Theory of Stochastic Processes
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A. LOGACHOV

LARGE DEVIATION PRINCIPLE FOR PROCESSES WITH POISSON NOISE TERM

Let $\tilde{\nu}_n(du,dt)$ be a centered Poisson measure with the parameter $n\Pi(du)dt$, and let $a_n(t,\omega)$ and $f_n(u,t,\omega)$ be stochastic processes. The large deviation principle for the sequence $\eta_n(t) = x_0 + \int_0^t a_n(s)ds + \frac{1}{\sqrt{n}\varphi(n)}\int_0^t \int f_n(u,s)\tilde{\nu}_n(du,ds)$ is proved. As examples, the large deviation principles for the normalized integral of a telegraph signal and for stochastic differential equations with periodic coefficients are obtained.

1. Introduction

In this paper, we are concerned with the large deviation principle (LDP) for the sequence of stochastic processes $\eta_n(t)$, $n \in N$, $t \in [0,1]$ that is defined on the stochastic basis $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)$ and admits the representation

$$\eta_n(t) = x_0 + \int_0^t a_n(s,\omega)ds + \frac{1}{\sqrt{n}\varphi(n)}\int_0^t \int f_n(u,s,\omega)\tilde{\nu}_n(du,ds), \tag{1}$$

where the martingale Poisson measure $\tilde{\nu}_n(du,dt)$ with the parameter $n\Pi(du)dt$, $u \in U$, is adapted to the flow of sigma algebras $\mathfrak{F}_t$; the random process $a_n(t,\omega)$ is measurable to the flow of sigma algebras $\mathfrak{F}_t$; $f_n(u,t,\omega)$ is a $\mathfrak{F}_t$-predictable random process; and the positive monotonically increasing function $\varphi(n)$ tends to $+\infty$.

We assume that the following conditions are satisfied almost surely $\exists\, \lambda > 1 : \forall\, n \in N$, $\forall\, t \in [0,1]$

$$\int \exp\{|f_n(u,t,\omega)|\} I(|f_n(u,t,\omega)| > 1)\Pi(du) \le \lambda, \quad \frac{1}{\lambda} \le \int f_n^2(u,t,\omega)\Pi(du) \le \lambda. \tag{2}$$

For simplicity, the symbol ω could often be omitted in notations. Sometimes, if it is obvious, we omit other function arguments too.

In a particular case, the process in (1) can be a solution of the stochastic equation with random coefficients, i.e.,

$$a_n(s,\omega) = A_n(s,\eta_n(s),\omega), \quad f_n(u,s,\omega) = F_n(u,s,\eta_n(s),\omega),$$

where the random functions $A_n(s,x,\omega)$ and $F_n(u,s,x,\omega)$ are $\mathfrak{F}_s$ measurable and can depend on the past of the process.

The LDP [1], [2] for a Poisson process in the space of functions without discontinuities of the second kind with a uniform metric and the Skorohod metric is known for the cases $\lim_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = 0$ and $\lim_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = k > 0$, and, in some topology space, for the case $\lim_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = \infty$. The LDP for processes (1) with continuous nonrandom coefficients $a(x)$, $f(u,x)$ is a well-known result, see [3]. When process (1) is a process with independent increments (i.e., $a_n(s,\omega) = a(s)$, $f_n(u,s,\omega) = f(u,s)$), the LDP follows from [4], [5]

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for the case $\lim\limits_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = 0$ and from [6] for the case $\lim\limits_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = k > 0$. A stochastic evolution equation of the jump type is considered in [7] for the case $\lim\limits_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = k > 0$.

The LDP for solutions of stochastic differential equations with an integral over the Poisson measure and the coefficients depending on n was obtained in [8] and [9]. We note that the conditions for the coefficients in (1) and the methods of proofs in [8], [9] are different from conditions and methods of this article. Specifically, one of the basic assumptions in [8] was the nondegenerate diffusion. In our case, the diffusion is identically zero. Moreover, the coefficients in (1) can be random and, respectively, measurable functions.

This paper is organized as follows. In Section 2 in Theorem 2.1, we prove the LDP if the absolutely continuous component and the characteristic of the martingale part in (1) converge to integrals of deterministic functions with some velocity. In Sections 3 and 4, we give the LDP for a normalized integral of the telegraph signal process and for solutions of differential equations with periodic coefficients, respectively.

We will use the notation $\mathbf{B}(X,\rho)$ for a Borel σ-algebra of sets, where (X,ρ) is a metric space.

Recall [10] that the family of probability measures P_n on the space (X,ρ) satisfies the LDP with the rate functional $S(x)$ and the normalizing function $\psi(n)$ if $\psi(n)\to\infty$ as $n\to\infty$, and the following conditions are satisfied:
i) the set $\Phi(x) = \{x : S(x) \le c\}$ is compact for all $c > 0$,
ii) $\overline{\lim\limits_{n\to\infty}} \frac{1}{\psi(n)} \ln P_n(F) \le -S(F)$ for all closed sets $F \in \mathbf{B}(X,\rho)$,
iii) $\underline{\lim\limits_{n\to\infty}} \frac{1}{\psi(n)} \ln P_n(G) \ge -S(G)$, for all open sets $G \in \mathbf{B}(X,\rho)$,
where $S(A) = \inf\limits_{x\in A} S(x)$.
We will use the following notations for functions $x(t)$ defined on $[0,1]$:

$D[0,1]$ – the space of right-continuous functions which have limits from the left and are continuous from the left at $t = 1$;

$AC_{x_0}[0,1]$ – the set of absolutely continuous functions such that $x(0) = x_0$;

$C[0,1]$ – the space of continuous functions;

$L_2[0,1]$ – the space of functions that have a 2-nd power finite integral.

We use the following notations for the sets: $I(A)$ is an indicator of the set A, and $\overline{A}$ is the complement of A.

2. Auxiliary results

We define a metric on the space $D[0,1]$ in the following way:

$$\rho(x,y) = \sup_{0\le t\le 1} \mid x(t) - y(t) \mid .$$

Theorem 2.1. *Let the following conditions hold:*

1) $\lim\limits_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = 0$,

2) there exists a nonrandom function $f(t) \in L_2[0,1]$ *such that, for all* $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\varphi^2(n)} \ln P\left(\sup_{t\in[0,1]} \left| \int_0^t \int f_n^2(u,s)\Pi(du)ds - \int_0^t f^2(s)ds \right| > \varepsilon \right) = -\infty,$$

3) there exists a nonrandom function $a(t) \in L_2[0,1]$ *such that, for all* $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\varphi^2(n)} \ln P\left(\sup_{t\in[0,1]} \left| \int_0^t a_n(s)ds - \int_0^t a(s)ds \right| > \varepsilon \right) = -\infty,$$

4) the function $f_n(u,t)$ *satisfies inequalities (2).*

Then the family of probability measures $P_n(A) = P\{\eta_n(\cdot) \in A\}$, $A \in \mathbf{B}(D[0,1],\rho)$ *satisfies the LDP on the space* $(D[0,1],\rho)$ *with the function* $\psi(n) = \varphi^2(n)$ *and the rate functional*

$$S(x) = \begin{cases} \frac{1}{2}\int\limits_0^1 \frac{(\dot{x}(t)-a(t))^2}{f^2(t)}dt, & \textit{if } x(\cdot) \in AC_{x_0}[0,1], \\ +\infty, & \textit{if } x(\cdot) \notin AC_{x_0}[0,1]. \end{cases} \tag{3}$$

This theorem is proved below after the proof of auxiliary statements.

