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THE VARIANCE OF THE NUMBER OF WINDINGS OF THE RANDOM FIELD ALONG THE PLANAR CURVE

This article is devoted to the study of the distribution of a winding number of a random vector field along the fixed plane curve. For some case of Gaussian homogeneous isotropic vector field, an explicit expression for the variance of the number of windings along the planar curve is given.

1. Introduction

A considerable attention, in particular, with regard for the study of turbulence, is paid to the investigation of the topological characteristics of random vector fields. In this work, we study the distribution of a winding number of a random vector field along the fixed plane curve. For this, we use an expression of a winding number of a vector field on a planar closed curve as a sum of indices of the critical points of the field and apply some results of the book [1] concerning the expectation of the number of points of a random vector field satisfying certain conditions. All necessary definitions and formulations of the statements we refer to are given in Section 3. In Section 2, we formulate our main result, which gives an expression for the expectation of the second moment of the index of a vector field on a planar closed curve. In Section 5, we apply it to the special case of a homogeneous isotropic vector field and obtain an explicit expression for the variance of the index of a curve with respect to this field. The proof of the main result is given in Section 4.

2. Main result

Here, we formulate the main result of the article. For this, we need the definition of the index of a continuous vector field with respect to the critical point. For reader's convenience, the definition of the index and its properties are given in Section 3.

Theorem 2.1. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a random field on $\mathbb{R}^2$, and let Γ be a fixed piecewise smooth simple (i.e., without self-intersections) closed curve in $\mathbb{R}^2$ that bounds some domain T.*

We assume that the field f satisfies the following conditions:

a) f and $\frac{\partial f_i}{\partial t_j}$ are continuous on T with probability 1 and have bounded moments of the order 4 on T;

b) for all $t \in T$, the density of the distribution $p_{f(t)}(x)$ of the random vector $f(t)$ is continuous at $x = 0$;

b′) for all $\tilde{t} \in \tilde{T} = \{(t^1, t^2) \in T^2 : t^1 \neq t^2\}$, the density of the distribution $p_{\tilde{f}(\tilde{t})}(x_1, x_2)$ of the random vector $\tilde{f}(\tilde{t}) = (f(t^1), f(t^2))$ is continuous at $(x_1, x_2) = (0, 0)$;

c) for all $t \in T$, the conditional density of the distribution $p_t(x \mid \nabla f(t))$ of the random vector $f(t)$ conditioned on $\nabla f(t)$ exists, is bounded, and is continuous at $x = 0$;

c′) for all $\tilde{t} \in \tilde{T}$, the conditional density of the distribution $p_{\tilde{t}}(\tilde{x} \mid \nabla \tilde{f}(\tilde{t}))$ of the random vector $\tilde{f}(\tilde{t})$ conditioned on $\nabla \tilde{f}(\tilde{t})$ exists, is bounded, and continuous at $\tilde{x} = (0, 0)$;

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d) for all $t \in T$, the conditional density of the distribution $p_t(z \mid f(t) = x)$ of $\nabla f(t)$ conditioned on $f(t) = x$ is continuous in z and x;

d′) for all $\tilde{t} \in \tilde{T}$, the conditional density of the distribution $p_{\tilde{t}}(z \mid \tilde{f}(\tilde{t}) = \tilde{x})$ of $\nabla \tilde{f}(\tilde{t})$ conditioned on $\tilde{f}(\tilde{t}) = \tilde{x}$ is continuous in z and $\tilde{x}$;

e) the modulus of continuity on T of each of the components f, ∇f satisfies the relation $\mathbb{P}(\omega(\eta) > \epsilon) = o(\eta^4)$, when $\eta \downarrow 0$ for each $\epsilon > 0$.

Let $\mathrm{ind}_\Gamma f$ be the index of a curve Γ with respect to the field f. Then the following relations hold:

$$\mathbb{E}(\mathrm{ind}_\Gamma f) = \int_T \mathbb{E}\{\det \nabla f(t) \mid f(t) = 0\} p_t(0) dt,$$

$$\mathbb{E}(\mathrm{ind}_\Gamma f)^2 = \int_{T^2} \mathbb{E}\{\det \nabla f(t^1) \det \nabla f(t^2) \mid f(t^1) = f(t^2) = 0\} p_{(t^1,t^2)}(0) dt^1 dt^2 + \\ + \int_T \mathbb{E}\{|\det \nabla f(t)| \mid f(t) = 0\} p_t(0) dt,$$

where $p_t(0)$ is the density of the distribution of $f(t)$ at 0, and $p_{(t^1,t^2)}(0)$ is the density of the distribution of the field $\tilde{f}(t^1, t^2) = (f(t^1), f(t^2))$ at 0.

Remark 2.1. The conditions of the theorem are easily checked for Gaussian fields. A sufficient condition for the continuity of a centered Gaussian field is, for example, its covariance function being Lipschitz. All the conditional distributions in the Gaussian case can be obtained from the theorem of normal correlation. So, in this case, the main point to check is that the corresponding covariance matrix is nondegenerate.

In Section 5, we will consider the following example of a homogeneous isotropic Gaussian vector field f satisfying the conditions of the theorem:

$$f(u) = \begin{pmatrix} f_1(u) \\ f_2(u) \end{pmatrix} = \begin{pmatrix} \int_{\mathbb{R}^2} \phi(u+v) W_1(dv) \\ \int_{\mathbb{R}^2} \phi(u+v) W_2(dv) \end{pmatrix},$$

where $\phi \in C_0^\infty(\mathbb{R}^2)$, and W_1 and W_2 are two independent Brownian sheets on $\mathbb{R}^2$.

3. Preliminaries

Here, we give some necessary definitions.

Definition 3.1 ([2], §1)**.** Let $\Phi : \Gamma \to \mathbb{R}^2$ be a continuous vector field on some continuous curve $\Gamma = \{(x(t), y(t)) \mid t \in [a, b]\}$. Assume that the vector-valued function $\Phi(t) = \Phi(x(t), y(t))$ is continuous and non-zero on $[a, b]$. For each $t \in [a, b]$, we define an angle between $\Phi(t)$ and $\Phi(a)$, counted from $\Phi(a)$ counterclockwise. This angle is a multiple-valued function of t. We denote, by $\theta(t)$, a continuous branch of this function that becomes zero at $t = a$ and call it the angle function of the field Φ on the curve Γ. The increment of the angle function $\theta(t)$ on the whole segment $[a, b]$, counted in the number of the whole rotations, i.e., the quantity $\gamma(\Phi, \Gamma) = \frac{1}{2\pi}(\theta(b) - \theta(a)) = \frac{1}{2\pi}\theta(b)$, is called the winding number of the field Φ on Γ.

Definition 3.2 ([2], §2)**.** Let Φ be a continuous vector field on a closed curve Γ with values in $\mathbb{R}^2$ without zero vectors. Divide Γ with the points M_1 and M_3 into the parts: $M_1 M_2 M_3$ and $M_3 M_4 M_1$, and each of these parts will be considered to be oriented according to the positive direction of the girdle of the closed contour Γ. Define the number of windings of the field Φ on Γ (or "the index of the curve Γ with respect to the field f") as the sum of the numbers of windings of Φ on the curves $M_1 M_2 M_3$ and $M_3 M_4 M_1$ (see Fig. 1).

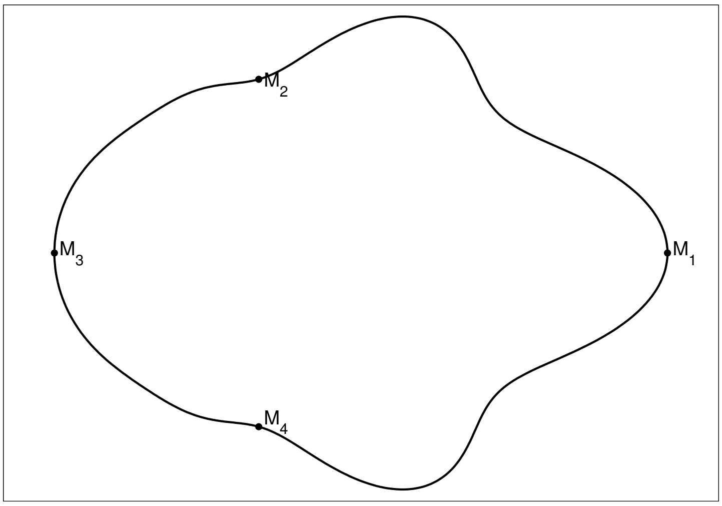


FIGURE 1

Definition 3.3 ([2], §3)**.** Assume that the vector field Φ is defined on the domain Ω and is continuous on this domain, maybe, except for some finite number of points. Those points where the field is discontinuous, as well as those where the field equals zero, are called critical points of the field Φ. A critical point M_0 is called isolated, if there are no other critical points in some its neighborhood.

Definition 3.4 ([2], §3)**.** Assume that the vector field Φ satisfies the assumptions of the previous definition. Let M_0 be an isolated critical point of the field Φ. It is easily seen that the field Φ has no zero vectors on any circle of a sufficiently small radius with the center at the point M_0. The number of windings of any such circle is called the index of the critical point M_0.

It can be shown that this number does not depend on the choice of a circle, and so the definition is correct.

Definition 3.5 ([2], §3)**.** Let Γ be a fixed 2-dimensional piecewise smooth curve. Assume that the vector field Φ on the closed domain $\overline{\Omega}$ with the boundary Γ has a finite number of critical points inside Γ. The sum of indices of these points is called the algebraic number of critical points of the field Φ inside $\overline{\Omega}$.

The following result concerning the algebraic number of critical points holds true. It reduces finding the number of windings of a vector field to computing the sum of the indices of critical points of this field.

Proposition 3.1 ([2], §3)**.** *The algebraic number of critical points of a continuous vector field Φ inside the closed domain with boundary Γ is equal to the number of windings of the field Φ along Γ.*

To compute the indices of critical points, we will use the following result that generalizes the statement that, for any nondegenerate linear vector field $\Phi_0(x, y) = (a_1x + b_1y, a_2x + b_2y)$, the index γ of the critical point $(0, 0)$ is $\gamma = \operatorname{sign}\Delta$, where $\Delta = \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$.

