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ON THE SPEED OF CONVERGENCE IN THE LOCAL LIMIT THEOREM FOR TRIANGULAR ARRAYS OF RANDOM VARIABLES

We establish the upper bound on the speed of convergence to the infinitely divisible limit density in the local limit theorem for triangular arrays of random variables $\{X_{k,n},\ k=1,..,a_n,\ n\in\mathbb{N}\}$.

1. Introduction

This paper is motivated by [5], where the local limit theorem for a triangular array of random variables $\{X_{k,n},\ k=1,..,a_n\}$, independent and identically distributed (i.i.d) in each series random variables, is established. Staying in frames of the situation studied in [5], we would like make a step further and obtain the information about the speed of convergence to the limit density.

In contrast to the local limit theorem for the normal law, there is not much known even about the local limit theorem for infinitely divisible limit densities. Of course, one can refer to Gnedenko's theorem on the necessary and sufficient conditions for the convergence to the stable law, see [3]. Under certain conditions the uniform convergence to the limit density was proved in [5], but to the best of author's knowledge, in the general case nothing is known about the speed of convergence.

To make the presentation self-contained, we quote below the necessary and sufficient conditions for convergence to the infinitely divisible law, see [2, Theorem 2, Chapter XVII §2].

Recall that a measure M on $\mathbb{R}$ is called *canonical* if $M(I)<\infty$ for any finite interval, and

$$M^+(x)=\int_x^{+\infty}\frac{1}{u^2}M(du)<+\infty,\quad M^-(x)=\int_{-\infty}^{-x}\frac{1}{u^2}M(dy)<+\infty,\quad x>0.$$

A sequence of canonical measures $\{M_n\}$ converges to a canonical measure *properly*, if $M_n(I)\to M(I)$ for any finite interval, and $M_n^+(x)\to M^+(x)$, $M_n^-(x)\to M^-(x)$ for every $x>0$. In this case we write $M_n\to M$.

Theorem 1.1. [2] *Let $\{X_{k,n}\ 1\le k\le a_n\}$ be such that $X_{k,n}$ are i.i.d. for any $1\le k\le a_n$, $a_n\to\infty$ as $n\to\infty$, and satisfy*

$$\lim_{n\to\infty}P\{|X_{1,n}|\ge\varepsilon\}=0. \tag{1.1}$$

Let $F_n(du)$ be the distribution function of $X_{1,n}$,

$$M_n(du):=a_nu^2F_n(du),\quad \beta_n:=\int_{\mathbb{R}}\sin uF_n(du).$$

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Table 1. Notation

Variable	Char. function	Probab. measure	Probab. density
$\xi_{1,n}$	$\theta_n(z)$	$G_n(du)$	$g_n(u)$
$X_{1,n}$	$\theta_n(z/b_n)$	$F_n(du) \equiv G_n(b_n du)$	$f_n(u) \equiv b_n g_n(b_n u)$
S_n	$\Phi_n(z) \equiv \theta_n^{a_n}(z/b_n)$	$P_n(dx)$	$p_n(x)$
ζ	$\Phi(z) = e^{-\psi(z)}$	$P(dx)$	$p(x)$

Then $S_n := X_{1,n} + ... + X_{a_n,n}$ *converges in distribution to some random variable* ζ *if and only if*

$$M_n \to M, \quad a_n\beta_n \to \beta, \quad \text{as} \quad n \to \infty, \tag{1.2}$$

for some $\beta \in \mathbb{R}$ *and some canonical measure* M. *In this case the characteristic function* $\Phi(z)$ *of* ζ *is given by*

$$\Phi(z) = \exp\left\{ i\beta z + \int_{\mathbb{R}} \frac{e^{izu} - 1 - iz\sin u}{u^2} M(du) \right\}. \tag{1.3}$$

The function

$$\psi(z) := -i\beta z + \int_{\mathbb{R}} \frac{1 - e^{izu} + iz\sin u}{u^2} M(du) = -i\beta z + \phi(z) \tag{1.4}$$

is called the *characteristic exponent* of the infinitely divisible variable ζ. Put

$$\psi_n(z) := -i\beta_n z + \int_{\mathbb{R}} \frac{1 - e^{izu} + iz\sin u}{u^2} M_n(du) = -i\beta_n z + \phi_n(z). \tag{1.5}$$

Remark 1.1. Of course, one can formulate Theorem 1.1 with the function $\mathbb{1}_{|u|\leq 1}$ instead of $\sin u$ under the integral, but for technical reasons we need the Lévy representation (1.3).