We will consider the stochastic process

$$\tilde{\eta}_n(t) = \frac{1}{\sqrt{n}\varphi(n)} \int\limits_0^t \int f_n(u,s)\tilde{\nu}_n(du,ds).$$

Lemma 2.1. *Let conditions 1 and 4 of Theorem 2.1 hold. Then, for all* $0 \le v < t \le 1,\ \delta > 0$, *there exists a constant* $N(\delta, t-v):\ \ \forall\ \ n > N(\delta, t-v)$

$$P(\sup_{r\in[v,t]} |\tilde{\eta}_n(r) - \tilde{\eta}_n(v)| \ge \delta) \le 2\exp\biggl\{-\frac{\varphi^2(n)\delta^2}{6\lambda(t-v)}\biggr\}.$$

Proof. For all $c > 0$,

$$P(\sup_{r\in[v,t]} |\tilde{\eta}_n(r) - \tilde{\eta}_n(v)| \ge \delta) = P\biggl(\sup_{r\in[v,t]}\biggl|\frac{1}{\sqrt{n}\varphi(n)}\int\limits_v^r\int f_n(u,s)\tilde{\nu}_n(du,ds)\biggr| \ge \delta\biggr) \le$$

$$\le P\biggl(\sup_{r\in[v,t]} \exp\biggl\{\frac{c}{\sqrt{n}}\int\limits_v^r\int f_n(u,s)\tilde{\nu}_n(du,ds)\biggr\} \ge \exp\{c\delta\varphi(n)\}\biggr) +$$

$$+P\biggl(\sup_{r\in[v,t]} \exp\biggl\{\frac{-c}{\sqrt{n}}\int\limits_v^r\int f_n(u,s)\tilde{\nu}_n(du,ds)\biggr\} \ge \exp\{c\delta\varphi(n)\}\biggr) = P_1 + P_2.$$

Let us estimate P_1. We have

$$P_1 = P\biggl(\sup_{r\in[v,t]} \exp\biggl\{\frac{c}{\sqrt{n}}\int\limits_v^r\int f_n(u,s)\tilde{\nu}_n(du,ds) \pm$$

$$\pm n\int\limits_v^r\int\biggl(\exp\biggl(\frac{cf_n(u,s)}{\sqrt{n}}\biggr) - \frac{cf_n(u,s)}{\sqrt{n}} - 1\biggr)\Pi(du)ds\biggr\} \ge \exp\{c\delta\varphi(n)\}\biggr). \tag{4}$$

Using the inequality $\frac{x^2}{2}\exp|x| \ge \exp(x) - x - 1$ and a decomposition of the function $\exp(x)$ in a Taylor series, we obtain

$$\sup_{r\in[v,t]} n\int\limits_v^r\int\biggl(\exp\biggl(\frac{cf_n(u,s)}{\sqrt{n}}\biggr) - \frac{cf_n(u,s)}{\sqrt{n}} - 1\biggr)\Pi(du)ds \le$$

$$\le \sup_{r\in[v,t]}\int\limits_v^r\int \frac{c^2 f_n^2(u,s)}{2}\exp\biggl(\frac{c|f_n(u,s)|}{\sqrt{n}}\biggr) I(|f_n(u,t)| \le 1)\Pi(du)ds +$$

$$+ \sup_{r\in[v,t]}\int\limits_v^r\int\biggl(\frac{c^2 f_n^2}{2} + \frac{c^3}{\sqrt{n}}\sum_{k=3}^{\infty}\frac{c^{k-3}|f_n^k|}{k!(\sqrt{n})^{k-3}}\biggr) I(|f_n| > 1)\Pi(du)ds = A_1 + A_2.$$

Now set $c = c_n = \frac{\varphi(n)\delta}{2\lambda(t-v)}$. Using conditions 1 and 4 of Theorem 2.1, we obtain the estimates

$$A_1 \le \sup_{r\in[v,t]} \int_v^r \int \frac{c_n^2 f_n^2(u,s)}{2} \exp\left(\frac{c_n}{\sqrt{n}}\right)\Pi(du)ds \le (t-v)\frac{c_n^2\lambda}{2}\exp\left(\frac{c_n}{\sqrt{n}}\right) \le \frac{\varphi^2(n)\delta^2}{6\lambda(t-v)},$$

$$A_2 \le \frac{\varphi^2(n)\delta^2}{8\lambda(t-v)} + \frac{\varphi^2(n)\delta^2}{24\lambda^2(t-v)^2}\sup_{r\in[v,t]}\int_v^r\int\left(\sum_{k=0}^{\infty}\frac{|f_n^k|}{k!}\right)I(|f_n|>1)\Pi(du)ds \le \frac{\varphi^2(n)\delta^2}{6\lambda(t-v)}$$

for sufficiently large n.

Substituting c_n in (4) and using the above estimates, we have

$$P_1 \le P\Bigg(\sup_{r\in[v,t]} \exp\Bigg\{\frac{c_n}{\sqrt{n}}\int_v^r\int f_n(u,s)\tilde{\nu}_n(du,ds) -$$

$$-n\int_v^r\int\left(\exp\left(\frac{c_n f_n(u,s)}{\sqrt{n}}\right) - \frac{c_n f_n(u,s)}{\sqrt{n}} - 1\right)\Pi(du)ds\Bigg\} \ge \exp\left\{\frac{\varphi^2(n)\delta^2}{2\lambda(t-v)} - \frac{\varphi^2(n)\delta^2}{3\lambda(t-v)}\right\}\Bigg).$$

Applying the Doob's inequality for stochastically continuous martingales to the martingale

$$\exp\left\{\frac{c_n}{\sqrt{n}}\int_v^r\int f_n(u,s)\tilde{\nu}_n(du,ds) - n\int_v^r\int\left(\exp\left(\frac{c_n f_n(u,s)}{\sqrt{n}}\right) - \frac{c_n f_n(u,s)}{\sqrt{n}} - 1\right)\Pi(du)ds\right\},$$

we get

$$P_1 \le \exp\left\{-\frac{\varphi^2(n)\delta^2}{6\lambda(t-v)}\right\}.$$

Similarly, P_2 can be estimated. □

Definition. (C-exponential tightness) Definition 3.2.2 and Theorem 3.2.3 in [11], [12]. A sequence of stochastic processes $X_n(\cdot)$ from D[0,1] is C-exponentially tight if, for each $\delta > 0$,

$$\overline{\lim_{n\to\infty}} \frac{1}{\varphi^2(n)} \ln P(\sup_{0\le s\le 1}|X_n(s) - X_n(s-)| \ge \delta) = -\infty.$$

Lemma 2.2. *The sequence of stochastic processes $\{\tilde{\eta}_n(\cdot)\}$ is C–exponentially tight.*

Proof. Using Lemma 2.1, we obtain

$$P(\sup_{0\le s\le 1}|\tilde{\eta}_n(s) - \tilde{\eta}_n(s-)| \ge \delta) \le \sum_{k=1}^{m} P\left(\sup_{\frac{k-1}{m}\le s\le \frac{k}{m}}\left|\tilde{\eta}_n(s) - \tilde{\eta}_n\left(\frac{k-1}{m}\right)\right| \ge \frac{\delta}{2}\right) \le$$

$$\le 2m\exp\left\{-\frac{\varphi^2(n)\delta^2 m}{24\lambda}\right\}$$

for all $\delta > 0$, $m \in N$.