Proposition 3.2 ([2], §6). *Let (x_0, y_0) be a fixed point in $\mathbb{R}^2$. Define the field $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\Phi(x, y) = (\phi(x, y), \psi(x, y))$, where ϕ and ψ are functions differentiable at (x_0, y_0), and $\phi(x_0, y_0) = \psi(x_0, y_0) = 0$. If*

$$\Delta = \det \begin{pmatrix} \phi'_x(x_0, y_0) & \phi'_y(x_0, y_0) \\ \psi'_x(x_0, y_0) & \psi'_y(x_0, y_0) \end{pmatrix} \neq 0,$$

then the critical point (x_0, y_0) of the field is isolated, and its index γ equals

$$\gamma = \operatorname{sign} \det \begin{pmatrix} \phi'_x(x_0, y_0) & \phi'_y(x_0, y_0) \\ \psi'_x(x_0, y_0) & \psi'_y(x_0, y_0) \end{pmatrix}.$$

We will also use the following result that concerns the expectation of the number of critical points of a random vector field that satisfy a certain condition.

Proposition 3.3 ([1], p. 284). *Let $T \subset \mathbb{R}^N$ be a compact set such that its boundary ∂T has finite Hausdorff measure H_{N-1}, and let $B \subset \mathbb{R}^K$ be an open subset of $\mathbb{R}^K$ such that ∂B has Hausdorff dimension $K - 1$. Let $f = (f_1, \dots, f_N)$ and $g = (g_1, \dots, g_K)$ be some N-parameter random fields on T with values in $\mathbb{R}^N$ and $\mathbb{R}^K$, respectively. We assume that the following conditions are satisfied:*

a) all the components of the random fields $f, \nabla f, g$ are continuous with probability 1 and have finite second moments;

b) for all $t \in T$, the marginal densities $p_t(x)$ of $f(t)$ are continuous at $x = u$;

c) the conditional densities $p_t(x \mid \nabla f(t), g(t))$ of the field $f(t)$ given $g(t)$ and $\nabla f(t)$ are bounded and continuous at $x = u$, uniformly in $t \in T$;

d) the conditional densities $p_t(z \mid f(t) = x)$ of $\det \nabla f(t)$ given $f(t) = x$ are continuous in z and x in neighborhoods of 0 and u, respectively, uniformly in $t \in T$;

e) the conditional densities $p_t(z \mid f(t) = x)$ of $g(t)$ given $f(t) = x$ are continuous for all z and for x in a neighborhood of u, uniformly in $t \in T$;

f) the following restriction on moments holds:

$$\sup_{t \in T} \max_{1 \leq i,j \leq N} \mathbb{E} \left\{ \left| \frac{\partial f_i}{\partial t_j}(t) \right|^N \right\} < \infty;$$

g) the modulus of continuity (with respect to the Euclidian norm) of each of the components of f, ∇f, and g satisfies the inequality

$$\mathbb{P}(\omega(\eta) > \epsilon) = o(\eta^N), \eta \downarrow 0$$

for each $\epsilon > 0$.

Then, if N_u means the number of points in T, for which $f(t) = u \in \mathbb{R}^n$ and $g(t) \in B$, the following relation holds:

$$\mathbb{E}N_u = \int_T \mathbb{E}\{|\det \nabla f(t)| \mathbb{1}_B(g(t)) \mid f(t) = u\} p_t(u) dt.$$

Remark 3.1. In our considerations (see Section 4), we will apply proposition 3.3 in the case $g(t) = \nabla f(t)$.

But the proof of this result in [1] used the existence of the joint distribution of the random variables $f(t), \det \nabla f(t)$, and $g(t)$, which does not hold in our situation. However, the proof can be easily made in the case $g(t) = \nabla f(t)$ as well. All the computations in [1] stay nearly the same. Indeed, let, as in [1], $\delta_\epsilon : \mathbb{R}^N \to \mathbb{R}$ be a function defined by $\delta_\epsilon = C_\epsilon^{-1} \mathbb{1}_{B_\epsilon}$, where C_ϵ is a volume of B_ϵ.

Denote, by $p_t(x, \nabla y)$, the joint distribution density of $f(t)$ and $\nabla f(t)$ and, by $p_t(\nabla y)$, the distribution density of $\nabla f(t)$. Let $p_t(x)$ be the distribution density of $f(t)$, and

$p_t(x|\nabla y)$ be the conditional distribution density of $f(t)$ given $\nabla f(t)$. Then, for example, the computations at page 272 of [1],

$$\mathbb{E}N_\epsilon(T) = \int\limits_T \int\limits_{\mathbb{R}^N\times\mathbb{R}^{N^2}\times B} \delta_\epsilon(x)|\det\nabla y|p_t(x,\nabla y,v)dxd\nabla ydv =$$

$$= \int\limits_T dt \int\limits_{\mathbb{R}^{N^2}\times B} |\det\nabla y|p_t(\nabla y,v)d\nabla ydv \times \int\limits_{\mathbb{R}^N} \delta_\epsilon(x)p_t(x\mid\nabla y,v)dx,$$

now appear as follows:

$$\mathbb{E}N_\epsilon(T) = \int\limits_T \int\limits_{\mathbb{R}^N\times B} \delta_\epsilon(x)|\det\nabla y|p_t(x,\nabla y)dxd\nabla y =$$

$$= \int\limits_T dt \int\limits_B |\det\nabla y|p_t(\nabla y)d\nabla y \times \int\limits_{\mathbb{R}^N} \delta_\epsilon(x)p_t(x\mid\nabla y)dx.$$

Later, we will search for the explicit expression for the expectation and the variance of the index of a curve with respect to a random homogeneous isotropic vector field.

Definition 3.6 ([3])**.** Let $U = \{U(x) \mid x\in\mathbb{R}^2\}$ be a vector-valued random field on $\mathbb{R}^2$. For any transformation $T:\mathbb{R}^2\to\mathbb{R}^2$, we denote, by UT, the random field defined by $UT(x) = U(T(x))$. The field U is called homogeneous if the field UT has the same distribution as U for any shift transformation $T:\mathbb{R}^2\to\mathbb{R}^2$, i.e., for any transformation of the form $T(x) = T_z(x) = x+z$, where $z\in\mathbb{R}^2$ is a constant vector. The field U is called invariant with respect to rotations, if the random field $GU(G^Tx), x\in\mathbb{R}^2$ has the same distribution as U, for all rotation matrices $G = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}$. A homogeneous random field, which is invariant under rotations, is called polarized. If a polarized vector field is invariant under reflections as well, it is called isotropic.

We now describe some properties of such fields, following the article [Zirbel, Woyczyńsky]. It is easily derived from the conditions of a random field being homogeneous and isotropic that the covariance matrix of the components of the field has a special form described by the following result.

Proposition 3.4 ([3])**.** *A matrix-valued function b is a covariance matrix of the components of some homogeneous isotropic vector field f, if and only if it has the form*

$$b_{ij}(z) = \delta_{ij}b_N(\|z\|) + \frac{z^iz^j}{\|z\|^2}(b_L(\|z\|) - b_N(\|z\|)).$$

Moreover, $b(0) = b_N(0)I, b_L(0) = b_N(0)$, and the functions b_N, b_L are of the form

$$b_L(r) = \int\limits_0^\infty J_1'(r\alpha)\Phi_P(d\alpha) + \int\limits_0^\infty \frac{J_1(r\alpha)}{r\alpha}\Phi_S(d\alpha),$$

$$b_N(r) = \int\limits_0^\infty \frac{J_1(r\alpha)}{r\alpha}\Phi_P(d\alpha) + \int\limits_0^\infty J_1'(r\alpha)\Phi_S(d\alpha),$$

where Φ_P, Φ_S are some positive finite masures on $[0,\infty)$.

If Φ_P and Φ_S have finite fourth moments, then the functions b_L, b_N are four times continuously differentiable, and

$$b_L(r) = b_0 - \frac{1}{2}\beta_L r^2 + O(r^4), r\to 0,$$

$$b_N(r) = b_0 - \frac{1}{2}\beta_N r^2 + O(r^4), r \to 0,$$

where

$$\beta_L = \frac{3}{8}\int_0^\infty \alpha^2 \Phi_P(d\alpha) + \frac{1}{8}\int_0^\infty \alpha^2 \Phi_S(d\alpha)$$

and

$$\beta_N = \frac{1}{8}\int_0^\infty \alpha^2 \Phi_P(d\alpha) + \frac{3}{8}\int_0^\infty \alpha^2 \Phi_S(d\alpha).$$

Remark 3.2. We denote, by $J_n(x)$, the Bessel function of the n-th order,

$$x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0.$$

Later, we will use the following properties of these functions that can be found in [4]:

$$J_0(x) = J_1'(x) + \frac{J_1(x)}{x},$$

$$J_0'(x) = -J_1(x),$$

$$J_0(br) = \frac{1}{2\pi}\int_0^{2\pi} e^{ibr\sin\theta} d\theta.$$

4. Proof of the main result

Here, we prove Theorem 2.1.

Let N_+ be the number of critical points of f inside T (i.e., points $t \in T$ where $f(t) = 0$) with the index 1, N_- be the number of critical points inside T with the index -1:

$$N_+ = \#\{t \in T : f(t) = 0, \mathrm{ind}_t f = 1\}, N_- = \#\{t \in T : f(t) = 0, \mathrm{ind}_t f = -1\}.$$

It is known that the field f has no critical points with $\det \nabla f(t) = 0$ with probability 1 (lemma 11.2.11 from [1]).

