the summation is performed over bounded subsets of $\mathbb{Z}^d$ containing a site x.

The sequence $\rho = \{\rho(\omega_X; z) = \rho(\omega_X), X \subset \mathbb{Z}^d, |X| < \infty\}$ of the thermodynamic limit of the grand canonical Gibbs finite-volume correlation functions satisfies the Kirkwood–Salzburg (KS) equation (its derivation can be found in [1])

$$\rho = zK\rho + z\alpha,$$

where $\alpha(\omega_X) = \delta_{|X|,1}$, $\delta_{k,l} = 1, k = l$, and $\delta_{k,l} = 0, k \neq l$. The KS operator K is given by

$$(KF)(\omega_X) = \sum_{Y \subset X^c} \int K(\omega_x|\omega_{X\backslash x}; \omega_Y)[F(\omega_{X\backslash x \cup Y}) - \int P(d\omega_x) F(\omega_{X\cup Y})] P(d\omega_Y),$$

where the summation is performed over all bounded subsets of X^c, the integrations are performed over the Cartesian $|Y|$-fold product $\Omega^{|Y|}$ of the measure space Ω, for $X = x$, the first term in the square bracket corresponding to $Y = \emptyset$ is equal to zero, and

$$K(\omega_x|\omega_{X\backslash x}; \omega_\emptyset) = e^{-\beta W(\omega_x|\omega_{X\backslash x})}, \qquad W(\omega_x|\omega_{X\backslash x}) = U(\omega_X) - U(\omega_{X\backslash x}).$$

The KS kernels are determined in the following way ($X \cap Y = \emptyset$):

$$K(\omega_x|\omega_{X\backslash x}; \omega_Y) = e^{-\beta W(\omega_x|\omega_{X\backslash x})} K_x(\omega_X; \omega_Y),$$

(2)
$$K_x(\omega_X; \omega_Y) = \sum_n \sum_{\cup Y_j = Y, Y_j \neq \emptyset} \prod_{j=1}^n (e^{-\beta W(\omega_X; \omega_{Y_j}|x)} - 1),$$

where the summation is performed over all sets Y_j which cover Y, and

$$W(\omega_X; \omega_Y|x) = \sum_{x\in Z\subseteq X} u_{Z\cup Y}(\omega_{Z\cup Y}).$$

The KS kernels are determined from the equality

$$e^{-\beta \tilde{W}(\omega_X;\omega_Y|x)} = \prod_{\emptyset\neq S\subseteq Y} (1 + (e^{-\beta W(\omega_X;\omega_S|x)} - 1)) = \sum_{S\subseteq Y} K_x(\omega_X; \omega_S), \quad K_x(\omega_X; \omega_\emptyset) = 1,$$

where

(3)
$$\tilde{W}(\omega_X; \omega_Y|x) = \sum_{x\in Z\subseteq X} \sum_{\emptyset\neq S\subseteq Y} u_{Z\cup S}(\omega_{Z\cup S}) = \sum_{\emptyset\neq S\subseteq Y} W(\omega_X; \omega_S|x),$$

and

$$W(\omega_x|\omega_{X\backslash x},\omega_Y)=W(\omega_x|\omega_{X\backslash x})+\tilde{W}(\omega_X;\omega_Y|x).$$

In the case of a pair interaction with a pair potential $u_{x,y}$, the KS kernel is given by

$$K_x(\omega_X;\omega_Y)=\prod_{y\in Y}(e^{-\beta u_{x,y}(\omega_x,\omega_y)}-1).$$

One expects to find the unique solution of the KS equation in the Banach space $\mathbb{E}_{\xi,f}$, which is a linear space of the sequences of measurable functions $G_X(\omega_X)$ with the norm

$$||G||_{\xi,f}=\max_X \xi^{-|X|} ess\sup_{w_X}\exp\{-\sum_{x\in X}f(\omega_x)\}|G(\omega_X)|, \quad f(\omega)=\gamma\beta v^{1+\zeta}(\omega), \quad \zeta\geq 0.$$