Thus,

$$\overline{\lim_{n\to\infty}} \frac{1}{\varphi^2(n)} \ln P(\sup_{0\le s\le 1}|\tilde{\eta}_n(s) - \tilde{\eta}_n(s-)| \ge \delta) \le -\frac{\delta^2 m}{24\lambda}$$

for all $\delta > 0$, $m \in N$.

Passing to the limit $m \to +\infty$, we have

$$\overline{\lim_{n\to\infty}} \frac{1}{\varphi^2(n)} \ln P(\sup_{0\le s\le 1}|\tilde{\eta}_n(s) - \tilde{\eta}_n(s-)| \ge \delta) = -\infty$$

for all $\delta > 0$.

□

We now consider the exponential martingale

$$\alpha(t) = \exp\Big\{ \frac{\varphi(n)}{\sqrt{n}} \int_0^t \int c(s) g_n(u,s) \tilde{\nu}_n(du,ds) -$$

$$- n \int_0^t \int \Big(\exp\Big(\frac{\varphi(n)c(s)g_n(u,s)}{\sqrt{n}} \Big) - \frac{\varphi(n)c(s)g_n(u,s)}{\sqrt{n}} - 1 \Big) \Pi(du)ds \Big\},$$

where $c(s)$ is a nonrandom bounded function.

Lemma 2.3. *Let the function $g_n(u,t)$ satisfy inequalities (2), $\sup_{s\in[0,t]} |c(s)| \le \tilde{c}$. Then*

$$E\alpha^2(t) \le \exp(12\lambda \tilde{c}^2 \varphi^2(n) t)$$

for sufficiently large n.

Proof. Applying the Itô formula, we get

$$E\alpha^2(t) = 1 + En \int_0^t \int \alpha^2(s) \Big(\exp\Big(\frac{\varphi(n)c(s)g_n(u,s)}{\sqrt{n}} \Big) - 1 \Big)^2 \Pi(du)ds.$$

Using inequalities (2), condition 1 of Theorem 2.1, and a decomposition of the function $\exp(x)$ in a Taylor series, we have

$$n \int \Big(\exp\Big(\frac{\varphi(n)c(s)g_n(u,s)}{\sqrt{n}} \Big) - 1 \Big)^2 \Pi(du) \le n \int \Big(\sum_{k=1}^{\infty} \frac{\varphi^k(n)|g_n^k(u,s)|\tilde{c}^k}{k!(\sqrt{n})^k} \Big)^2 \Pi(du) =$$

$$= \tilde{c}^2 \int \varphi^2(n) g_n^2(u,s) \Big(\sum_{k=1}^{\infty} \frac{\varphi^{k-1}(n)|g_n^{k-1}(u,s)|\tilde{c}^{k-1}}{k!(\sqrt{n})^{k-1}} \Big)^2 I(|g_n(u,s)| \le 1)\Pi(du) +$$

$$+ 4\tilde{c}^2 \int \varphi^2(n) \Big(\sum_{k=1}^{\infty} \frac{\varphi^{k-1}(n)|g_n^k(u,s)|}{2^k k! \Big(\frac{\sqrt{n}}{2\tilde{c}} \Big)^{k-1}} \Big)^2 I(|g_n(u,s)| > 1)\Pi(du) \le$$

$$\le \tilde{c}^2 \int \varphi^2(n) g_n^2(u,s) \exp\{2\}\Pi(du) + 4\tilde{c}^2 \int \varphi^2(n) \exp\{|g_n(u,s)|\} I(|g_n(u,s)| > 1)\Pi(du) \le$$

$$\le 12\varphi^2(n)\lambda\tilde{c}^2$$

for sufficiently large n.

Thus,

$$E\alpha^2(t) \le 1 + 12\varphi^2(n)\lambda\tilde{c}^2 E \int_0^t \alpha^2(s)ds.$$

Applying Gronwall's inequality [10], we obtain

$$E\alpha^2(t) \le \exp(12\lambda\tilde{c}^2\varphi^2(n)t).$$

□

Lemma 2.4. *Let the conditions of Theorem 2.1 hold, then the family of probability measures*

$$P_{n,m}(A) = P\{(\tilde{\eta}_n(t_1), ..., \tilde{\eta}_n(t_m)) \in A\},$$

$A \in \mathbf{B}(R^m)$, $0 = t_0 < t_1 < t_2 < ... < t_m \leq 1$ satisfies the LDP on the space R^m with the function $\psi(n) = \varphi^2(n)$ and the rate functional

$$S_1(x) = \frac{x_1^2}{2\int\limits_0^{t_1} f^2(s)ds} + \sum_{k=2}^{m} \frac{(x_k - x_{k-1})^2}{2\int\limits_{t_{k-1}}^{t_k} f^2(s)ds}, \quad x = (x_1, ..., x_m). \tag{5}$$

Proof. We will use the Gartner–Ellis theorem [13] to prove this lemma. To apply the Gartner-Ellis theorem, we need to find

$$\lim_{n\to\infty} \frac{1}{\varphi^2(n)} \ln E \exp\Big\{\varphi^2(n)\sum_{k=1}^{m} a_k \tilde{\eta}_n(t_k)\Big\}$$

for $a \in R^m$.

Let us find the upper and lower bounds for $E\exp\Big\{\varphi^2(n)\sum\limits_{k=1}^{m} a_k\tilde{\eta}_n(t_k)\Big\}$. We have

$$E_1 = E\exp\Big\{\varphi^2(n)\sum_{k=1}^{m} a_k\tilde{\eta}_n(t_k)\Big\} = E\exp\Big\{\sum_{k=1}^{m}\frac{a_k\varphi(n)}{\sqrt{n}}\int\limits_0^{t_k}\int f_n(u,s)\tilde{\nu}_n(du,ds)\Big\} =$$

$$= E\exp\Big\{\sum_{k=1}^{m}\Big(\sum_{l=k}^{m} a_l\Big)\frac{\varphi(n)}{\sqrt{n}}\int\limits_{t_{k-1}}^{t_k}\int f_n(u,s)\tilde{\nu}_n(du,ds)\Big\}.$$

Let us denote, by $b_k = \sum\limits_{l=k}^{m} a_l$, $f_{n,m}(u,t) = f_n(u,t)\sum\limits_{k=1}^{m} b_k I(t \in [t_{k-1}, t_k))$ and by $M(0,t)$, the sequence of exponential martingales

$$\exp\Big\{\frac{\varphi(n)}{\sqrt{n}}\int\limits_0^t\int f_{n,m}(u,s)\tilde{\nu}_n(du,ds) -$$

$$-n\int\limits_0^t\int\Big(\exp\Big(\frac{\varphi(n)f_{n,m}(u,s)}{\sqrt{n}}\Big) - \frac{\varphi(n)f_{n,m}(u,s)}{\sqrt{n}} - 1\Big)\Pi(du)ds\Big\}.$$