So, with probability 1, we have

$$N_+ = \#\{t \in T \mid f(t) = 0, \det \nabla f(t) > 0\},$$

$$N_- = \#\{t \in T \mid f(t) = 0, \det \nabla f(t) < 0\},$$

and

$$\mathrm{ind}_\Gamma f = N_+ - N_-.$$

Denote $\tilde{T} = \{(t^1, t^2) \in T^2 : t^1 \neq t^2\}, T_2^\delta = \{(t^1, t^2) \in T^2 : \|t^1 - t^2\| \geq \delta\}$. Define the vector fields $\tilde{f}$, $\tilde{g}$ on $\tilde{T}$ by $\tilde{f}(\tilde{t}) = (f(t^1), f(t^2)), \tilde{g}(\tilde{t}) = (\nabla f(t^1), \nabla f(t^2))$, where $\tilde{t} = (t^1, t^2) \in \tilde{T}$. We also define a vector field g on T by $g(t) = \nabla f(t), t \in T$. Let

$$B_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \cong \mathbb{R}^4 : ad - bc > 0 \right\},$$
$$B_{-1} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \cong \mathbb{R}^4 : ad - bc < 0 \right\}.$$

Our proof is analogous to that of Theorem 11.5.1 in [1]. We fix $\delta > 0$ and a pair (s_1, s_2), where $s_i \in \{1, -1\}$, and apply Proposition 3.3 to $\tilde{f}, \tilde{g}, T_2^\delta, B = \prod_{i=1}^{2} B_{s_i}$. The conditions of this proposition are trivially checked in our case. We get

$$\begin{aligned}
&\mathbb{E}\#\{(t^1,t^2)\in T_2^\delta : f(t^i)=0, g(t^i)\in B_{s_i}, i=1,2\} = \\
&\qquad = \mathbb{E}\#\{\tilde{t}\in T_2^\delta : \tilde{f}(\tilde{t})=0, \tilde{g}(\tilde{t})\in \prod_{i=1}^{2} B_{s_i}\} = \\
&\qquad = \int\limits_{T_2^\delta} \mathbb{E}\{|\det\nabla\tilde{f}(\tilde{t})| \mathbb{1}_{\prod_{i=1}^{2} B_{s_i}}(\tilde{g}(\tilde{t})) \mid \tilde{f}(\tilde{t})=0\} p_{\tilde{t}}(0)d\tilde{t} = \\
&\qquad = \int\limits_{T_2^\delta} \mathbb{E}\{|\det\nabla\tilde{f}(\tilde{t})| \prod_{i=1}^{2} \mathbb{1}_{\operatorname{sign}\det\nabla f(t^i)=s_i} \mid \tilde{f}(\tilde{t})=0\} p_{\tilde{t}}(0)d\tilde{t} = \\
&\qquad = s_1 s_2 \int\limits_{T_2^\delta} \mathbb{E}\{\prod_{i=1}^{2} \det\nabla f(t^i) \mathbb{1}_{\operatorname{sign}\det\nabla f(t^i)=s_i} \mid \tilde{f}(\tilde{t})=0\} p_{\tilde{t}}(0)d\tilde{t}.
\end{aligned}$$

When $\delta\to 0$, we have, with probability 1,

$$\begin{aligned}
&\#\{(t^1,t^2)\in T_2^\delta : f(t^i)=0, g(t^i)\in B_{s_i}, i=1,2\} \to \\
&\qquad \to \#\{(t^1,t^2)\in \tilde{T} : f(t^i)=0, g(t^i)\in B_{s_i}, i=1,2\},
\end{aligned}$$

and this convergence is monotone. So, by monotone convergence,

$$\begin{aligned}
&\mathbb{E}\#\{(t^1,t^2)\in T_2^\delta : f(t^i)=0, g(t^i)\in B_{s_i}, i=1,2\} \to \\
&\qquad \to \mathbb{E}\#\{(t^1,t^2)\in \tilde{T} : f(t^i)=0, g(t^i)\in B_{s_i}, i=1,2\}.
\end{aligned}$$

On the other hand, we have (also by monotone convergence):

$$\begin{aligned}
&\int\limits_{T_2^\delta} \mathbb{E}\{|\det\nabla\tilde{f}(\tilde{t})| \prod_{i=1}^{2} \mathbb{1}_{\operatorname{sign}\det\nabla f(t^i)=s_i, i=1,2} \mid \tilde{f}(\tilde{t})=0\} p_{\tilde{t}}(0)d\tilde{t} \to \\
&\qquad \to \int\limits_{T^2} \mathbb{E}\{|\det\nabla\tilde{f}(\tilde{t})| \prod_{i=1}^{2} \mathbb{1}_{\operatorname{sign}\det\nabla f(t^i)=s_i} \mid \tilde{f}(\tilde{t})=0\} p_{\tilde{t}}(0)d\tilde{t}.
\end{aligned}$$

Therefore, setting $\delta\to 0$, we get

$$\begin{aligned}
&\mathbb{E}\#\{(t^1,t^2)\in \tilde{T} : f(t^i)=0, g(t^i)\in B_{s_i}, i=1,2\} = \\
&\qquad = s_1 s_2 \int\limits_{T^2} \mathbb{E}\{\prod_{i=1}^{2} \det\nabla f(t^i) \mathbb{1}_{\operatorname{sign}\det\nabla f(t_i)=s_i} \mid \tilde{f}(\tilde{t})=0\} p_{\tilde{t}}(0)d\tilde{t},
\end{aligned}$$

or

$$\begin{aligned}
&s_1 s_2 \mathbb{E}\#\{(t^1,t^2)\in \tilde{T} : f(t^i)=0, g(t^i)\in B_{s_i}\} = \\
&\qquad = \int\limits_{\tilde{T}} \mathbb{E}\{\prod_{i=1}^{2} \det\nabla f(t^i) \mathbb{1}_{\operatorname{sign}\det\nabla f(t^i)=s_i} \mid \tilde{f}(\tilde{t})=0\} p_{\tilde{t}}(0)d\tilde{t}.
\end{aligned}$$

Now, summing over all four sequences (s_1, s_2) with $s_i = \pm 1$, we get

$$\sum_{s_1,s_2:s_i=\pm 1} s_1 s_2 \mathbb{E}\#\{(t^1,t^2) \in \tilde{T} : f(t^i) = 0, g(t^i) \in B_{s_i}\} =$$

$$= \sum_{s_1,s_2:s_i=\pm 1} \int_{\tilde{T}} \mathbb{E}\{\prod_{i=1}^{2} [\det \nabla f(t^i) \mathbb{1}_{\operatorname{sign}\det \nabla f(t^i)=s_i}] \mid \tilde{f}(\tilde{t}) = 0\} p_{\tilde{t}}(0) d\tilde{t}.$$

On the left-hand side, we have

$$\mathbb{E} \sum_{s_1,s_2:s_i=\pm 1} s_1\ s_2 \#\{(t^1,t^2) \in \tilde{T} : f(t^i) = 0, \operatorname{sign}\det \nabla f(t^i) = s_i\} =$$

$$= \mathbb{E}\{(N_+(N_+ - 1) + N_-(N_- - 1) - 2N_+N_-\} = \mathbb{E}(N_+ - N_-)^2 - \mathbb{E}(N_+ + N_-).$$

On the right-hand side, we get

$$\sum_{s_1,s_2:s_i=\pm 1} \int_{\tilde{T}} \mathbb{E}\{\prod_{i=1}^{2} [\det \nabla f(t^i) \mathbb{1}_{\operatorname{sign}\det \nabla f(t^i)=s_i}] \mid \tilde{f}(\tilde{t}) = 0\} p_{\tilde{t}}(0) d\tilde{t} =$$

$$= \int_{\tilde{T}} \mathbb{E}\{\prod_{i=1}^{2} [\det \nabla f(t^i)] \mid \tilde{f}(\tilde{t}) = 0\} p_{\tilde{t}}(0) d\tilde{t}.$$

So, we get $\mathbb{E}(N_+ - N_-)^2 - \mathbb{E}(N_+ + N_-) = \int_{\tilde{T}} \mathbb{E}\{\prod_{i=1}^{2} [\det \nabla f(t^i)] \mid \tilde{f}(\tilde{t}) = 0\} p_{\tilde{t}}(0) d\tilde{t}$. The similar application of Proposition 3.3 to the fields f, g on T gives

$$\mathbb{E}N_+ = \int_T \mathbb{E}\{|\det \nabla f(t)| \mathbb{1}_{\det \nabla f(t) > 0} \mid f(t) = 0\} p_t(0) dt,$$

$$\mathbb{E}N_- = \int_T \mathbb{E}\{|\det \nabla f(t)| \mathbb{1}_{\det \nabla f(t) < 0} \mid f(t) = 0\} p_t(0) dt,$$

$$\mathbb{E}(N_+ + N_-) = \int_T \mathbb{E}\{|\det \nabla f(t)| \mid f(t) = 0\} p_t(0) dt.$$

Finally, we get

$$\mathbb{E}(\operatorname{ind}_\Gamma f) = \mathbb{E}(N_+ - N_-) = \int_T \mathbb{E}\{\det \nabla f(t) \mid f(t) = 0\} p_t(0) dt$$

and

$$\mathbb{E}(\operatorname{ind}_\Gamma f)^2 = \mathbb{E}(N_+ - N_-)^2 =$$

$$= \int_{\tilde{T}} \mathbb{E}\{\prod_{i=1}^{2} [\det \nabla f(t^i)] \mid \tilde{f}(\tilde{t}) = 0\} p_{\tilde{t}}(0) d\tilde{t} + \int_T \mathbb{E}\{|\det \nabla f(t)| \mid f(t) = 0\} p_t(0) dt.$$

This proves our statement.

5. Examples

As an application of the obtained result, we consider the case of a homogeneous isotropic two-dimensional Gaussian vector field. To simplify our calculations, we will consider only the case $b_L(r) \equiv b_N(r)$ or, in terms of Proposition 3.4, $\Phi_P = \Phi_S$. So, the correlation function of the components of such a field has the form

$$b_{ij}(z) = \delta_{ij} b_N(\|z\|), z \in \mathbb{R}^2.$$

The above-mentioned field

$$f(u)=\begin{pmatrix}f_1(u)\\ f_2(u)\end{pmatrix}=\begin{pmatrix}\int\limits_{\mathbb{R}^2}\phi(u+v)W_1(dv)\\ \int\limits_{\mathbb{R}^2}\phi(u+v)W_2(dv)\end{pmatrix},$$

where $\phi(x)=h(\|x\|), x\in\mathbb{R}^2$, and $h\in C_0^\infty(\mathbb{R})$ is an even function, gives an example of such a field.