Sometimes, especially when the convergence in Theorem 1.1 is that to a stable law (cf. [3]), it is more convenient to consider the random variables in the form

$$X_{k,n} = \frac{\xi_{k,n}}{b_n}, \tag{1.6}$$

where the variables $\xi_{k,n}$, $1 \leq k \leq a_n$, are i.i.d. for each n, and the sequence $(b_n)_{n\geq 1}$ satisfies certain growth assumptions. In what follows we assume that the random variables $X_{k,n}$ are of the form (1.6). We assume that the conditions of Theorem 1.1 hold true, and thus $S_n = \frac{\xi_{1,n}+..+\xi_{k,n}}{b_n}$ converges weakly as $n \to \infty$ to some infinitely divisible random variable ζ. Under some conditions on the sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$, and on the distribution of $\xi_{1,n}$ (cf. [5]), S_n and ζ possess transition probability densities, and the local limit theorem takes place. Taking this result as the starting point we derive in Theorem 2.1 the speed of convergence to the limit density, and illustrated our result by examples.

In order to make the presentation as transparent as possible, we write the main notation in Table 1. Finally, denote by $\hat{m}$ the symmetrization of the measure m, i.e. $\hat{m}(A) := \frac{m(A)+m(-A)}{2}$ for any Borel set $A \in \mathbb{R}$.

2. Main result

We assume that the assumptions below hold true.

A. For any $n \geq 1$ the variable $\xi_{1,n}$ possesses the density $g_n(x)$.

B. $\exists \alpha > 0$ such that $\operatorname{Re}\psi(z) \geq c|z|^\alpha$ for $|z|$ large enough.

C. $\forall \delta > 0$ we have $N(\delta) := \sup_{n\geq 1,\, |z|\geq\delta} |\theta_n(z)| < 1$.

D. $\sup_{n\ge 1}\int_{\mathbb{R}} g_n^2(x)dx<\infty$.
E. $b_n\to\infty$, $\frac{\ln b_n}{a_n}\to 0$ as $n\to\infty$.
F. For $n\ge 1$ one of the conditions below is satisfied:
a) there exist $c(\delta)>0$, and $0<\kappa<2$, such that

$$\int_{\mathbb{R}} \tfrac{(1-\cos(zu))}{u^2}\hat{M}_n(du)\ge c(\delta)|z|^\kappa \quad \text{for all } |z|\le \delta b_n, \tag{2.1}$$

b) $\hat{M}\le \hat{M}_n$ on $\mathbb{R}$.
G. $\exists\delta>0$ such that $\inf\limits_{n\ge 1,\,|z|\le\delta}|\mathrm{Re}\,\theta_n(z)|>0$.

Remark 2.1. Conditions **A** and **C**–**E** are taken from [5]. Condition **B** is different from those assumed in [5]. Namely, in [5] a version of the Kallenberg condition (see [4]) for the sequence of measures $\hat{M}_n$ is assumed, which in fact implies **B**.

Let

$$\begin{aligned}\gamma_n' &:= \sup_{z\in\mathbb{R}}\frac{|\mathrm{Re}\,\phi(z)-\mathrm{Re}\,\phi_n(z)|}{1+z^2},\quad \gamma_n'':=\sup_{z\in\mathbb{R}}\frac{|\mathrm{Im}\,\phi(z)-\mathrm{Im}\,\phi_n(z)|}{1+z^2},\\ \chi_n &:= |a_n\beta_n-\beta|,\end{aligned} \tag{2.2}$$

where β_n has the same meaning as in Theorem 1.1.