Then E_1 can be rewritten in the form

$$E_1 = EM(0,t_m)\exp\Big\{n\int\limits_0^{t_m}\int\Big(\exp\Big(\frac{\varphi(n)f_{n,m}}{\sqrt{n}}\Big) - \frac{\varphi(n)f_{n,m}}{\sqrt{n}} - 1\Big)\Pi(du)ds\Big\}.$$

Using a decomposition of the function $\exp(x)$ in a Taylor series, we obtain

$$\frac{x^2}{2} - \sum_{k=3}^{\infty}\frac{|x|^k}{k!} \leq \exp(x) - 1 - x \leq \frac{x^2}{2} + \sum_{k=3}^{\infty}\frac{|x|^k}{k!}.$$

By this inequality,

$$\int\limits_0^{t_m}\int\Big(\varphi^2(n)\frac{f_{n,m}^2}{2} - \sum_{j=3}^{\infty}\frac{\varphi^j(n)|f_{n,m}^j|}{j!(\sqrt{n})^{j-2}}\Big)\Pi(du)ds \leq$$

$$\leq n\int\limits_0^{t_m}\int\Big(\exp\Big(\frac{\varphi(n)f_{n,m}}{\sqrt{n}}\Big) - \frac{\varphi(n)f_{n,m}}{\sqrt{n}} - 1\Big)\Pi(du) \leq$$

$$\leq \int\limits_0^{t_m}\int\Big(\varphi^2(n)\frac{f_{n,m}^2}{2} + \sum_{j=3}^{\infty}\frac{\varphi^j(n)|f_{n,m}^j|}{j!(\sqrt{n})^{j-2}}\Big)\Pi(du)ds. \tag{6}$$

Let us estimate $A_m = \int\limits_0^{t_m} \int \sum\limits_{j=3}^{\infty} \frac{\varphi^j(n)|f_{n,m}^j|}{j!(\sqrt{n})^{j-2}} \Pi(du)ds$.

Condition 1 of Theorem 2.1 implies that $\exists\ N(a) : \forall\ n > N(a),\ \forall\ k$ the inequality $\frac{b_k \varphi(n)}{\sqrt{n}} < 1$ is true. Thus, using condition 4 of Theorem 2.1, we obtain, for $n > N(a)$, the inequality

$$A_m = \sum_{k=1}^{m} \int\limits_{t_{k-1}}^{t_k} \int \Big(\sum_{j=3}^{\infty} \frac{\varphi^j(n)|b_k^j||f_n^j|}{j!(\sqrt{n})^{j-2}} \Big) \Pi(du)ds \le$$

$$\le \sum_{k=1}^{m} \frac{|b_k|^3 \varphi^3(n)}{\sqrt{n}} \int\limits_{t_{k-1}}^{t_k} \int \Big(\sum_{j=3}^{\infty} \frac{|f_n^j|}{j!} \Big) \Pi(du)ds \le$$

$$\le \sum_{k=1}^{m} \frac{|b_k|^3 \varphi^3(n)}{\sqrt{n}} \int\limits_{t_{k-1}}^{t_k} \int \Big(f_n^2 \exp\{1\} I(|f_n| \le 1) + \exp\{|f_n|\} I(|f_n| > 1) \Big) \Pi(du)ds \le$$

$$\le \sum_{k=1}^{m} (e+1)\lambda(t_k - t_{k-1}) \frac{|b_k|^3 \varphi^3(n)}{\sqrt{n}} \le 4 \frac{b^3 \varphi^3(n)\lambda}{\sqrt{n}}, \tag{7}$$

where $b = \max\limits_k |b_k|$.

We denote

$$B_n^\varepsilon = \Big\{ \omega : \sup_{t \in [0,1]} \Big| \int\limits_0^t \int f_n^2(u,t) \Pi(du)ds - \int\limits_0^t f^2(s)ds \Big| \le \varepsilon \Big\}.$$

Now, we find the upper bound for E_1. Using inequality (7), we obtain

$$E_1 \le EM(0,t_m) \exp\Big\{ \int\limits_0^{t_m} \int \varphi^2(n) \frac{f_{n,m}^2}{2} \Pi(du)ds + \frac{4b^3 \varphi^3(n)\lambda}{\sqrt{n}} \Big\} =$$

$$= EM(0,t_m) \exp\Big\{ \int\limits_0^{t_m} \int \varphi^2(n) \frac{f_{n,m}^2}{2} \Pi(du)ds + \frac{4b^3 \varphi^3(n)\lambda}{\sqrt{n}} \Big\} (I(B_n^\varepsilon) + I(\overline{B_n^\varepsilon})).$$

Using condition 4 of Theorem 2.1 and the equality $EM(0,t_m) = 1$, we get

$$E_1 \le EM(0,t_m) \exp\Big\{ \sum_{k=1}^{m} \Big(\int\limits_{t_{k-1}}^{t_k} b_k^2 \varphi^2(n) \frac{f^2(s)}{2} ds + \varepsilon b_k^2 \varphi^2(n) \Big) + \frac{4b^3 \varphi^3(n)\lambda}{\sqrt{n}} \Big\} I(B_n^\varepsilon) +$$

$$+ EM(0,t_m) \exp\Big\{ \frac{4b^3 \varphi^3(n)\lambda}{\sqrt{n}} + \sum_{k=1}^{m} \lambda b_k^2 \varphi^2(n)(t_k - t_{k-1}) \Big\} I(\overline{B_n^\varepsilon}) \le$$

$$\le \exp\Big\{ m\varepsilon b^2 \varphi^2(n) + \frac{4b^3 \varphi^3(n)\lambda}{\sqrt{n}} + \sum_{k=1}^{m} \int\limits_{t_{k-1}}^{t_k} b_k^2 \varphi^2(n) \frac{f^2(s)}{2} ds \Big\} +$$

$$+ \exp\Big\{ \lambda b^2 \varphi^2(n) + \frac{4b^3 \varphi^3(n)\lambda}{\sqrt{n}} \Big\} EM(0,t_m) I(\overline{B_n^\varepsilon}).$$

Using the Cauchy–Schwarz inequality, we obtain

$$EM(0,t_m) I(\overline{B_n^\varepsilon}) \le (EM^2(0,t_m) P(\overline{B_n^\varepsilon}))^{1/2}.$$

Condition 2 of Theorem 2.1 implies that

$$P(\overline{B_n^\varepsilon}) \le \exp\{-18\lambda b^2 \varphi^2(n)\}, \tag{8}$$

for sufficiently large n. Applying Lemma 2.3, we get

$$EM^2(0,t_m) \le \exp(12\lambda b^2\varphi^2(n)). \tag{9}$$

Using inequalities (8), (9) and condition 1 of Theorem 2.1, we have

$$\exp\left\{\lambda b^2\varphi^2(n) + \frac{4b^3\varphi^3(n)\lambda}{\sqrt{n}}\right\}EM(0,t_m)I(\overline{B_n^\varepsilon}) \le \exp(-\lambda b^2\varphi^2(n))$$

for sufficiently large n.