Theorem 5.1. *Let f be a centered Gaussian isotropic vector field on $\mathbb{R}^2$ with the components correlation function $b_{ij}(z)=\delta_{ij}b_N(\|z\|)$, where $b_N:[0,+\infty)\to\mathbb{R}$ has the form*

$$b_N(r)=\int\limits_0^\infty J_1'(r\alpha)\Phi(d\alpha)+\int\limits_0^\infty\frac{J_1(r\alpha)}{r\alpha}\Phi(d\alpha)=\int\limits_0^\infty J_0(r\alpha)\Phi(d\alpha),$$

Φ is a positive finite measure on $[0,\infty)$ with the finite fourth moment, and $\Phi\neq c\delta_0$, i.e., Φ is not concentrated on $\{0\}$. Then the variance of the number of windings of f along the closed piecewise smooth curve Γ that bounds an open domain T equals

$$\mathbb{E}(\mathrm{ind}_\Gamma f)^2=$$
$$=\int\limits_{T^2}\frac{2b_N'(\|t^1-t^2\|)}{\|t^1-t^2\|}\left\{\frac{b_N(\|t^1-t^2\|)b_N'(\|t^1-t^2\|)^2}{b_N(0)^2-b_N(\|t^1-t^2\|)^2}+b_N''(\|t^1-t^2\|)\right\}p_{(t^1,t^2)}(0)dt^1dt^2-$$
$$-b_N''(0)p(0)S(T),$$

where $S(T)$ is an area of T. Here $p_{(t^1,t^2)}(0)=\frac{1}{(2\pi)^2(b_N(0)^2-b_N(\|t^1-t^2\|)^2)}$, and $p_t(0)=p(0)\equiv\frac{1}{2\pi b_N(0)^2}$ for any $t\in T$.

Remark 5.1. It is easily seen that, for any homogeneous Gaussian vector field f and the curve Γ satisfying the conditions of Theorem 2.1, $\mathbb{E}(\mathrm{ind}_\Gamma f)=0$. To see this, it is sufficient to prove that $\mathbb{E}\{\det\nabla f(t)\mid f(t)=0\}=0$. Set $b_{ij}(t,s)=b_{ij}(s-t)=\mathbb{E}(f_i(t)f_j(s))$. We have

$$\mathbb{E}\{\det\nabla f(t)\mid f(t)=0\}=\mathbb{E}(\frac{\partial f_1}{\partial t_1}(t)\frac{\partial f_2}{\partial t_2}(t)-\frac{\partial f_1}{\partial t_2}(t)\frac{\partial f_2}{\partial t_1}(t))=$$
$$=-\frac{\partial b_{12}}{\partial t_1\partial t_2}(0)+\frac{\partial b_{12}}{\partial t_2\partial t_1}(0)=0.$$

Remark 5.2. It can be shown that the given above example of the field

$$f(u)=\begin{pmatrix}f_1(u)\\ f_2(u)\end{pmatrix}=\begin{pmatrix}\int\limits_{\mathbb{R}^2}\phi(u+v)W_1(dv)\\ \int\limits_{\mathbb{R}^2}\phi(u+v)W_2(dv)\end{pmatrix},$$

where $\phi(x)=h(\|x\|), x\in\mathbb{R}^2$, satisfies the conditions of the theorem 5.1. Indeed, for the Fourier transform

$$\hat{b}(k)=\frac{1}{2\pi}\int\limits_{\mathbb{R}^2}b(z)e^{-ikz}dz$$

of the function

$$b(z)=b_N(\|z\|)=\int\limits_{\mathbb{R}^2}\phi(u)\phi(u+z)du$$

we get

$$\hat{b}(k)=\frac{1}{2\pi}\hat{\phi}(k)^2.$$

As $\phi(\cdot)\in S(\mathbb{R}^2)$, we also get $\hat{\phi}(\cdot)\in S(\mathbb{R}^2)$, where we denote by $S(\mathbb{R}^2)$ the Schwartz space. It is clear (as $\phi(x)=h(\|x\|)$) that $\hat{\phi}(k)$ depends only on the $\|k\|$. Thus, the same is true for $\hat{b}(k)$, that is, $\hat{b}(k) = a(\|k\|)$. So, we obtain (using integration in polar coordinates and the third property of the Bessel functions, mentioned in remark 3.2):

$$b_N(\|z\|) = b(z) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \hat{b}(k)e^{ikz}dk = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} a(\|k\|)e^{ikz}dk = \int\limits_0^\infty \alpha a(\alpha)J_0(\alpha\|z\|)d\alpha.$$

Therefore, the function $b_N(r)$ has the desired form

$$b_N(r) = \int\limits_0^\infty J_0(r\alpha)\Phi(d\alpha),$$

and the measure $\Phi(d\alpha) = \alpha a(\alpha)d\alpha$ has all the moments, not only the fourth, as the function $a(\alpha)$ is rapidly decreasing when $\alpha \to \infty$.

Proof of Theorem 5.1: First, we check that our field satisfies the conditions of Theorem 2.1. Let us check condition e) on the modulus of continuity of the field and its derivatives. In our case, we can state even more: the modulus of continuity of each of the components f, ∇f satisfies the relation $\mathbb{P}(\omega(\eta) > \epsilon) = o(\eta^N)$ when $\eta \downarrow 0$ for any $\epsilon > 0$ and for each $N > 0$. To prove this, we need the following statement.

Proposition 5.1. *Let h be a centered Gaussian random vector field on a closed domain $T \subset \mathbb{R}^2$, such that its covariance function $K(t,s) = \mathbb{E}f(t)f(s)$ is twice continuously differentiable on $T \times T$. Then the field h has a continuous modification, and its modulus of continuity $\omega(\eta)$ satisfies the condition*

$$\mathbb{P}(\omega(\eta) > \epsilon) = o(\eta^N), \eta \downarrow 0$$

for each $\epsilon > 0$ and each $N > 0$.

This statement is, in fact, proved on page 268 of [1]. The proof is based on the application of the Borell–Tsirelson inequality (Theorem 2.1.1 in [1]) to the field H on $T \times T$ defined by $H(s,t) = h(t) - h(s)$.

It is easily seen that, with our assumptions, the correlation functions of all the fields $\frac{\partial f_i}{\partial t_j}$ are twice continuously differentiable on $T \times T$ (as the function b_N is four times continuously differentiable); so, we can apply our proposition 5.1 to them. Thus, we get that the fields considered are indeed continuous with probability 1, and their moduli of continuity satisfy the conditions of Theorem 2.1.

For jointly Gaussian random variables, the theorem of normal correlation implies that all the conditional distributions are Gaussian. So, to check the other conditions of our theorem, we have to prove the nondegeneracy of the correlation matrix of the variables in question. This is the statement of the following proposition, for which we have a rather long proof included in Propositions 5.3-5.10.

Proposition 5.2. *In the assumptions of Theorem 5.1, for any two points $t^1 \neq t^2$, the joint distribution of Gaussian variables $f_i(t^j)$, $\frac{\partial f_k}{\partial t^l_m}(t^l), i, j, k, l, m = 1, 2$, is nondegenerate.*

In our case, for any two points $t^1 \neq t^2$, the random vectors

$$(f_1(t^j), \frac{\partial f_1}{\partial t^l_m}(t^l), j, l, m = 1, 2) \quad \text{and} \quad (f_2(t^j), \frac{\partial f_2}{\partial t^l_m}(t^l), j, l, m = 1, 2)$$

are independent and identically distributed. So, it suffices to check that the correlation matrix of the random variables

$$f_1(t^j), \tfrac{\partial f_1}{\partial t^l_m}(t^l), j, l, m = 1, 2$$

is nondegenerate. This can be done by explicitly computing the correlation matrix of these Gaussian variables. As our field is isotropic, it suffices to perform these computations for the particular case

$$t^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} r \\ 0 \end{pmatrix},$$

where $r > 0$. We have the following auxiliary propositions. We omit some of the proofs, as they include only standard technical considerations.

Proposition 5.3. *The correlation matrix of the random variables*

$$f_1(t^j),\ \tfrac{\partial f_1}{\partial t^l_m}(t^l),\ j,l,m = 1,2,$$

where

$$t^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} r \\ 0 \end{pmatrix}, \quad r > 0,$$

is equal to

$$\begin{pmatrix} b_N(0) & b_N(r) & 0 & \frac{\partial b_N(r)}{\partial r} & 0 & 0 \\ b_N(r) & b_N(0) & -\frac{\partial b_N(r)}{\partial r} & 0 & 0 & 0 \\ 0 & -\frac{\partial b_N(r)}{\partial r} & \beta_N & -\frac{\partial^2 b_N(r)}{\partial r^2} & 0 & 0 \\ \frac{\partial b_N(r)}{\partial r} & 0 & -\frac{\partial^2 b_N(r)}{\partial r^2} & \beta_N & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_N & -\frac{\partial b_N(r)}{\partial r}/r \\ 0 & 0 & 0 & 0 & -\frac{\partial b_N(r)}{\partial r}/r & \beta_N \end{pmatrix}.$$

Proposition 5.4. *The determinant of the correlation matrix of the random variables*

$$f_1(t^j), \tfrac{\partial f_1}{\partial t^l_m}(t^l), j,l,m = 1,2$$

is equal to

$$\begin{aligned} &\frac{1}{r^2}(\beta_N b_N(0) - b'_N(r)^2 + b_N(r)b''_N(r) + \beta_N b_N(r) + b_N(0)b''_N(r))\times \\ &\qquad \times (\beta_N b_N(0) - b'_N(r)^2 + b_N(r)b''_N(r) - \beta_N b_N(r) - b_N(0)b''_N(r))\times \\ &\qquad\qquad \times (\beta_N r - b'_N(r))(\beta_N r + b'_N(r)) = \\ &= \frac{1}{r^2}((\beta_N b_N(0) - b'_N(r)^2 + b_N(r)b''_N(r))^2 - (\beta_N b_N(r) + b_N(0)b''_N(r))^2)\times \\ &\qquad\qquad \times ((\beta_N r)^2 - b'_N(r)^2), \end{aligned}$$

where $r = \|t^1 - t^2\|$.

Proposition 5.5. *For any* $x > 0$, *we have*

$$|J'_1(x)| < \tfrac{1}{2}, \text{ and } |J_1(x)| < \tfrac{x}{2}.$$

The inequality $|J'_1(x)| < \frac{1}{2}$ can be proved using standard methods. The inequality $|J_1(x)| < \frac{x}{2}$ is an easy consequence (as $J_1(0) = 0$).