Fix $\delta>0$ for which the above conditions hold true. For some fixed $0<\epsilon<1$ put

$$\rho_{\epsilon,\delta}(n):=\max\Big(\chi_n,\gamma_n',\gamma_n'',a_n^{-1},e^{a_n(\ln N(\delta)+\epsilon)}e^{-(1-\epsilon)\mathrm{Re}\,\psi(\delta b_n)}\Big), \tag{2.3}$$

where $N(\delta)$ is defined in **C**.

Theorem 2.1. *Suppose that conditions* (1.1), (1.2), *and* ***A***–***G*** *are satisfied. Then the distributions of* S_n *and* ζ *possess, respectively, the densities* $p_n(x)$ *and* $p(x)$, *and*

$$\sup_{x\in\mathbb{R}}|p_n(x)-p(x)|\le C\rho_{\epsilon,\delta}(n),\quad n\to\infty, \tag{2.4}$$

where $\rho_{\epsilon,\delta}(n)$ *is given by* (2.3).

One can simplify the expression for the speed of convergence at the expense of some additional assumptions on a_n and b_n. We say that a sequence $(c_n)_{n\ge1}$ satisfies condition **H**, if there exist some constants $a,b>0$ such that

$$0<\liminf_{n\to\infty}\frac{c_n}{n^a}\le\limsup_{n\to\infty}\frac{c_n}{n^b}<\infty.$$

Corollary 2.1. *Suppose that conditions of Theorem 2.1 hold true, and assume in addition that the sequences* $(a_n)_{n\ge1}$ *and* $(b_n)_{n\ge1}$ *satisfy* ***H***. *Then*

$$\sup_{x\in\mathbb{R}}|p_n(x)-p(x)|\le C\max\Big(\gamma_n',\gamma_n'',\chi_n,a_n^{-1}\Big). \tag{2.5}$$

Corollary 2.2. *Suppose that conditions* ***A***–***F*** *and* ***H*** *hold true, the densities* $p_n(x)$ *and* $p(x)$ *are symmetric, and*

$$\Phi_n(z)\ge\Phi(z)\quad\forall n\ge1, \tag{2.6}$$

uniformly in $\{z:\ |z|\le\delta b_n\}$. *Then*

$$\sup_{x\in\mathbb{R}}|p_n(x)-p(x)|\le C\Big(\gamma_n'+r(n)\Big), \tag{2.7}$$

where $r(n)=o(n^{-k})$ *as* $n\to\infty$ *for any* $k\ge1$.

Remark 2.2. As one can expect, the oscillation of measures involved in γ_n' and γ_n'' can play the crucial role in the estimation of the speed of convergence. For example, it might be insufficient to know the behaviour of such a "rough estimate" for γ_n' as below:

$$\sup_{z\in\mathbb{R}} \frac{\left|\int_{\mathbb{R}}(1\wedge |uz|^2)(M_n(du)-M(du))\right|}{1+z^2},$$

in particular, when the densities $(g_n)_{n\geq 1}$ have oscillations. Such a situation is illustrated in Example 4.1.

3. Proofs

Proof of Theorem 2.1. Recall that the densities $p_n(x)$ and $p(x)$ can be written as the inverse Fourier transforms of the respective characteristic functions:

$$p(x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{-izx}\Phi(z)dz = (2\pi)^{-1}\int_{\mathbb{R}} e^{-izx-\psi(z)}dz, \tag{3.1}$$

$$p_n(x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{-izx}\Phi_n(z)dz. \tag{3.2}$$

By (3.1) and (3.2) we have

$$\begin{aligned}\Delta_n :&= 2\pi\sup_{x\in\mathbb{R}}|p_n(x)-p(x)| \leq \int_{\mathbb{R}}|\Phi_n(z)-\Phi(z)|dz\\ &\leq \int_{-\delta b_n}^{\delta b_n}|\Phi_n(z)-\Phi(z)|dz + \int_{|z|>\delta b_n}|\Phi_n(z)|dz + \int_{|z|>\delta b_n}|\Phi(z)|dz\\ &=: I_1(n)+I_2(n)+I_3(n),\end{aligned}$$

where $\delta>0$. We estimate the terms $I_k(n)$, $k=1,2,3$, separately.