The unbounded character of the functions $W(\omega_x|\omega_{X\backslash x})$, $W(\omega_X;\omega_Y|x)$ is an obstruction for a proof that the KS operator is bounded in $\mathbb{E}_{\xi,f}$ and finding the solution of the KS equation as a convergent series in powers of the activity z if x is fixed. To eliminate this obstruction, one has either to symmetrize the KS equation as in [1],[4] (see Remark 1) or to establish the special ordering in x: the ordered KS operator and the equation are such that the site x depends on ω_X as the first site (in the lexicographic order) determined through the ordering condition $v(\omega_x)=\max_{y\in X}v(\omega_y)$. The symmetrization is more complicated than the ordering. For classical lattice oscillator systems and an even function v, the ordering condition is equivalent to $|q_x|=\max_{y\in X}|q_y|$. The analog of this condition was proposed in [5] for the ordering of the KS equation corresponding to the integer-valued Ising model with a pair potential. In this paper, we establish that the ordered KS operator is bounded in $\mathbb{E}_{\xi,f}$ for a positive γ if $\zeta>0$.

The ordering and the first inequality (1) yield the important inequality

$$|W(\omega_x|\omega_{X\backslash x})|\leq\sum_{x\in Z\subseteq X}|u_Z(\omega_Z)|\leq 2\sum_{x\in Z\subseteq X}J_Z\sum_{y\in Z}v(\omega_y)\leq 2||J||_1v(\omega_x), \tag{4}$$

where $||J||_1=\max_x\sum_{x\in Y}J_Y|Y|$, which is applied to the case of non-positive potentials.

Let $W^{\pm},\tilde{W}^{\pm}$ correspond to the potentials $u^{\pm}$. Then the inequality

$$\sum_{j=1}^n W^-(\omega_X;\omega_{Y_j}|x)\leq\tilde{W}^-(\omega_X;\omega_Y|x), \qquad \cup_{j=1}^n Y_j=Y$$

is true, since the sum in (3) for $W^{\pm},\tilde{W}^{\pm}$ contains the sets $Y_1,...,Y_n$. The last inequality and (4) give

$$|K_x(\omega_X;\omega_Y)|\leq\exp\{\beta\tilde{W}^-(\omega_X;\omega_Y|x)-\beta\kappa||J||\sum_{y\in Y}v(\omega_y)\}K_x^-(\omega_X;\omega_Y), \tag{5}$$

where

$$K_x^-(\omega_X;\omega_Y)=\exp\{\beta\kappa||J||\sum_{y\in Y}v(\omega_y)\}\sum_n\sum_{\cup Y_j=Y,Y_j\neq\emptyset}\prod_{j=1}^n\tilde{K}_x(\omega_X;\omega_{Y_j})$$

$$\tilde{K}_x(\omega_X;\omega_Y)=1-e^{-\beta W^-(\omega_X;\omega_Y|x)}+1-e^{-\beta W^+(\omega_X;\omega_Y|x)},$$

and $\kappa=0(1)$ corresponds to positive (non-positive) potentials. Here, we used the inequality $|e^{-a}-e^{-b}|\leq 1-e^{-a}+1-e^{-b}, a,b\geq 0$. In addition, we have (due to (1) and the ordering)

$$\tilde{W}^-(\omega_X;\omega_Y|x)=\sum_{\emptyset\neq S\subseteq Y}W^-(\omega_X;\omega_S|x)\leq\sum_{x\in Z\subseteq X\cup Y}J_Z\sum_{y\in Z}v(\omega_y)\leq$$

$$\leq \sum_{x\in Z\subseteq X\cup Y} J_Z[|Z|v(\omega_x)+\sum_{y\in Y}v(\omega_y)] \leq ||J||_1 v(\omega_x)+||J||\sum_{y\in Y}v(\omega_y). \tag{6}$$