Proposition 5.6. *For any* $x > 0$, *we have*

$$(\frac{1}{2} - J'_1(x))(1 + J_0(x)) > J_1(x)^2,$$

and

$$(\frac{1}{2} + J'_1(x))(1 - J_0(x)) > J_1(x)^2.$$

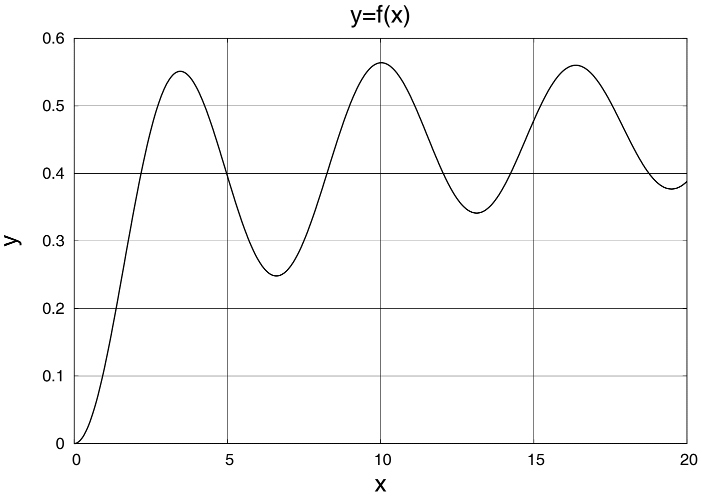
Proof. We put

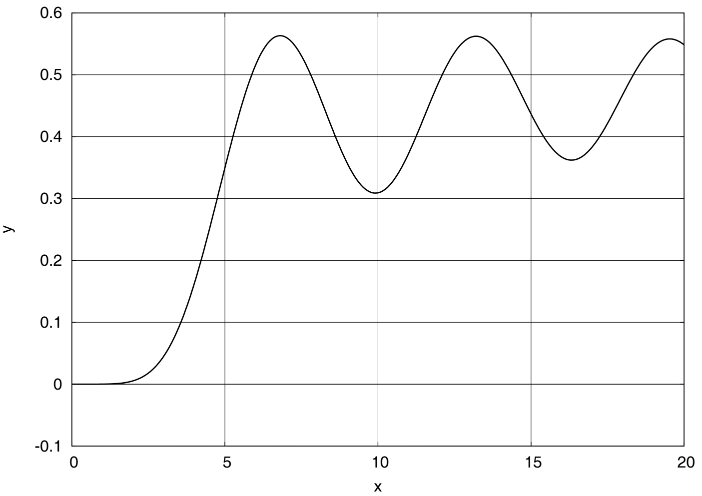
$$f(x) = (\frac{1}{2} - J_1'(x))(1 + J_0(x)) - J_1(x)^2$$

and

$$g(x) = (\frac{1}{2} + J_1'(x))(1 - J_0(x)) - J_1(x)^2.$$

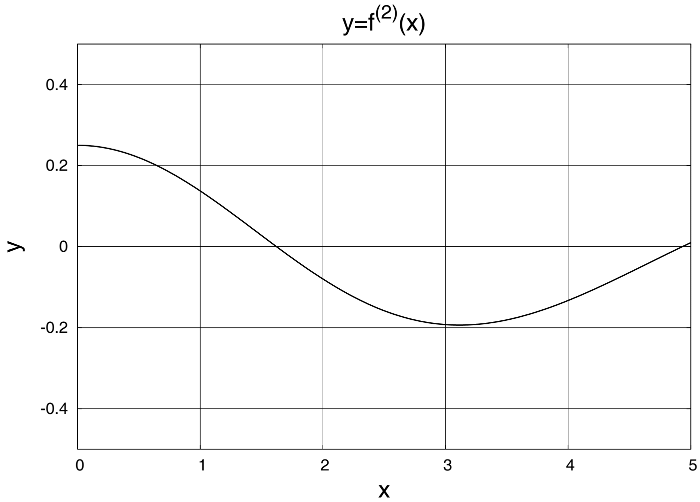
The plots of the corresponding functions look like

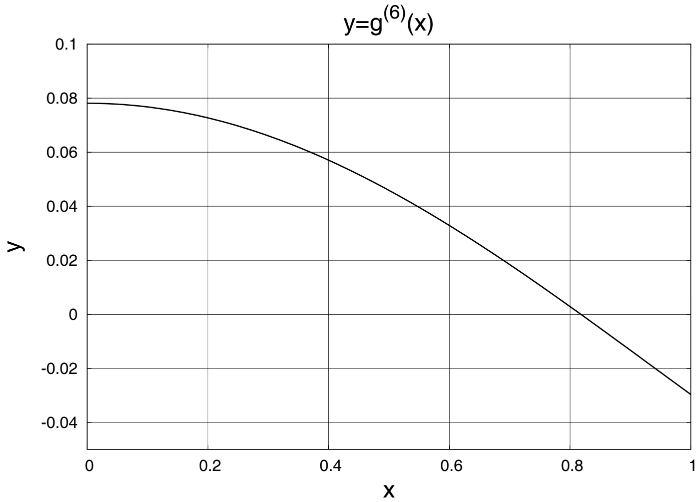




For $x \in [0.5, 20]$, it is seen from the plots (and, of course, can be verified strictly) that $f(x) > 0$ and $g(x) > 0$. So, we need to verify our inequalities for small values of x (i.e., $0 < x < 0.5$) and for the large ones (say, for $x > 20$).

Performing the symbolic differentiation with the computer algebra system Maxima, we obtain $f(0) = f'(0) = 0$, $f^{(2)}(x) > 0$ on $[0, 1]$, $g(0) = g^{(1)}(0) = g^{(2)}(0) = g^{(3)}(0) = g^{(4)}(0) = g^{(5)}(0) = 0$ and $g^{(6)}(x) > 0$ on $[0, 0.8]$ (of course, all of these can be proved strictly). The plots of the corresponding derivatives look like





So, we conclude that $f(x) > 0$ on $(0, 1]$ and $g(x) > 0$ on $(0, 0.8]$.
For values of x larger than 20, we can use the following proposition.

Proposition 5.7. *For any $x > 20$, we have*

$$|J_0(x)| < 0.3, |J_1'(x)| < 0.3, J_1(x)^2 < 0.05$$

Using this proposition, we obtain, for $x > 20$,

$$(\frac{1}{2} - |J_1'(x)|)(1 - |J_0(x)|) - J_1(x)^2 > 0.2 \times 0.7 - 0.05 > 0.$$

Therefore, for $x > 20$, we have $f(x) > 0$ and $g(x) > 0$. □

Proposition 5.8. *For any $\alpha > 0, \beta > 0$, we have the following inequalities:*

$$\frac{1}{2}(\alpha^2 + \beta^2) - 2\alpha\beta J_1(\alpha)J_1(\beta) - J_0(\alpha)\beta^2 J_1'(\beta) - \\ - J_0(\beta)\alpha^2 J_1'(\alpha) - \alpha^2 J_1'(\alpha) - \beta^2 J_1'(\beta) + \frac{1}{2}\alpha^2 J_0(\beta) + \frac{1}{2}\beta^2 J_0(\alpha) > 0$$

and

$$\frac{1}{2}(\alpha^2+\beta^2) - 2\alpha\beta J_1(\alpha)J_1(\beta) - J_0(\alpha)\beta^2 J_1'(\beta) - \\ - J_0(\beta)\alpha^2 J_1'(\alpha) + \alpha^2 J_1'(\alpha) + \beta^2 J_1'(\beta) - \frac{1}{2}\alpha^2 J_0(\beta) - \frac{1}{2}\beta^2 J_0(\alpha) > 0.$$

Proof. We can rewrite our first inequality in the form

$$\alpha^2(\frac{1}{2} - J_0(\beta)J_1'(\alpha) - J_1'(\alpha) + \frac{1}{2}J_0(\beta)) + \\ + \beta^2(\frac{1}{2} - J_0(\alpha)J_1'(\beta) - J_1'(\beta) + \frac{1}{2}J_0(\alpha)) - \\ - 2\alpha\beta J_1(\alpha)J_1(\beta) > 0.$$

We put

$$a = \frac{1}{2} - J_0(\beta)J_1'(\alpha) - J_1'(\alpha) + \frac{1}{2}J_0(\beta),$$
$$b = -2J_1(\alpha)J_1(\beta),$$
$$c = \frac{1}{2} - J_0(\alpha)J_1'(\beta) - J_1'(\beta) + \frac{1}{2}J_0(\alpha).$$

It is sufficient to prove that the quadratic form $ax^2 + bxy + cy^2$ has no zeros, except $x = y = 0$. To do this, we will show that $b^2 - 4ac < 0$. We have from Proposition 5.6:

$$(\tfrac{1}{2} - J_1'(\alpha))(1 + J_0(\alpha)) > J_1(\alpha)^2, \text{ and } (\tfrac{1}{2} - J_1'(\beta))(1 + J_0(\beta)) > J_1(\beta)^2.$$

Muliplying these inequalities, we get

$$(\frac{1}{2} - J_1'(\alpha))(\frac{1}{2} - J_1'(\beta))(1 + J_0(\alpha))(1 + J_0(\beta)) > J_1(\alpha)^2 J_1(\beta)^2.$$

After some technical manipulations, we get exactly $b^2 - 4ac < 0$. So, the first inequality from our proposition is proved. The second one is proved in the same way, using the inequalities

$$(\tfrac{1}{2} + J_1'(\alpha))(1 - J_0(\alpha)) > J_1(\alpha)^2, \text{ and } (\tfrac{1}{2} + J_1'(\beta))(1 - J_0(\beta)) > J_1(\beta)^2,$$

from the same proposition 5.6. □

Proposition 5.9. *For any $r > 0$, the following inequality holds:*

$$|\beta_N b_N(0) - b_N'(r)^2 + b_N(r)b_N''(r)| > |b_N(0)b_N''(r) + \beta_N b_N(r)|.$$

Proof. We have $b_N'(r) = -\int\limits_0^\infty \alpha J_1(r\alpha)\Phi(d\alpha)$. Therefore,

$$b_N'(r)^2 = \int\limits_0^\infty\int\limits_0^\infty \alpha\beta J_1(r\alpha)J_1(r\beta)\Phi(d\alpha)\Phi(d\beta).$$