Estimation of I_1. Observe, that

$$\begin{aligned}|1-e^{x+iy}| &= |1+e^{2x}-2e^x\cos y|^{1/2}\\ &= |(1-e^x)^2+2e^x(1-\cos y)|^{1/2}\\ &\leq |(1-e^x)^2+e^xy^2|^{1/2}\\ &\leq |1-e^x|+e^{x/2}|y|\\ &\leq e^{x_+}(|x|+|y|),\end{aligned} \tag{3.3}$$

where $x,y\in\mathbb{R}$, and $x_+ := \max(x,0)$. Denote

$$H_n(z) := \psi(z) + a_n\ln\theta_n\big(\frac{z}{b_n}\big). \tag{3.4}$$

By (3.3) we get

$$I_1(n) \leq \int_{-\delta b_n}^{\delta b_n} e^{-\mathrm{Re}\,\psi(z)+(\mathrm{Re}\,H_n(z))_+}(|\mathrm{Re}\,H_n(z)|+|\mathrm{Im}\,H_n(z)|)dz.$$

Since $\ln(1-z)\leq -z$ for $z\in(0,1)$, then

$$\mathrm{Re}H_n(z) = \mathrm{Re}\,\psi(z) + a_n\ln\big|\theta_n\big(\frac{z}{b_n}\big)\big| \leq \mathrm{Re}\,\psi(z) - a_n\big(1-|\theta_n\big(\frac{z}{b_n}\big)|\big). \tag{3.5}$$

Observe, that

$$\Big|\theta_n\big(\frac{z}{b_n}\big)\Big| = \int_{\mathbb{R}}\cos(zu)\hat{F}_n(du) = a_n^{-1}\int_{\mathbb{R}}\frac{\cos(zu)}{u^2}\hat{M}_n(du). \tag{3.6}$$

Therefore, by (3.5) and **F** we have for all n large enough and $|z| \le \delta b_n$

$$\begin{aligned} \operatorname{Re}\psi(z) - \big(\operatorname{Re} H_n(z)\big)_+ &\ge a_n\big(1 - |\theta_n\big(\frac{z}{b_n}\big)|\big) \\ &= a_n \int_{\mathbb{R}} (1-\cos(uz))\hat F_n(du) \\ &\ge c(\delta)|z|^{\kappa}, \end{aligned} \tag{3.7}$$

if **F**.a) holds true, or

$$\operatorname{Re}\psi(z) - \big(\operatorname{Re} H_n(z)\big)_+ = \operatorname{Re}\psi(z), \tag{3.8}$$

if **F**.b) is satisfied. On the other hand, for $z \in (0,1)$ we have

$$|\ln z + 1 - z| \le \sum_{k=2}^{\infty} \frac{(1-z)^k}{k} \le \frac{(1-z)^2}{2} \sum_{k=0}^{\infty} (1-z)^k \le \frac{(1-z)^2}{2z}.$$