Thus,

$$E_1 \le 2\exp\left\{m\varepsilon b^2\varphi^2(n) + \frac{4b^3\varphi^3(n)\lambda}{\sqrt{n}} + \sum_{k=1}^{m}\int_{t_{k-1}}^{t_k} b_k^2\varphi^2(n)\frac{f^2(s)}{2}ds\right\}. \tag{10}$$

Now, we find the lower bound for E_1. Using inequalities (6) and (7), we get

$$E_1 \ge \exp\left\{-m\varepsilon b^2\varphi^2(n) - \frac{4b^3\varphi^3(n)\lambda}{\sqrt{n}} + \sum_{k=1}^{m}\int_{t_{k-1}}^{t_k} b_k^2\varphi^2(n)\frac{f^2(s)}{2}ds\right\}EM(0,t_m)I(B_n^\varepsilon)$$

for sufficiently large n.

We now find the lower bound for $EM(0,t_m)I(B_n^\varepsilon)$. Using inequalities (8), (9) and the equality $EM(0,t_m) = 1$, we get

$$EM(0,t_m)I(B_n^\varepsilon) = EM(0,t_m) - EM(0,t_m)I(\overline{B_n^\varepsilon}) \ge 1 - \exp\{-4\lambda b^2\varphi^2(n)\}. \tag{11}$$

In view of inequality (11), we obtain

$$E_1 \ge \frac{1}{2}\exp\left\{-m\varepsilon b^2\varphi^2(n) - \frac{4b^3\varphi^3(n)\lambda}{\sqrt{n}} + \sum_{k=1}^{m}\int_{t_{k-1}}^{t_k} b_k^2\varphi^2(n)\frac{f^2(s)}{2}ds\right\} \tag{12}$$

for sufficiently large n.

From inequalities (10), (12) and condition 1 of Theorem 2.1, we have

$$\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k} b_k^2\frac{f^2(s)}{2}ds - m\varepsilon b^2 \le \lim_{n\to\infty}\frac{1}{\varphi^2(n)}\ln E\exp\left\{\varphi^2(n)\sum_{k=1}^{m}a_k\tilde{\eta}_n(t_k)\right\} \le$$

$$\le \sum_{k=1}^{m}\int_{t_{k-1}}^{t_k} b_k^2\frac{f^2(s)}{2}ds + m\varepsilon b^2. \tag{13}$$

Inequality (13) holds for all $\varepsilon > 0$. Therefore, passing to the limit $\varepsilon \to 0$, we get

$$\lim_{n\to\infty}\frac{1}{\varphi^2(n)}\ln E\exp\left\{\varphi^2(n)\sum_{k=1}^{m}a_k\tilde{\eta}_n(t_k)\right\} = \sum_{k=1}^{m}\int_{t_{k-1}}^{t_k} b_k^2\frac{f^2(s)}{2}ds.$$

Let us find the Legendre transform for the function $\Lambda(a) = \sum_{k=1}^{m}\left(\sum_{l=k}^{m}a_l\right)^2\int_{t_{k-1}}^{t_k}\frac{f^2(s)}{2}ds$,

$$\Lambda^*(x) = \sup_{a\in R^m}\left(\sum_{k=1}^{m}a_kx_k - \sum_{k=1}^{m}\left(\sum_{l=k}^{m}a_l\right)^2\int_{t_{k-1}}^{t_k}\frac{f^2(s)}{2}ds\right).$$

Denote $F(a) = \sum\limits_{k=1}^{m} a_k x_k - \sum\limits_{k=1}^{m} \left(\sum\limits_{l=k}^{m} a_l \right)^2 \int\limits_{t_{k-1}}^{t_k} \frac{f^2(s)}{2} ds.$

We find the supremum of the function $F(a)$ by elementary calculations:

$$\Lambda^*(x) = \frac{x_1^2}{2 \int\limits_0^{t_1} f^2(s) ds} + \sum_{k=2}^{m} \frac{(x_k - x_{k-1})^2}{2 \int\limits_{t_{k-1}}^{t_k} f^2(s) ds}.$$

□

Lemma 2.5. *Let the conditions of Theorem 2.1 hold. Then the family of probability measures $P\{\tilde{\eta}_n(\cdot) \in A\}$, $A \in \mathbf{B}(D[0,1], \rho)$ satisfies the LDP on the space $(D[0,1], \rho)$ with the function $\psi(n) = \varphi^2(n)$ and the rate functional*

$$S_2(x) = \begin{cases} \frac{1}{2} \int\limits_0^1 \frac{\dot{x}(t)^2}{f^2(t)} dt, & \text{if } x(\cdot) \in AC_0[0,1], \\ +\infty, & \text{if } x(\cdot) \notin AC_0[0,1]. \end{cases} \tag{14}$$

Proof. From Lemma 2.1, it follows that the sequence of stochastic processes $\{\tilde{\eta}_n(\cdot)\}$ is exponentially tight. In Lemma 2.2, we proved that this sequence is C-exponentially tight. Therefore, from Theorem 4.30 [12] and Lemma 2.4, we obtain

$$S_2(x) = \sup_{\{t_k \in T_0\}} S_1((x(t_1), ..., x(t_m))) = \sup_{\{t_k \in T_0\}} \left(\frac{x^2(t_1)}{2 \int\limits_0^{t_1} f^2(s) ds} + \sum_{k=2}^{m} \frac{(x(t_k) - x(t_{k-1}))^2}{2 \int\limits_{t_{k-1}}^{t_k} f^2(s) ds} \right),$$

where T_0 is a dense subset of $[0,1]$.

From conditions 2 and 4 of Theorem 2.1, it follows that $\frac{1}{\lambda} \leq f^2(s) \leq \lambda$ almost everywhere. Therefore,

$$\frac{1}{2\lambda} \sup_{\{t_k \in T_0\}} \left(\frac{x^2(t_1) - 0}{t_1 - 0} + \sum_{k=2}^{m} \frac{(x(t_k) - x(t_{k-1}))^2}{t_k - t_{k-1}} \right) \leq S_2(x) \leq$$

$$\leq \frac{\lambda}{2} \sup_{\{t_k \in T_0\}} \left(\frac{x^2(t_1) - 0}{t_1 - 0} + \sum_{k=2}^{m} \frac{(x(t_k) - x(t_{k-1}))^2}{t_k - t_{k-1}} \right). \tag{15}$$

From Theorem 7 in [14], we get

$$\sup_{\{t_k \in T_0\}} \left(\frac{x^2(t_1) - 0}{t_1 - 0} + \sum_{k=2}^{m} \frac{(x(t_k) - x(t_{k-1}))^2}{t_k - t_{k-1}} \right) < \infty$$

only for $x(\cdot) \in AC_0[0,1]$, whose derivative belongs to the space $L_2[0,1]$. Therefore, from (15), we have $S_2(x) = +\infty$ for functions $x(\cdot) \notin AC_0[0,1]$.