From (4-6), we derive

$$|K(\omega_x|\omega_{X\backslash x};\omega_Y)| \leq e^{\beta 3\kappa ||J||_1 v(\omega_x)} K_x^-(\omega_X;\omega_Y). \tag{7}$$

For the norm of the ordered KS operator, we have the inequality

$$||K||_{\xi,f} \leq (\xi^{-1}+N_0)\max_X ess\sup_{\omega_X} e^{-f(\omega_x)}\bar{K}_x(\omega_X),$$

where

$$\bar{K}_x(\omega_X)=\sum_{Y\subseteq X^c}\xi^{|Y|}\int |K(\omega_x|\omega_{X\backslash x};\omega_Y)|P'(d\omega_Y),$$

$$P'(d\omega_Y)=\exp\{f(\omega_Y)\}P(d\omega_Y), \qquad f(\omega_Y)=\sum_{x\in Y}f(\omega_x).$$

Further, we will rely on the inequality

$$\int\prod_{j=1}^n \tilde{K}_x(\omega_X;\omega_{Y_j})P'(d\omega_Y)\leq$$

$$\leq \prod_{j=1}^n I_n(X;Y_j)\int \exp\{\frac{1}{n}\sum_{j=1}^n f(\omega_{Y_j})\}P'(d\omega_Y)\leq N_3^{|Y|}\prod_{j=1}^n I_n(X;Y_j),$$

where

$$I_n(X;Y)=ess\sup_{\omega_Y} e^{-\frac{1}{n}f(\omega_Y)}\tilde{K}_x(\omega_X;\omega_Y), \quad N_3=\int e^{2f(\omega)}P(d\omega).$$

Here, we used the inequality

$$\frac{1}{n}\sum_{j=1}^n f(\omega_{Y_j})\leq f(\omega_Y),$$

which results from $f(\omega_{Y_j})\leq f(\omega_Y)$. Then (7) and the last inequalities give

$$e^{-3\beta\kappa||J||_1 v(\omega_x))}\bar{K}_x(\omega_X)\leq \sum_{Y\subset X^c}(N_3\xi)^{|Y|}\sum_n\sum_{\cup Y_j=Y,Y_j\neq\emptyset}\prod_{j=1}^n I_n(X;Y_j)\leq$$

$$\leq 1+\sum_{n\geq 1}\frac{1}{n!}(\sum_{Y\neq\emptyset,Y\subset X^c}(N_3\xi)_*(|Y|)I_n(X;Y))^n, \tag{8}$$

where $\xi_*(a)$ equals ξ^a and 1, if $\xi\geq 1$ and $\xi<1$, respectively. Bound (8) is an analog of bound (2.67) (there is no index n in I_n in it) from the fourth section in [6] for the KS kernels for a lattice gas. From the inequality

$$|\tilde{K}_x(\omega_X;\omega_Y)|\leq \beta|W|(\omega_X;\omega_Y|x), \qquad |W|=W^+ + W^-,$$

we deduce

$$I_n(X;Y)\leq \beta ess\sup_{\omega_Y} e^{-\frac{1}{n}f(\omega_Y)}|W|(\omega_X;\omega_Y|x).$$

But, due to (1) and the ordering,

$$|W|(\omega_X;\omega_Y|x)\leq \sum_{x\in Z\subseteq X}|u_{Z\cup Y}|(\omega_{Z\cup Y})\leq 2\sum_{x\in Z\subseteq X}J_{Z\cup Y}\sum_{y\subseteq Z\cup Y}v(\omega_y)\leq$$

$$\leq 2\sum_{x\in Z\subseteq X}J_{Z\cup Y}[\sum_{y\subseteq Y}v(\omega_y)+|Z|v(\omega_x)].$$