Then by (3.6) and **G** we derive

$$\begin{aligned} |\operatorname{Re} H_n(z)| &\le \Big|\operatorname{Re}\psi(z) - a_n\big(1 - |\theta_n(z/b_n)|\big)\Big| + a_n\Big|\big(1-|\theta_n(z/b_n)|\big) - \ln|\theta_n(z/b_n)|\Big| \\ &\le \Big|\int_{\mathbb{R}} (1-\cos(zu))u^{-2}(\hat M(du) - \hat M_n(du))\Big| \\ &\quad + 2^{-1} a_n \Big(\int_{\mathbb{R}} (1-\cos(zu))\hat F_n(du)\Big)^2 \cdot \Big(\int_{\mathbb{R}} \cos(zu)\hat F_n(du)\Big)^{-1} \\ &\le c_1 \Big(\gamma_n'(1+z^2) + (2a_n)^{-1}\big(\operatorname{Re}\psi_n(z)\big)^2\Big) \\ &\le c_1 \Big(\gamma_n'(1+z^2) + (2a_n)^{-1}\Big((1+z^2)\gamma_n' + (1+z^2)\Big)^2\Big) \\ &\le c_2(1+z^2)^2(\gamma_n' + a_n^{-1}). \end{aligned} \tag{3.9}$$

Next we estimate $|\operatorname{Im} H_n(z)|$. Observe that for $z = x + iy$, where $x, y \in \mathbb{R}$,

$$\operatorname{Im}\ln z = Arg\, z = \arctan\frac{y}{x},$$

and for all $x \in \mathbb{R}$ we have $|\arctan x - x| \le c_3|x|^3$, where $c_3 > 0$ is some constant. Therefore,

$$\begin{aligned} |\operatorname{Im} H_n(z)| &= \Big|\operatorname{Im}\psi(z) + a_n \operatorname{Im}\,\ln\theta_n\big(z/b_n\big)\Big| \\ &\le \Big| -\beta z + \int_{\mathbb{R}} \frac{z\sin u - \sin(zu)}{u^2} M(du) + a_n \arctan\frac{\operatorname{Im}\theta_n(z/b_n)}{\operatorname{Re}\theta_n(z/b_n)}\Big| \\ &\le \Big| -\beta z + z\int_{\mathbb{R}} \frac{\sin u}{u^2} M_n(du)\Big| + \Big|\int_{\mathbb{R}} \frac{z\sin u - \sin(zu)}{u^2}(M_n - M)(du)\Big| \\ &\quad + \Big|\int_{\mathbb{R}} \frac{\sin(zu) - z\sin u}{u^2} M_n(du)\Big|\Big|\frac{1}{\operatorname{Re}\theta_n(z/b_n)} - 1\Big| \\ &\quad + \Big|\int_{\mathbb{R}} z\frac{\sin u}{u^2} M_n(du)\Big|\Big|\frac{1}{\operatorname{Re}\theta_n(z/b_n)} - 1\Big| + a_n\Big|\arctan\frac{\operatorname{Im}\theta_n(z/b_n)}{\operatorname{Re}\theta_n(z/b_n)} - \frac{\operatorname{Im}\theta_n(z/b_n)}{\operatorname{Re}\theta_n(z/b_n)}\Big| \\ &\le I_{11} + I_{12} + I_{13} + I_{14} + I_{15}. \end{aligned} \tag{3.10}$$

Since $M_n \to M$ and $a_n\beta_n \to \beta$ (cf. (1.2)), we obtain

$$I_{11}(n) \le |z||\beta - a_n\beta_n| = |z|\chi_n. \tag{3.11}$$

For $I_{12}(n)$ we have

$$I_{12}(n) \le \big|\operatorname{Im}\phi_n(z) - \operatorname{Im}\phi(z)\big| \le c_4(1+z^2)\gamma_n''. \tag{3.12}$$

Using **G**, we derive

$$I_{13}(n) \le c_5 a_n^{-1}|\operatorname{Im}\phi_n(z)|\operatorname{Re}\phi_n(z) \le c_6 a_n^{-1}(1+z^2)^2. \tag{3.13}$$

Analogously,

$$I_{14}(n) \le c_7\beta_n a_n^{-1}|z|\operatorname{Re}\phi_n(z) \le c_8 a_n^{-1}|z|(1+z^2). \tag{3.14}$$