Let $x(\cdot) \in AC_0[0,1]$, $\dot{x}(\cdot) \in L_2[0,1]$. Then

$$S_2(x) = \sup_{\{t_k \in T_0\}} \sum_{k=1}^{m} \frac{\left(\int\limits_{t_{k-1}}^{t_k} \dot{x}(s) ds \right)^2}{2 \int\limits_{t_{k-1}}^{t_k} f^2(s) ds}.$$

The inequality

$$\frac{a^2}{c} + \frac{b^2}{d} \geq \frac{(a+b)^2}{c+d}$$

holds for all a, b, $c > 0$, $d > 0$. Thus,

$$S_2(x) = \lim_{\max(t_k - t_{k-1}) \to 0} \sup_{\{t_k \in T_0\}} \sum_{k=1}^{m} \frac{\left(\int\limits_{t_{k-1}}^{t_k} \dot{x}(s)ds\right)^2}{2 \int\limits_{t_{k-1}}^{t_k} f^2(s)ds}.$$

From Theorem 5 in [14], it follows that almost all points of [0,1] are the Lebesgue points of the functions $\dot{x}^2(\cdot)$ and $f^2(\cdot)$. If we choose the set $T_0 \subseteq [0,1]$ as the set of Lebesgue points of the functions $\dot{x}^2(\cdot)$ and $f^2(\cdot)$, then we have

$$\lim_{(t_k - t_{k-1}) \to 0} \frac{1}{t_k - t_{k-1}} \int\limits_{t_{k-1}}^{t_k} |\dot{x}^2(s) - \dot{x}^2(t_{k-1})| ds = 0, \tag{16}$$

$$\lim_{(t_k - t_{k-1}) \to 0} \frac{1}{t_k - t_{k-1}} \int\limits_{t_{k-1}}^{t_k} |f^2(s) - f^2(t_{k-1})| ds = 0 \tag{17}$$

for $\{t_k\} \in T_0$.

Thus, from (16), (17) and the fact that $\frac{1}{\lambda} \leq f^2(\cdot) \leq \lambda$ almost everywhere, we obtain

$$0 \leq \lim_{(t_k - t_{k-1}) \to 0} \frac{1}{t_k - t_{k-1}} \int\limits_{t_{k-1}}^{t_k} \left| \frac{\dot{x}^2(t_{k-1})}{f^2(t_{k-1})} - \frac{\dot{x}^2(s)}{f^2(s)} \right| ds =$$

$$= \lim_{(t_k - t_{k-1}) \to 0} \frac{1}{t_k - t_{k-1}} \int\limits_{t_{k-1}}^{t_k} \left| \frac{f^2(s)(\dot{x}^2(t_{k-1}) - \dot{x}^2(s)) - \dot{x}^2(s)(f^2(t_{k-1}) - f^2(s))}{f^2(t_{k-1}) f^2(s)} \right| ds \leq$$

$$\leq \lim_{(t_k - t_{k-1}) \to 0} \frac{\lambda^2}{t_k - t_{k-1}} \left(\int\limits_{t_{k-1}}^{t_k} |\dot{x}^2(t_{k-1}) - \dot{x}^2(s)| ds + \int\limits_{t_{k-1}}^{t_k} \dot{x}^2(s) |f^2(t_{k-1}) - f^2(s)| ds \right)$$

$$\leq \lim_{(t_k - t_{k-1}) \to 0} \frac{\lambda^2 \dot{x}^2(t_{k-1})}{t_k - t_{k-1}} \int\limits_{t_{k-1}}^{t_k} |f^2(t_{k-1}) - f^2(s)| ds = 0. \tag{18}$$

Using (16)–(18), we obtain

$$S_2(x) = \lim_{\max(t_k - t_{k-1}) \to 0} \sup_{\{t_k \in T_0\}} \frac{1}{2} \sum_{k=1}^{m} \frac{\left(\frac{1}{t_k - t_{k-1}} \int\limits_{t_{k-1}}^{t_k} \dot{x}(s)ds \right)^2 (t_k - t_{k-1})}{\frac{1}{t_k - t_{k-1}} \int\limits_{t_{k-1}}^{t_k} f^2(s)ds} =$$

$$= \lim_{\max(t_k - t_{k-1}) \to 0} \sup_{\{t_k \in T_0\}} \frac{1}{2} \sum_{k=1}^{m} \frac{\dot{x}^2(t_{k-1})(t_k - t_{k-1})}{f^2(t_{k-1})} =$$

$$= \lim_{\max(t_k - t_{k-1}) \to 0} \sup_{\{t_k \in T_0\}} \frac{1}{2} \sum_{k=1}^{m} \int\limits_{t_{k-1}}^{t_k} \left(\frac{\dot{x}^2(t_{k-1})}{f^2(t_{k-1})} - \frac{\dot{x}^2(s)}{f^2(s)} \right) ds + \frac{1}{2} \int\limits_{0}^{1} \frac{\dot{x}^2(s)}{f^2(s)} ds =$$

$$= \frac{1}{2} \int\limits_{0}^{1} \frac{\dot{x}^2(s)}{f^2(s)} ds.$$

□

Proof of Theorem 2.1. We consider a sequence of stochastic processes

$$\hat{\eta}_n(t) = x_0 + \int_0^t a(s)ds + \tilde{\eta}_n(t). \tag{19}$$

We now use Theorem 3.1 in [10] (contraction principle) in order to find the rate functional for sequence (19). We define an operator $A_{x_0} : x(\cdot) \to y(\cdot)$ that has the form

$$y(t) = x_0 + \int_0^t a(s)ds + x(t)$$

as a continuous mapping of $(D[0,1],\rho)$ in $(D[0,1],\rho)$.

There exists an inverse operator $A_{x_0}^{-1}$, which is given by

$$x(t) = y(t) - x_0 - \int_0^t a(s)ds.$$

Therefore, sequence (19) has the rate functional

$$S_3(y) = \inf_{x:A(x)=y} S_2(x) = \begin{cases} \frac{1}{2}\int_0^1 \frac{(\dot{y}(t)-a(t))^2}{f^2(t)}dt, & \text{if } y(\cdot) \in AC_{x_0}[0,1], \\ +\infty, & \text{if } y(\cdot) \notin AC_{x_0}[0,1]. \end{cases}$$

From condition 3 of Theorem 2.1, it follows that

$$\lim_{n\to\infty} \frac{1}{\varphi^2(n)} \ln P(\rho(\eta_n, \hat{\eta}_n) > \varepsilon) =$$

$$= \lim_{n\to\infty} \frac{1}{\varphi^2(n)} \ln P\left(\sup_{t\in[0,1]} \left| \int_0^t a_n(s)ds - \int_0^t a(s)ds \right| > \varepsilon \right) = -\infty$$

for all $\varepsilon > 0$.

Hence, from Theorem 4.2.13 in [13], it follows that the sequence $\{\eta_n(\cdot)\}$ satisfies the same LDP as the sequence $\{\hat{\eta}_n(\cdot)\}$. □

3. LDP for the normalized integral of the telegraph signal process

In this section, we will use Theorem 2.1 in order to obtain the LDP for the normalized integral of the telegraph signal process.