Using the equality $b_N''(r) = -\int\limits_0^\infty \beta^2 J_0(r\beta)\Phi(d\beta)$, we obtain

$$b_N(r)b_N''(r) = -\int\limits_0^\infty \beta^2 J_0(r\alpha)J_1'(r\beta)\Phi(d\alpha)\Phi(d\beta) = \\ = -\frac{1}{2}\int\limits_0^\infty\int\limits_0^\infty J_0(r\alpha)\beta^2 J_1'(r\beta)\Phi(d\alpha)\Phi(d\beta) - \frac{1}{2}\int\limits_0^\infty\int\limits_0^\infty J_0(r\beta)\alpha^2 J_1'(r\alpha)\Phi(d\alpha)\Phi(d\beta).$$

As for the term $\beta_N b_N(0)$, we get from $\beta_N = \frac{1}{2}\int\limits_0^\infty \alpha^2\Phi(d\alpha)$ and $b_N(0) = \int\limits_0^\infty \Phi(d\beta)$ that

$$\beta_N b_N(0) = \frac{1}{2}\int\limits_0^\infty \alpha^2\Phi(d\alpha)\Phi(d\beta) = \frac{1}{2}\int\limits_0^\infty \frac{\alpha^2+\beta^2}{2}\Phi(d\alpha)\Phi(d\beta).$$

In the same way, we get

$$b_N(0)b_N''(r) = -\int\limits_0^\infty \frac{\alpha^2 J_1'(r\alpha)+\beta^2 J_1'(r\beta)}{2}\Phi(d\alpha)\Phi(d\beta).$$

So, we need to prove the following inequality:

$$\begin{aligned}
&|\frac{1}{2}\int\limits_0^\infty \frac{\alpha^2+\beta^2}{2}\Phi(d\alpha)\Phi(d\beta) - \int\limits_0^\infty\int\limits_0^\infty \alpha\beta J_1(r\alpha)J_1(r\beta)\Phi(d\alpha)\Phi(d\beta) - \\
&\quad -\frac{1}{2}\int\limits_0^\infty\int\limits_0^\infty J_0(r\alpha)\beta^2 J_1'(r\beta)\Phi(d\alpha)\Phi(d\beta) - \frac{1}{2}\int\limits_0^\infty\int\limits_0^\infty J_0(r\beta)\alpha^2 J_1'(r\alpha)\Phi(d\alpha)\Phi(d\beta)| > \\
&\qquad > |-\int\limits_0^\infty\int\limits_0^\infty \frac{\alpha^2 J_1'(r\alpha)+\beta^2 J_1'(r\beta)}{2}\Phi(d\alpha)\Phi(d\beta) + \\
&\qquad + \frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty \alpha^2 J_0(r\beta)\Phi(d\alpha)\Phi(d\beta) + \frac{1}{2}\int\limits_0^\infty\int\limits_0^\infty \beta^2 J_0(r\alpha)\Phi(d\alpha)\Phi(d\beta)|.
\end{aligned}$$

It suffices to prove that, for every $\alpha, \beta > 0$, the two following inequalities hold:

$$\begin{aligned}
&\Big\{\frac{1}{2}(\alpha^2+\beta^2) - 2\alpha\beta J_1(r\alpha)J_1(r\beta) - \\
&\qquad -J_0(r\alpha)\beta^2 J_1'(r\beta) - J_0(r\beta)\alpha^2 J_1'(r\alpha) - \alpha^2 J_1'(r\alpha)\Big\} \pm \\
&\qquad\qquad \pm \Big\{-\alpha^2 J_1'(r\alpha) - \beta^2 J_1'(r\beta) + \frac{1}{2}\alpha^2 J_0(r\beta) + \frac{1}{2}\beta^2 J_0(r\alpha)\Big\} > 0.
\end{aligned}$$

We can put $r=1$, as the multiplication by r^2 and setting $\hat{\alpha} = r\alpha$, $\hat{\beta} = r\beta$ reduce our inequality to the case $r = 1$. So, we need to prove that

$$\begin{aligned}
&\Big\{\frac{1}{2}(\alpha^2+\beta^2) - 2\alpha\beta J_1(\alpha)J_1(\beta) - \\
&\qquad - J_0(\alpha)\beta^2 J_1'(\beta) - J_0(\beta)\alpha^2 J_1'(\alpha) - \alpha^2 J_1'(\alpha)\Big\} \pm \\
&\qquad\qquad \pm \Big\{-\alpha^2 J_1'(\alpha) - \beta^2 J_1'(\beta) + \frac{1}{2}\alpha^2 J_0(\beta) + \frac{1}{2}\beta^2 J_0(\alpha)\Big\} > 0.
\end{aligned}$$

These are exactly the inequalites from Proposition 5.8. □

Proposition 5.10. *For any $r > 0$, $\beta_N \cdot r > |b_N'(r)|$.*

Proof. We have $\beta_N = \frac{1}{2}\int\limits_0^\infty \alpha^2\Phi(d\alpha)$ and $b_N'(r) = -\alpha\int\limits_0^\infty J_1(r\alpha)\Phi(d\alpha)$.

So, it suffices to prove that $|\alpha J_1(r\alpha)| < \frac{1}{2}\alpha^2 r$, or $|J_1(r\alpha)| < \frac{1}{2}\alpha r$, which is known from Proposition 5.5. □

Using Proposition 5.10 with propositions 5.9 and 5.4 proves Proposition 5.2.
The latter allows to check the conditions b', c', d' of Theorem 2.1 for our case.
We now apply Theorem 2.1. It suffices to prove that

$$\mathbb{E}(\det \nabla f(t^1) \det \nabla f(t^2) \mid f(t^1) = f(t^2) = 0) = \\ = \frac{2b'_N(\|t^1-t^2\|)}{\|t^1-t^2\|}\left\{\frac{b_N(\|t^1-t^2\|)b'_N(\|t^1-t^2\|)^2}{b_N(0)^2 - b_N(\|t^1-t^2\|)^2} + b''_N(\|t^1-t^2\|)\right\}$$

and $\mathbb{E}(|\det \nabla f(t)| \mid f(t) = 0) = \beta_N$, where β_N is defined in Proposition 3.4.

Denote, by (X_{ij}, Y_{ij}), a conditional distribution $(\frac{\partial f_i}{\partial t_j}(t^1), \frac{\partial f_i}{\partial t_j}(t^2))$ given $f(t^1) = f(t^2) = 0$. It follows from the theorem of normal correlation that (X_{ij}, Y_{ij}) is again a Gaussian vector. We put $u_{ijkl} = \mathbb{E}(\frac{\partial f_i}{\partial t_k}(t^1)\frac{\partial f_j}{\partial t_l}(t^2) \mid f(t^1) = f(t^2) = 0)$.

Note that the following three lemmas (Lemma 5.1,5.2, 5.3) hold for any centered homogeneous Gaussian vector field.

Lemma 5.1. *The following relation holds:*

$$\mathbb{E}(\det \nabla f(t^1) \det \nabla f(t^2) \mid f(t^1) = f(t^2) = 0) = \\ = \frac{1}{2}\sum_{i,j,k,l=1,2}(-1)^{i+j+k+l}u_{ijkl}u_{3-i,3-j,3-k,3-l} + \\ + \mathbb{E}(\det \nabla f(t^1) | f(t^1) = f(t^2) = 0)\mathbb{E}(\det \nabla f(t^2) \mid f(t^1) = f(t^2) = 0).$$

Proof. In our notation, we have

$$\mathbb{E}(\det \nabla f(t^1) \det \nabla f(t^2) \mid f(t^1) = f(t^2) = 0) = \\ = \mathbb{E}\{(X_{11}X_{22} - X_{12}X_{21})(Y_{11}Y_{22} - Y_{12}Y_{21})\} = \\ = \mathbb{E}X_{11}X_{22}Y_{11}Y_{22} - \mathbb{E}X_{12}X_{21}Y_{11}Y_{22} - \mathbb{E}X_{11}X_{22}Y_{12}Y_{21} + \mathbb{E}X_{12}X_{21}Y_{12}Y_{21}.$$

The application of Wick's formula ([1], Lemma 11.6.1) gives:

$$\mathbb{E}(\det \nabla f(t^1) \det \nabla f(t^2) \mid f(t^1) = f(t^2) = 0) = \\ = \underbrace{\mathbb{E}X_{11}X_{22}\mathbb{E}Y_{11}Y_{22} - \mathbb{E}X_{12}X_{21}\mathbb{E}Y_{11}Y_{22} - \mathbb{E}X_{11}X_{22}\mathbb{E}Y_{12}Y_{21} + \mathbb{E}X_{12}X_{21}\mathbb{E}Y_{12}Y_{21}}_{\mathbb{E}(\det \nabla f(t^1)|f(t^1)=f(t^2)=0)\mathbb{E}(\det \nabla f(t^2)|f(t^1)=f(t^2)=0)} + \\ + \mathbb{E}X_{11}Y_{11}\mathbb{E}X_{22}Y_{22} + \mathbb{E}X_{11}Y_{22}\mathbb{E}X_{22}Y_{11} - \mathbb{E}X_{12}Y_{11}\mathbb{E}X_{21}Y_{22} - \mathbb{E}X_{12}Y_{22}\mathbb{E}X_{21}Y_{11} - \\ - \mathbb{E}X_{11}Y_{12}\mathbb{E}X_{22}Y_{21} - \mathbb{E}X_{11}Y_{21}\mathbb{E}X_{22}Y_{12} + \mathbb{E}X_{12}Y_{12}\mathbb{E}X_{21}Y_{21} + \mathbb{E}X_{12}Y_{21}\mathbb{E}X_{21}Y_{12} = \\ = \mathbb{E}(\det \nabla f(t^1) \mid f(t^1) = f(t^2) = 0) \times \mathbb{E}(\det \nabla f(t^2) \mid f(t^1) = f(t^2) = 0) + \\ + u_{1111}u_{2222} + u_{1212}u_{2121} - u_{1121}u_{2212} - u_{1222}u_{2111} - \\ - u_{1112}u_{2221} - u_{1211}u_{2122} + u_{1122}u_{2211} + u_{1221}u_{2112}.$$

□

Denote, by K, the covariance matrix $K = cov(f_1(t^1), f_2(t^1), f_1(t^2), f_2(t^2))$.