Finally, for I_{15} we derive

$$\begin{aligned} I_{15} &\le a_n c_3\Big|\frac{\operatorname{Im}\theta_n(z/b_n)}{\operatorname{Re}\theta_n(z/b_n)}\Big|^3 \le c_9 a_n\Big|\int_{\mathbb{R}}\sin(zu)F_n(du)\Big|^3 \\ &\le c_9 a_n^{-2}\Big|\int_{\mathbb{R}}\frac{\sin(zu)-z\sin u}{u^2}M_n(du) + z\int_{\mathbb{R}}\frac{\sin u}{u^2}M_n(du)\Big|^3 \\ &= c_9 a_n^{-2}\Big|\operatorname{Im}\phi_n(z) - z\beta_n\Big|^3 \\ &\le c_{10}a_n^{-2}(1+z^2)^3. \end{aligned}$$

Thus, we arrive at

$$\begin{aligned} I_1(n) &\le c_{11}\max(\kappa_n,\gamma_n',\gamma_n'',a_n^{-1})\int_0^{\delta b_n} e^{-c(\delta)z^\kappa}(1+z^2)^3dz \\ &\le c_{12}\max(\kappa_n,\gamma_n',\gamma_n'',a_n^{-1}). \end{aligned} \tag{3.15}$$

Estimation of I_2. We have by **C** and **D**

$$\begin{aligned} I_2(n) &= \int_{|z|\ge\delta b_n}\Big|\theta_n\Big(\frac{z}{b_n}\Big)\Big|^{a_n}dz = b_n\int_{|x|\ge\delta}|\theta_n(x)|^{a_n}dx \\ &\le b_n N(\delta)^{a_n-2}\int_{|x|\ge\delta}|\theta_n(x)|^2dx \\ &\le b_n N(\delta)^{a_n-2}\sup_{n\ge1}\int_{\mathbb{R}} g_n^2(x)dx \\ &= c_{13}b_n e^{a_n\ln N(\delta)}. \end{aligned}$$

Take $\varepsilon > 0$ such that $\ln N(\delta)+\varepsilon < 0$. By **E**, we have $\frac{\ln b_n}{a_n} \to 0$ as $n\to\infty$. Thus, without loss of generality we may assume that $\frac{\ln b_n}{a_n} \le \varepsilon$ for all $n \ge 1$, which implies

$$b_n e^{a_n\ln N(\delta)} \le e^{-a_n|\ln N(\delta)+\varepsilon|}.$$

Estimation of I_3. For any $\epsilon > 0$ we have

$$I_3(n) \le c_{14}\int_{|z|\ge\delta b_n} e^{-\operatorname{Re}\psi(z)}dz \le c_{15}(\epsilon)e^{-(1-\epsilon)\operatorname{Re}\psi(\delta b_n)}.$$

Summarizing the estimates for $I_i(n)$, $i = 1,2,3$, we arrive at $\Delta_n \le C\rho_{\epsilon,\delta}(n)$. □

Proof of Corollaries 2.1 and 2.2. Clearly, the proofs are obtained as slight modifications of the proof of Theorem 2.1. Since a_n and b_n satisfy condition **H**, the terms $I_2(n)$ and $I_3(n)$ decay as $o(n^{-k})$, $n\to\infty$, for any $k \ge 0$. This implies the statement of Corollary 2.1. To complete the proof of Corollary 2.2, we need to estimate more precisely $I_1(n)$. Let us look more closely on the properties of the function $H_n(z)$ from (3.4). Since both

$p_n(x)$ and $p(x)$ are symmetric, the function $H_n(z)$ is real-valued. Further, condition (2.6) implies that $H_n(z) \geq 0$. Therefore, instead of (3.9) we get

$$H_n(z) \leq \psi(z) + a_n \ln \theta_n(z/b_n) \leq \gamma_n'(1+z^2),$$

which implies

$$I_1(n) \leq C\gamma_n'.$$

□

4. Examples

Example 4.1. Let $(\xi_n)_{n\geq 1}$ be i.i.d. random variables with probability density

$$g(u) = c_\alpha \frac{(1-\cos u)}{|u|^{1+\alpha}}, \quad u \in \mathbb{R}, \quad 0 < \alpha < 2.$$