Lemma 3.1. *Let the following conditions hold:*

1) stochastic processes $\xi_n(t)$, $n \in N$ are almost surely continuous,

2) the family of probability measures $P_n(A) = P\{\xi_n(\cdot) \in A\}$, $A \in \mathbf{B}(D[0,1],\rho)$ satisfies the LDP on the space $(D[0,1],\rho)$ with the normalizing function $\psi(n)$ and some rate functional $S(x)$.

Then the family of probability measures $P_n(B) = P\{\xi_n(\cdot) \in B\}$, $B \in \mathbf{B}(C[0,1],\rho)$ satisfies the LDP on the space $(C[0,1],\rho)$ with the normalizing function $\psi(n)$ and the rate functional $S(x)$.

Proof. It follows obviously from the facts that the stochastic processes $\xi_n(t)$ are almost surely continuous and $(C[0,1],\rho)$ is a closed subspace. □

Theorem 3.1. *Let a positive monotonically increasing function $\varphi(n)$ tend to $+\infty$ and satisfy the condition $\lim\limits_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = 0$. Let the sequence of processes $\zeta_n(t)$, $n \in N$ be defined on a stochastic basis $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)$, and let it have the form*

$$\zeta_n(t) = \frac{\sqrt{\lambda n}}{\varphi(n)} \int_0^t (-1)^{\nu(ns)} ds, \quad t \in [0,1],$$

where the Poisson process $\nu(nt)$ with parameter λnt is adapted to the flow of sigma algebras $\mathfrak{F}_t$.

Then the family of probability measures $P_n(B) = P\{\zeta_n(\cdot) \in B\}$, $B \in \mathbf{B}(C[0,1],\rho)$ satisfies the LDP on the space $(C[0,1],\rho)$ with the function $\psi(n) = \varphi^2(n)$ and the rate functional

$$S_4(x) = \begin{cases} \frac{1}{2}\int\limits_0^1 \dot{x}(t)^2 dt, & \text{if } x(\cdot) \in AC_0[0,1], \\ +\infty, & \text{if } x(\cdot) \notin AC_0[0,1]. \end{cases} \tag{20}$$

Proof. Let us consider a sequence of stochastic processes

$$\theta_n(t) = \frac{1}{2\sqrt{\lambda n}\varphi(n)}\left((-1)^{\nu(nt)} - 1 + 2\lambda n \int_0^t (-1)^{\nu(ns)} ds\right).$$

Using the Itô formula on $\frac{\cos(\pi\nu(nt))}{2\sqrt{\lambda n}\varphi(n)}$, we have that the process $\theta_n(t)$ can be written as

$$\theta_n(t) = -\frac{1}{\sqrt{\lambda n}\varphi(n)} \int_0^t (-1)^{\nu(ns)} d\tilde{\nu}(ns),$$

where $\tilde{\nu}(nt) = \nu(nt) - \lambda nt$.

The process $\nu(nt)$ could be represented as $\int\limits_0^t \int \nu_n(du, ds)$, where the martingale Poisson measure $\tilde{\nu}_n(du, dt)$ has parameter $n\Pi(du)dt$ and $\int \Pi(du) = \lambda$. For this reason,

$$\theta_n(t) = -\frac{1}{\sqrt{\lambda n}\varphi(n)} \int_0^t \int (-1)^{\nu(ns)} \tilde{\nu}_n(du, dt).$$

Therefore, the conditions of Theorem 2.1 hold with the functions $f^2(t) = f_n^2(t) \equiv 1$, $a(t) \equiv 0$, and the family of probability measures $\tilde{P}_n(A) = P\{\theta_n(\cdot) \in A\}$, where $A \in \mathbf{B}(D[0,1],\rho)$ satisfies the LDP on the space $(D[0,1],\rho)$ with the function $\psi(n) = \varphi^2(n)$ and the rate functional(20).

Let us prove that the family of probability measures $P\{\zeta_n(\cdot) \in A\}$ satisfies the same LDP on the space $(D[0,1],\rho)$.

We have

$$\overline{\lim_{n\to\infty}} \frac{1}{\varphi^2(n)} \ln P(\rho(\zeta_n, \theta_n) > \delta) =$$

$$= \overline{\lim_{n\to\infty}} \frac{1}{\varphi^2(n)} \ln P\left(\sup_{t\in[0,1]}\left|\frac{1}{2\sqrt{\lambda n}\varphi(n)}((-1)^{\nu(nt)} - 1)\right| > \delta\right) = -\infty,$$

for all $\delta > 0$.

Hence, from Theorem 4.2.13 in [13], it follows that the family of probability measures $P\{\zeta_n(\cdot) \in A\}$ satisfies the LDP on the space $(D[0,1],\rho)$ with the rate functional(20). The processes $\zeta_n(t)$ are almost surely continuous for all n. Thus, from Lemma 3.1, it follows that the family of probability measures $P\{\zeta_n(\cdot) \in A\}$ satisfies the same LDP on the space $(C[0,1],\rho)$. □

4. LDP for solutions of differential equations with periodic coefficients

In this section, we will use theorem 2.1 in order to obtain the LDP for solutions of differential equations with periodic coefficients. A similar result for diffusion processes is obtained in [15]. The theorem on the LDP for solutions of differential equations with periodic coefficients, which contain an integral over the Poisson measure is obtained in [8], but with a nondegenerate diffusion.

We will consider the stochastic processes $\eta_n(t)$, $n \in N$, $t \in [0,1]$, which are defined on the stochastic basis $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)$ and can be represented as

$$\eta_n(t) = x_0 + \int_0^t a(\gamma(n)\eta_n(s))ds + \frac{1}{\sqrt{n}\varphi(n)} \int_0^t \int f(u, \gamma(n)\eta_n(s))\tilde{\nu}_n(du, ds),$$

where the martingale Poisson measure $\tilde{\nu}_n(du, dt)$ with the parameter $n\Pi(du)dt$, $u \in U$ is adapted to the flow of sigma algebras $\mathfrak{F}_t$; the nonrandom functions $a(x)$ and $\tilde{f}(x) = \int f^2(u,x)\Pi(du)$ are continuously differentiable and periodic with period 1; and the positive monotonically increasing function $\varphi(n)$ tends to $+\infty$.

We will use the notation

$$q = \frac{1}{\int_0^1 \frac{1}{\tilde{f}(x)}dx}, \quad v = \frac{\int_0^1 \frac{a(x)}{\tilde{f}(x)}dx}{\int_0^1 \frac{1}{\tilde{f}(x)}dx}.$$

Theorem 4.1. *Let the following conditions hold:*

1.1) $\lim\limits_{n\to\infty} \frac{\varphi(n)}{\sqrt{n}} = 0$, *1.2)* $\lim\limits_{n\to\infty} \frac{\gamma(n)}{\sqrt{n}\varphi(n)} = 0$, *1.3)* $\lim\limits_{n\to\infty} \frac{\varphi(n)}{\sqrt{\gamma(n)}} = 0$,

exist $\lambda > 1$ *such that*

2.1) $|f^2(u,x)| + |\tilde{f}(x)| + |\tilde{f}'(x)| + |a(x)| + |a'(x)| \le \lambda$,

2.2) $\frac{1}{\lambda} \le \int f^2(u,x)\Pi(du)$,

for all u, x.