Lemma 5.2.

$$u_{ijkl} = \mathbb{E}(\frac{\partial f_i}{\partial t_k}(t^1)\frac{\partial f_j}{\partial t_l}(t^2)) + \begin{pmatrix} 0 & 0 & \frac{\partial b_{i1}}{\partial t_k} & \frac{\partial b_{i2}}{\partial t_k}\end{pmatrix} K^{-1}\begin{pmatrix}\frac{\partial b_{1j}}{\partial t_l} \\ \frac{\partial b_{2j}}{\partial t_l} \\ 0 \\ 0\end{pmatrix} = \\ = \mathbb{E}(\frac{\partial f_i}{\partial t_k}(t^1)\frac{\partial f_j}{\partial t_l}(t^2)) + \begin{pmatrix}\frac{\partial b_{i1}}{\partial t_k} & \frac{\partial b_{i2}}{\partial t_k}\end{pmatrix} K_{12}^{-1}\begin{pmatrix}\frac{\partial b_{1j}}{\partial t_l} \\ \frac{\partial b_{2j}}{\partial t_l}\end{pmatrix}.$$

Here, K_{12}^{-1} is a submatrix of K^{-1}, formed with the intersection of its two last rows and two first columns, and $b_{ij} = b_{ij}(t^2 - t^1) = \mathbb{E} f_i(t^1) f_j(t^2)$.

Proof. Here, we calculate the conditional expectation

$$\mathbb{E}\{X_1 X_2 \mid Y_1 = Y_2 = Y_3 = Y_4 = 0\}$$

for jointly Gaussian centered random variables $X_1, X_2, Y_1, Y_2, Y_3, Y_4$. We can write:

$$X_1 = \sum_{i=1}^{4} \alpha_i Y_i + X_{1\perp}, \quad X_2 = \sum_{i=1}^{4} \beta_i Y_i + X_{2\perp},$$

where $X_{1\perp}, X_{2\perp}$ are independent of Y_1, Y_2, Y_3, Y_4.

Denote $X_{i\shortparallel} = X_i - X_{i\perp}, i = 1, 2, C = cov(Y_1, Y_2, Y_3, Y_4)$.

Then

$$\mathbb{E}\{X_1 X_2 | Y_1 = Y_2 = Y_3 = Y_4 = 0\} = \mathbb{E} X_1 X_2 - \mathbb{E} X_{1\shortparallel} X_{2\shortparallel}.$$

But

$$\mathbb{E} X_{1\shortparallel} X_{2\shortparallel} = \mathbb{E} X_{1\shortparallel} X_2 = \sum_{i=1}^{4} \alpha_i \mathbb{E} Y_i X_2 = \begin{pmatrix} \mathbb{E} Y_1 X_2 \\ \mathbb{E} Y_2 X_2 \\ \mathbb{E} Y_3 X_2 \\ \mathbb{E} Y_4 X_2 \end{pmatrix}^T \alpha,$$

where $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$. As $\alpha = C^{-1} \begin{pmatrix} \mathbb{E} X_1 Y_1 \\ \mathbb{E} X_1 Y_2 \\ \mathbb{E} X_1 Y_3 \\ \mathbb{E} X_1 Y_4 \end{pmatrix}$, we get the statement. □

Lemma 5.3.

$$\mathbb{E}(\det \nabla f(t^1) \mid f(t^1) = f(t^2) = 0) =$$
$$= \begin{pmatrix} \frac{\partial b_{11}}{\partial t_2} & \frac{\partial b_{12}}{\partial t_2} \end{pmatrix} K_{22}^{-1} \begin{pmatrix} \frac{\partial b_{21}}{\partial t_1} \\ \frac{\partial b_{22}}{\partial t_1} \end{pmatrix} - \begin{pmatrix} \frac{\partial b_{11}}{\partial t_1} & \frac{\partial b_{12}}{\partial t_1} \end{pmatrix} K_{22}^{-1} \begin{pmatrix} \frac{\partial b_{21}}{\partial t_2} \\ \frac{\partial b_{22}}{\partial t_2} \end{pmatrix}.$$

Here, K_{22}^{-1} is a submatrix of the matrix K^{-1} formed by the intersection of its two last rows and two last columns.

Proof.

$$\mathbb{E}(\det \nabla f(t^1) \mid f(t^1) = f(t^2) = 0) =$$
$$= \mathbb{E}(\frac{\partial f_1}{\partial t_1} \frac{\partial f_2}{\partial t_2}(t^1) - \frac{\partial f_1}{\partial t_2} \frac{\partial f_2}{\partial t_1}(t^1) \mid f(t^1) = f(t^2) = 0) =$$

$$
\begin{aligned}
&= \mathbb{E}\frac{\partial f_1}{\partial t_1}\frac{\partial f_2}{\partial t_2}(t^1) - \\
&- \begin{pmatrix} \mathbb{E}\frac{\partial f_1}{\partial t_1}(t^1)f_1(t^1) & \mathbb{E}\frac{\partial f_1}{\partial t_1}(t^1)f_2(t^1) & \mathbb{E}\frac{\partial f_1}{\partial t_1}(t^1)f_1(t^2) & \mathbb{E}\frac{\partial f_1}{\partial t_1}(t^1)f_2(t^2) \end{pmatrix} K^{-1} \cdot \\
&\cdot \begin{pmatrix} \mathbb{E}\frac{\partial f_2}{\partial t_2}(t^1)f_1(t^1) \\ \mathbb{E}\frac{\partial f_2}{\partial t_2}(t^1)f_2(t^1) \\ \mathbb{E}\frac{\partial f_2}{\partial t_2}(t^1)f_1(t^2) \\ \mathbb{E}\frac{\partial f_2}{\partial t_2}(t^1)f_2(t^2) \end{pmatrix} - \mathbb{E}\frac{\partial f_1}{\partial t_2}\frac{\partial f_2}{\partial t_1}(t^1) + \\
&+ \begin{pmatrix} \mathbb{E}\frac{\partial f_1}{\partial t_2}(t^1)f_1(t^1) & \mathbb{E}\frac{\partial f_1}{\partial t_2}(t^1)f_2(t^1) & \mathbb{E}\frac{\partial f_1}{\partial t_2}(t^1)f_1(t^2) & \mathbb{E}\frac{\partial f_1}{\partial t_2}(t^1)f_2(t^2) \end{pmatrix} K^{-1} \cdot \\
&\cdot \begin{pmatrix} \mathbb{E}\frac{\partial f_2}{\partial t_2}(t^1)f_1(t^1) \\ \mathbb{E}\frac{\partial f_2}{\partial t_2}(t^1)f_2(t^1) \\ \mathbb{E}\frac{\partial f_2}{\partial t_1}(t^1)f_1(t^2) \\ \mathbb{E}\frac{\partial f_2}{\partial t_1}(t^1)f_2(t^2) \end{pmatrix} = \\
&= \mathbb{E}\frac{\partial f_1}{\partial t_1}\frac{\partial f_2}{\partial t_2}(t^1) - \\
&- \begin{pmatrix} 0 & 0 & \mathbb{E}\frac{\partial f_1}{\partial t_1}(t^1)f_1(t^2) & \mathbb{E}\frac{\partial f_1}{\partial t_1}(t^1)f_2(t^2) \end{pmatrix} K^{-1} \begin{pmatrix} 0 \\ 0 \\ \mathbb{E}\frac{\partial f_2}{\partial t_2}(t^1)f_1(t^2) \\ \mathbb{E}\frac{\partial f_2}{\partial t_2}(t^1)f_2(t^2) \end{pmatrix} - \\
&- \mathbb{E}\frac{\partial f_1}{\partial t_2}\frac{\partial f_2}{\partial t_1}(t^1) + \\
&+ \begin{pmatrix} 0 & 0 & \mathbb{E}\frac{\partial f_1}{\partial t_2}(t^1)f_1(t^2) & \mathbb{E}\frac{\partial f_1}{\partial t_2}(t^1)f_2(t^2) \end{pmatrix} K^{-1} \begin{pmatrix} 0 \\ 0 \\ \mathbb{E}\frac{\partial f_2}{\partial t_1}(t^1)f_1(t^2) \\ \mathbb{E}\frac{\partial f_2}{\partial t_1}(t^1)f_2(t^2) \end{pmatrix} = \\
&= \mathbb{E}(\frac{\partial f_1}{\partial t_1}\frac{\partial f_2}{\partial t_2}(t^1) - \mathbb{E}(\frac{\partial f_1}{\partial t_2}\frac{\partial f_2}{\partial t_1}(t^1) + \\
&+ \begin{pmatrix} \mathbb{E}\frac{\partial f_1}{\partial t_2}(t^1)f_1(t^2) \\ \mathbb{E}\frac{\partial f_1}{\partial t_2}(t^1)f_2(t^2) \end{pmatrix} K_{22}^{-1} \begin{pmatrix} \mathbb{E}\frac{\partial f_2}{\partial t_1}(t^1)f_1(t^2) \\ \mathbb{E}\frac{\partial f_2}{\partial t_1}(t^1)f_2(t^2) \end{pmatrix} - \\
&- \begin{pmatrix} \mathbb{E}\frac{\partial f_2}{\partial t_1}(t^1)f_1(t^2) \\ \mathbb{E}\frac{\partial f_2}{\partial t_1}(t^1)f_2(t^2) \end{pmatrix} K_{22}^{-1} \begin{pmatrix} \mathbb{E}\frac{\partial f_2}{\partial t_2}(t^1)f_1(t^2) \\ \mathbb{E}\frac{\partial f_2}{\partial t_2}(t^1)f_2(t^2) \end{pmatrix} = \\
&= (-\frac{\partial b_{12}}{\partial t_1 \partial t_2}(0) + \frac{\partial b_{12}}{\partial t_2 \partial t_1}(0)) + \\
&+ \begin{pmatrix} -\frac{\partial b_{11}}{\partial t_2} & \frac{\partial b_{12}}{\partial t_2} \end{pmatrix} K_{22}^{-1} \begin{pmatrix} -\frac{\partial b_{21}}{\partial t_1} \\ -\frac{\partial b_{22}}{\partial t_1} \end{pmatrix} - \begin{pmatrix} -\frac{\partial b_{11}}{\partial t_1} & \frac{\partial b_{12}}{\partial t_1} \end{pmatrix} K_{22}^{-1} \begin{pmatrix} \frac{-\partial b_{21}}{\partial t_2} \\ \frac{-\partial b_{22}}{\partial t_2} \end{pmatrix} = \\
&= \begin{pmatrix} \frac{\partial b_{11}}{\partial t_2} & \frac{\partial b_{12}}{\partial t_2} \end{pmatrix} K_{22}^{-1} \begin{pmatrix} \frac{\partial b_{21}}{\partial t_1} \\ \frac{\partial b_{22}}{\partial t_1} \end{pmatrix} - \begin{pmatrix} \frac{\partial b_{11}}{\partial t_1} & \frac{\partial b_{12}}{\partial t_1} \end{pmatrix} K_{22}^{-1} \begin{pmatrix} \frac{\partial b_{21}}{\partial t_2} \\ \frac{\partial b_{22}}{\partial t_2} \end{pmatrix}.
\end{aligned}
$$

$\square$

In the following lemmas, we substantially use the form $b_{ij}(z) = \delta_{ij} b_N(\|z\|)$ of the covariance function of the components. The proofs of these statements are made by direct computations with the use of above-mentioned expressions from [3]. We denote $r = \|t^1 - t^2\|$. As the considered random field is isotropic, the value of the expression $\mathbb{E}\{\det \nabla f(t^1) \det \nabla f(t^2) \mid f(t^1) = f(t^2) = 0\}$ depends only on $r = \|t^1 - t^2\|$. So, in

Lemmas 5.4-5.8 we put

$$t^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} r \\ 0 \end{pmatrix},$$

and this will not influence the final result.