Then one can check (using Theorem 1.1 with $a_n = n$ and $b_n = n^{1/\alpha}$) that

$$S_n := \frac{\xi_1 + .. + \xi_n}{n^{1/\alpha}} \Rightarrow \zeta,$$

where ζ is a symmetric α-stable distribution. In this case the respective measure $M(du)$ in (1.3) is equal to $c_\alpha |u|^{1-\alpha} du$, and after the appropriate choice of c_α we have $\psi(z) = |z|^\alpha$. For example, in the case $\alpha = 1$ we must chose $c_\alpha = 1/\pi$. Clearly, conditions **A**, **B**, **D** and **G** are satisfied. Condition **C** is the Cramer condition (cf. [6]) for the characteristic function of ξ_1, which is satisfied since the law of ξ_1 is absolutely continuous.

Let us check condition **F**. Consider

$$\int_0^\infty \frac{1-\cos(zu)}{u^{1+\alpha}} \cos(n^{1/\alpha} u) du = |z|^\alpha \int_0^\infty \frac{1-\cos u}{u^{1+\alpha}} \cos(n^{1/\alpha} u/z) du.$$

We need to estimate from above

$$I(\alpha, k) := \int_0^\infty \frac{1-\cos v}{v^{1+\alpha}} \cos(kv) dv.$$

Note that for $\alpha = 1$ we have (cf. [1, p.28])

$$I(1,k) = \frac{\pi}{2}(1-|k|)_+. \tag{4.1}$$

It is also possible to calculate $I(\alpha, k)$ for $\alpha \in (0,2)\backslash\{1\}$. Integrating by parts, we get for any $k > 0$

$$\begin{aligned} I(\alpha,k) &= \frac{\sin kv}{k} \cdot \frac{1-\cos v}{v^{1+\alpha}}\Big|_0^\infty - \frac{1}{k}\int_0^\infty \sin(kv) \frac{v \sin v - (1+\alpha)(1-\cos v)}{v^{2+\alpha}} dv \\ &= -\frac{1}{k}\int_0^\infty \sin(kv) \frac{v \sin v - (1+\alpha)(1-\cos v)}{v^{2+\alpha}} dv. \end{aligned}$$

The integrals

$$I_1(\alpha,k) := \int_0^\infty \frac{\sin(kv)\sin v}{v^{1+\alpha}} dv$$

and

$$I_2(\alpha,k) := \int_0^\infty \frac{\sin(kv)\sin^2(v/2)}{v^{2+\alpha}} dv.$$

can be calculated explicitly, see [1, p.77–78]. In such a way, we have

$$I_1(\alpha,k) = \frac{\pi}{4} \frac{|k+1|^\alpha - |k-1|^\alpha}{\Gamma(1+\alpha)\sin(\pi\alpha/2)} \sim c_{\alpha,1}|k|^{\alpha-1}, \tag{4.2}$$

$$I_2(\alpha,k) = 2^{-2}\Gamma(-1-\alpha)\cos(\pi\alpha/2)\big[2|k|^{\alpha+1} - |k+1|^{\alpha+1} - |k-1|^{\alpha+1}\big] \sim c_{\alpha,2}|k|^{\alpha-1}, \tag{4.3}$$

as $k \to \infty$, where $c_{\alpha,1} := \frac{\pi}{2\Gamma(1+\alpha)\sin(\pi\alpha/2)}$, $c_{\alpha,2} = 2^{-1}\alpha(\alpha+1)\Gamma(-1-\alpha)\cos(\pi\alpha/2)$. Thus, we obtain the exact expression for $I(\alpha, k)$, from which we derive

$$I(\alpha, k) \sim c_{3,\alpha} k^{\alpha-1}, \quad k \to \infty,$$

where $c_{3,\alpha} = c_{1,\alpha} + 2(\alpha+1)c_{2,\alpha}$. Finally, for $|z| \le \delta n^{1/\alpha}$ with $\delta > 0$ is small enough, we get

$$a_n \int_{\mathbb{R}} (1-\cos(zu))\hat{F}_n(du) = |z|^\alpha \left(1 - 2c_\alpha I\Big(\alpha, \frac{n^{1/\alpha}}{|z|}\Big)\right) \ge c(\delta)|z|^\alpha, \quad \alpha \in (0,2),$$

where $c(\delta) > 0$ is some constant.