With regard for the equality

$$ess\sup_{\omega} e^{-n^{-1}\beta\gamma v^{1+\zeta}(\omega)}v(\omega) = (n\beta^{-1}\gamma^{-1})^{\frac{1}{1+\zeta}}\eta_\zeta, \qquad \eta_\zeta = \sup_{v\geq 0} e^{-v^{1+\zeta}}v,$$

we obtain

$$I_n(X;Y) \leq 2\beta \sum_{x\in Z\subseteq X} J_{Z\cup Y}[(n\beta^{-1}\gamma^{-1})^{\frac{1}{1+\zeta}}\eta_\zeta|Y| + |Z|v(\omega_x)]$$

and

$$\sum_{Y\neq\emptyset, Y\subset X^c} (N_3\xi)_*(|Y|)I_n(X;Y) \leq 2\beta||J||_{*2}[(n\beta^{-1}\gamma^{-1})^{\frac{1}{1+\zeta}}\eta_\zeta + v(\omega_x)], \tag{9}$$

where

$$||J||_{*2} = \max_x \sum_{x\in Y} J_Y|Y|(N_3\xi)_*(|Y|).$$

The last inequality shows that the power n on the left-hand side of (9) does not exceed

$$(4\beta||J||_{*2})^n[(n\beta^{-1}\gamma^{-1})^{\frac{n}{1+\zeta}}\eta_\zeta^n + v^n(\omega_x)].$$

Hence, relations (8) and (9) yield

$$\bar{K}_x(\omega_X) \leq e^{\beta(3\kappa||J||_1+4||J||_{*2})v(\omega_x)}(1 + \sum_{n\geq 1}\frac{1}{n!}[4\beta^{\frac{\zeta}{1+\zeta}}(n\gamma^{-1})^{\frac{1}{1+\zeta}}\eta_\zeta||J||_{*2}]^n).$$

The series on the right-hand side converges if $\zeta > 0$. Now it is not difficult to calculate the norm of the KS operator with the help of the formula

$$\sup_v \exp\{-\beta\gamma[v^{1+\zeta} + av]\} = \exp\{\beta\zeta\gamma(\frac{a}{1+\zeta})^{\frac{1+\zeta}{\zeta}}]\}.$$

For $\zeta > 0$, we have

$$||K||_{\xi,f} \leq \tag{10}$$

$$\leq (\xi^{-1}+N_0)\exp\{\beta\zeta\gamma(\frac{3\kappa||J||_1 + 4||J||_{*2}}{\gamma(1+\zeta)})^{\frac{1+\zeta}{\zeta}}\}(1+\sum_{n\geq 1}\frac{1}{n!}[4\beta^{\frac{\zeta}{1+\zeta}}(n\gamma^{-1})^{\frac{1}{1+\zeta}}\eta_\zeta||J||_{*2}]^n).$$

We recall that the case $\kappa = 0$ or $\kappa = 1$ corresponds to positive or non-positive potentials. Hence, we have proved the following proposition.

Theorem. *If $\zeta > 0$ and $||J||_1, ||J||_{*2} < \infty$, then the norm $||K||_{\xi,f}$ of the ordered KS operator K in the Banach space $\mathbb{E}_{\xi,f}$ is finite. It does not exceed the right-hand side of (10), and the series*

$$\rho = \sum_{n\geq 0} z^{n+1}K^n\alpha$$

determines the unique solution of the ordered KS equation in $\mathbb{E}_{f,\xi}$, which is a holomorphic function in z in the disc $|z| < ||K||^{-1}_{\xi,f}$.

Remark 1. *In [4], we wrote the sum in (2) for $K_x(\omega_X;\omega_Y)$ restricted by the condition of non- intersections of Y_j with Y_k, i.e., we dealt with the reduced KS kernels for positive infinite-range potentials. The partial result of [4] concerning these potentials can be obtained with the help of the bounds from this paper. In [4], we proved that the symmetrized KS operator is bounded in the Banach space $\mathbb{E}_{\xi,f}$ for finite-range superstable potentials.*

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