Lemma 5.4.

$$\mathbb{E}(\frac{\partial f_i}{\partial t_k}(t^1)\frac{\partial f_j}{\partial t_l}(t^2)) = -\delta_{ij}\delta_{kl}\left\{\delta_{k1}\frac{\partial^2 b_N(r)}{\partial r^2} + \delta_{k2}\frac{1}{r}\frac{\partial b_N(r)}{\partial r}\right\}.$$

Lemma 5.5. *The matrix $K = cov(f_1(t^1), f_2(t^1), f_1(t^2), f_2(t^2))$ has the form*

$$K = \begin{pmatrix} b_N(0) & 0 & b_N(r) & 0 \\ 0 & b_N(0) & 0 & b_N(r) \\ b_N(r) & 0 & b_N(0) & 0 \\ 0 & b_N(r) & 0 & b_N(0) \end{pmatrix}.$$

Lemma 5.6. *Matrices K_{22}^{-1}, K_{12}^{-1} are of the form*

$$K_{22}^{-1} = \frac{b_N(0)}{b_N(0)^2 - b_N(r)^2}I, \; K_{12}^{-1} = -\frac{b_N(0)}{b_N(0)^2 - b_N(r)^2}I,$$

where I is a 2×2 unit matrix.

Lemma 5.7. *The following relations hold:*

$$\begin{aligned} \frac{\partial}{\partial t_1}b_{11}(r,0) =& \frac{\partial}{\partial t_1}b_{22}(r,0) = b_N'(r), \\ \frac{\partial}{\partial t_2}b_{11}(r,0) =& \frac{\partial}{\partial t_1}b_{12}(r,0) = \frac{\partial}{\partial t_2}b_{12}(r,0) = \\ = \frac{\partial}{\partial t_1}b_{21}(r,0) =& \frac{\partial}{\partial t_2}b_{21}(r,0) = \frac{\partial}{\partial t_2}b_{22}(r,0) = 0. \end{aligned}$$

Lemma 5.8.

$$u_{ijkl} = -\delta_{ij}\delta_{kl}\left\{\delta_{k1}\left[\frac{\partial^2 b_N(r)}{\partial r^2} + \frac{b_N(r)}{b_N(0)^2 - b_N(r)^2}\left(\frac{\partial b_N(r)}{\partial r}\right)^2\right] + \delta_{k2}\frac{1}{r}\frac{\partial b_N(r)}{\partial r}\right\}.$$

The proof of this lemma is based on Lemmas 5.2, 5.4, 5.6, and 5.7.

Lemma 5.9.

$$\mathbb{E}(\det \nabla f(t^1) \mid f(t^1) = f(t^2) = 0) = 0.$$

The proof of this lemma is based on Lemmas 5.3, 5.6, and 5.7. Note that, although Lemmas 5.6 and 5.7 are proved for the special case

$$t^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} r \\ 0 \end{pmatrix},$$

the result of Lemma 5.9 holds for any two points $t^1 \neq t^2$. The same is true for the next lemma.

Lemma 5.10.

$$\begin{aligned} &\mathbb{E}(\det \nabla f(t^1) \det \nabla f(t^2) \mid f(t^1) = f(t^2) = 0) = \\ &\qquad = \frac{2b_N'(\|t^1 - t^2\|)}{\|t^1 - t^2\|}\left\{\frac{b_N(\|t^1 - t^2\|)b_N'(\|t^1 - t^2\|)^2}{b_N(0)^2 - b_N(\|t^1 - t^2\|)^2} + b_N''(\|t^1 - t^2\|)\right\}. \end{aligned}$$

This lemma is proved with the use of the Lemmas 5.1, 5.8, and 5.9.
We now calculate $\mathbb{E}\{|\det \nabla f(t)| \mid f(t) = 0\}$.

Lemma 5.11.

$$\mathbb{E}\frac{\partial f_i}{\partial t_k}(t)\frac{\partial f_j}{\partial t_l}(t) = \delta_{ij}\delta_{kl}\beta_N,$$

where $\beta_N = -b_N''(0)$, as in Proposition 3.4

Lemma 5.12.

$$\mathbb{E}\{|\det \nabla f(t)| \mid f(t) = 0\} = \beta_N.$$

Proof. It is easily seen that all $\frac{\partial f_i}{\partial t_k}(t)$ are independent of $f(t)$. So, $\det \nabla f(t)$ has the form $\xi_1\xi_2 - \xi_3\xi_4$, where $\xi_i \in N(0, \beta_N), i = 1, 2, 3, 4$ are jointly independent Gaussian random variables (as can be seen from Lemma 5.11), and

$$\mathbb{E}\{|\det \nabla f(t)| \mid f(t) = 0\} = \mathbb{E}|\det \nabla f(t)| = \mathbb{E}|\xi_1\xi_2 - \xi_3\xi_4|.$$

Setting $\eta_i = \frac{1}{\sqrt{\beta_N}}\xi_i$, we get for the jointly independent standard Gaussian random variables $\eta_i, i = 1, 2, 3, 4$:

$$\mathbb{E}|\xi_1\xi_2 - \xi_3\xi_4| = \beta_N \mathbb{E}|\eta_1\eta_2 - \eta_3\eta_4|,$$

and taking

$$\zeta_1 = \frac{\eta_1 + \eta_2}{\sqrt{2}}, \zeta_2 = \frac{\eta_1 - \eta_2}{\sqrt{2}}, \zeta_3 = \frac{\eta_3 + \eta_4}{\sqrt{2}}, \zeta_4 = \frac{\eta_3 - \eta_4}{\sqrt{2}},$$

we get, for the jointly independent standard Gaussian random variables $\zeta_i, i = 1, 2, 3, 4$:

$$\mathbb{E}|\eta_1\eta_2 - \eta_3\eta_4| = \frac{1}{2}\mathbb{E}|\zeta_1^2 + \zeta_4^2 - \zeta_2^2 - \zeta_3^2|.$$

So, we get

$$\mathbb{E}|\zeta_1^2 + \zeta_4^2 - \zeta_2^2 - \zeta_3^2| = \int\limits_{\mathbb{R}^4} |x_1^2 + x_4^2 - x_2^2 - x_3^2| \frac{1}{\sqrt{2\pi}^4} e^{-\frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{2}} dx_1 dx_2 dx_3 dx_4 =$$

$$= (x_1 = r_1 \cos\phi_1, x_4 = r_1 \sin\phi_1, x_2 = r_2 \cos\phi_2, x_3 = r_2 \sin\phi_2) =$$

$$= \int\limits_0^\infty \int\limits_0^\infty |r_1^2 - r_2^2| e^{-\frac{r_1^2 + r_2^2}{2}} r_1 r_2 dr_1 dr_2 =$$

$$= (u = \frac{r_1^2}{2}, v = \frac{r_2^2}{2}) = 2\int\limits_0^\infty \int\limits_0^\infty |u - v| e^{-(u+v)} du dv = 2.$$

□

Thus, Theorem 5.1 is completely proved. □

Theorem 5.2. *Let $\{f_\sigma\}_{\sigma>0}$ be a family of random isotropic Gaussian vector fields on $\mathbb{R}^2$, defined by*

$$f_\sigma(t) = \begin{pmatrix} f_{1,\sigma}(t) \\ f_{2,\sigma}(t) \end{pmatrix} = \begin{pmatrix} \int\limits_{\mathbb{R}^2} \phi_\sigma(t + v) W_1(dv) \\ \int\limits_{\mathbb{R}^2} \phi_\sigma(t + v) W_2(dv) \end{pmatrix},$$

where $\phi_\sigma(x, y) = \sigma^2\phi(\sigma x, \sigma y)$, $\phi \in C_0^\infty(\mathbb{R}^2)$ is a nonnegative function with the property $\int\limits_{\mathbb{R}^2} \phi^2(u) du = 1$, W_1 and W_2 are two independent Brownian sheets on $\mathbb{R}^2$. Then the variance of the winding number of the curve

$$\frac{1}{\sigma}\Gamma = \left\{ \left(\frac{1}{\sigma}x, \frac{1}{\sigma}y \right) \mid (x, y) \in \Gamma \right\}$$

is constant (i.e., it does not depend on σ).

Proof. Consider a random field $\tilde{f}^\sigma(t) = f(\sigma t)$. It is clear that the number of windings of the field $\tilde{f}^\sigma$ along the curve $\frac{1}{\sigma}\Gamma$ coincides (with probability 1) with the number of windings of the field f along Γ: $\mathrm{ind}_\Gamma f = \mathrm{ind}_{\frac{1}{\sigma}\Gamma}\tilde{f}_\sigma$. On the other hand, the fields $\tilde{f}^\sigma$ and f_σ have the same distribution (since these fields are Gaussian and centered, this can be easily checked by comparing the correlation functions). This implies that the random variables $\mathrm{ind}_\Gamma f$ and $\mathrm{ind}_{\frac{1}{\sigma}\Gamma} f_\sigma$ have the same distribution. So, $\mathbb{E}(\mathrm{ind}_\Gamma f_\sigma)^2$ is constant. □

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