Let us calculate the order of convergence. From above, we have for $\alpha \in (0,2)$

$$|\operatorname{Re}\psi(z) - \operatorname{Re}\psi_n(z)| = \Big| \int_{\mathbb{R}} \frac{(1-\cos(zu))}{|u|^{1+\alpha}} \cos(n^{1/\alpha}u)du \Big| \le \frac{Cz^2}{n^{(2-\alpha)/\alpha}}.$$

Thus, by Corollary 2.1 we arrive at

$$\rho(n) \le \begin{cases} Cn^{-1}, & 0 < \alpha < 1, \\ Cn^{-\frac{2-\alpha}{\alpha}}, & 1 \le \alpha < 2. \end{cases} \tag{4.4}$$

Example 4.2. Suppose now that $\xi_{1,n}$ possesses the distribution density

$$g_n(u) := \tfrac{1}{2nu\sinh(u/n)} \mathbb{1}_{|u|\ge 1},$$

and $a_n = b_n = n$. Conditions **A**, **C**–**E** were already checked in [5], in particular, it was shown that S_n converges in distribution to a random variable ζ possessing a hyperbolic cosine distribution, i.e. the distribution density of ζ is $p(x) = \frac{1}{\pi\cosh x}$. Since in this case

$$a_n f_n(u) = \frac{1}{2u\sinh u} 1_{|u|\ge\frac{1}{n}} \uparrow \frac{1}{2u\sinh u} =: f(u) \quad \text{as } n \to \infty,$$

the function

$$\psi(z) = \int_{\mathbb{R}} (1-\cos(uz)) f(u)du$$

satisfies condition **B** with $\alpha = 1$. Let us check **F** for $\kappa = 1$. Since for $|z| \le 1$ we have $1 - \cos z \ge (1-\cos 1)z^2$, then estimating $\frac{u}{\sinh u}$ from below for small u by a constant, we get

$$\begin{aligned} a_n \int_{\mathbb{R}} (1-\cos(zu)) f_n(u)du &\ge n(1-\cos 1)\int_{|uz|\le 1} (zu)^2 f_n(u)du \\ &\ge (1-\cos 1)|z| \inf_{|z|\le\delta n} |z| \int_{1/n}^{1/|z|} \frac{u}{\sinh u}du \\ &\ge c_1|z| \inf_{|z|\le\delta n} |z| \left(\frac{1}{|z|} - \frac{1}{n}\right) \\ &\ge c_1(1-\delta)|z|, \end{aligned}$$

uniformly in $\{z : \ |z| \le \delta n\}$. Thus, condition **F** holds true.

It remains to check condition **G**. Let $|z| \le \delta$. Since the function $r\sinh(u/r)$ is increasing in r, we have by the dominated convergence theorem

$$\begin{aligned} \inf_{n\ge 1} |\theta_n(z)| = \inf_{n\ge 1} \Big| \int_{u\ge 1} \frac{\cos(zu)}{nu\sinh(u/n)}du \Big| = \lim_{n\to\infty} |\theta_n(z)| = \int_1^\infty \frac{\cos(zu)}{u^2}du \\ \ge \cos 1 \int_1^{1/\delta} \frac{du}{u^2}, \end{aligned}$$

which gives **G**.

Finally, using Corollary 2.1 we arrive at

$$\sup_{x\in\mathbb{R}} |p_n(x) - p(x)| \le \frac{C}{n}.$$

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