Then the family of probability measures $P_n(A) = P\{\eta_n(\cdot) \in A\}$, $A \in \mathbf{B}(D[0,1], \rho)$ *satisfies the LDP on the space* $(D[0,1], \rho)$ *with the function* $\psi(n) = \varphi^2(n)$ *and the rate functional*

$$S_5(x) = \begin{cases} \frac{1}{2q}\int_0^1 (\dot{x}(t) - v)^2 dt, & \text{if } x(\cdot) \in AC_{x_0}[0,1], \\ +\infty, & \text{if } x(\cdot) \notin AC_{x_0}[0,1]. \end{cases}$$

Proof. From the fact that the function $\tilde{f}(x)$ is periodic and continuous, it follows that the integral

$$\int f^2(u, \gamma(n)\eta_n(t))\Pi(du)$$

is bounded.

Therefore, from condition 2.1 we have that the integral

$$\int \exp\{|f(u,\gamma(n)\eta_n(t))|\} I(|f(u,\gamma(n)\eta_n(t))| > 1)\Pi(du)$$

is bounded.

Hence, we will assume that λ is chosen in such way that the integrals above are less than λ.

Thus, conditions 1 and 4 of Theorem 2.1 are satisfied, and it is sufficient to prove that

$$\lim_{n\to\infty} \frac{1}{\varphi^2(n)} \ln P\left(\sup_{t\in[0,1]} \left| \int_0^t (a(\gamma(n)\eta_n(s)) - v)ds \right| > \varepsilon \right) = -\infty, \tag{21}$$

$$\lim_{n\to\infty}\frac{1}{\varphi^2(n)}\ln P\Big(\sup_{t\in[0,1]}\Big|\int_0^t\Big(\int f^2(u,\gamma(n)\eta_n(s))\Pi(du)-q\Big)ds\Big|>\varepsilon\Big)=-\infty. \tag{22}$$

Let us prove (21). We use the notation

$$\theta_1(x)=\frac{a(x)-v}{\tilde f(x)}.$$

From the definition of constant v, it follows that

$$\int_0^1\theta_1(x)dx=0. \tag{23}$$

We denote

$$H_1(x)=\int_0^x\int_0^r\theta_1(s)dsdr.$$

Applying the Itô formula to the function $H_1(\gamma(n)\eta_n(t))$, we have

$$\int_0^{\gamma(n)\eta_n(t)}\int_0^r\theta_1(s)dsdr=\int_0^{\gamma(n)x_0}\int_0^r\theta_1(s)dsdr+\gamma(n)\int_0^t\Big(\int_0^{\gamma(n)\eta_n(r)}\theta_1(s)ds\Big)a(\gamma(n)\eta_n(r))dr+$$

$$+n\int_0^t\int\Big[H_1\Big(\gamma(n)\eta_n(r)+\frac{\gamma(n)f(u,\gamma(n)\eta_n(r))}{\sqrt{n}\varphi(n)}\Big)-H_1(\gamma(n)\eta_n(r))-$$

$$-\frac{\gamma(n)f(u,\gamma(n)\eta_n(r))}{\sqrt{n}\varphi(n)}\int_0^{\gamma(n)\eta_n(r)}\theta_1(s)ds\Big]\Pi(du)dr+$$

$$+\int_0^t\int\Big[H_1\Big(\gamma(n)\eta_n(r)+\frac{\gamma(n)f(u,\gamma(n)\eta_n(r))}{\sqrt{n}\varphi(n)}\Big)-H_1(\gamma(n)\eta_n(r))\Big]\tilde\nu_n(du,dr). \tag{24}$$

Applying Taylor's formula to the function $H_1\Big(\gamma(n)\eta_n(r)+\frac{\gamma(n)f(u,\gamma(n)\eta_n(r))}{\sqrt{n}\varphi(n)}\Big)$, we get

$$H_1\Big(\gamma(n)\eta_n(r)+\frac{\gamma(n)f(u,\gamma(n)\eta_n(r))}{\sqrt{n}\varphi(n)}\Big)=H_1(\gamma(n)\eta_n(r))+$$

$$+\frac{\gamma(n)f(u,\gamma(n)\eta_n(r))}{\sqrt{n}\varphi(n)}\int_0^{\gamma(n)\eta_n(r)}\theta_1(s)ds+\theta_1(\gamma(n)\eta_n(r))\frac{\gamma^2(n)f^2(u,\gamma(n)\eta_n(r))}{2n\varphi^2(n)}+\rho_n(u,r).$$

Using Taylor's formula with the remainder term in the Lagrange form, we obtain

$$|\rho_n(u,r)|\le\frac{\gamma^3(n)f^3(u,\gamma(n)\eta_n(r))}{n^{3/2}\varphi^3(n)}\sup_x|\theta_1'(x)|.$$

From condition 2.1, it follows that

$$|\rho_n(u,r)|\le\frac{3\lambda^7\gamma^3(n)f^2(u,\gamma(n)\eta_n(r))}{n^{3/2}\varphi^3(n)}, \tag{25}$$

for all u, r.
Hence,

$$n\int_0^t\int\Big[H_1\Big(\gamma(n)\eta_n(r)+\frac{\gamma(n)f(u,\gamma(n)\eta_n(r))}{\sqrt{n}\varphi(n)}\Big)-H_1(\gamma(n)\eta_n(r))-$$

$$-\frac{\gamma(n)f(u,\gamma(n)\eta_n(r))}{\sqrt{n}\varphi(n)}\int\limits_0^{\gamma(n)\eta_n(r)}\theta_1(s)ds\Big]\Pi(du)dr=$$

$$=\frac{\gamma^2(n)}{2\varphi^2(n)}\int\limits_0^t(a(\gamma(n)\eta_n(r))-v)dr+n\int\limits_0^t\int\rho_n(u,r)\Pi(du)dr. \tag{26}$$

Applying Taylor's formula to the function $H_1\Big(\gamma(n)\eta_n(r)+\frac{\gamma(n)f(u,\gamma(n)\eta_n(r))}{\sqrt{n}\varphi(n)}\Big)$ again, we get

$$H_1\Big(\gamma(n)\eta_n(r)+\frac{\gamma(n)f(u,\gamma(n)\eta_n(r))}{\sqrt{n}\varphi(n)}\Big)=H_1(\gamma(n)\eta_n(r))+$$
$$+\frac{\gamma(n)f(u,\gamma(n)\eta_n(r))}{\sqrt{n}\varphi(n)}\int\limits_0^{\gamma(n)\eta_n(r)}\theta_1(s)ds+\tilde{\rho}_n(u,r). \tag{27}$$

From condition 2.1 and Taylor's formula with the remainder term in the Lagrange form, it follows that

$$|\tilde{\rho}_n(u,r)|\le\frac{\lambda^5\gamma^2(n)f^2(u,\gamma(n)\eta_n(r))}{n\varphi^2(n)}, \tag{28}$$

for all u, r.

Thus, from (24), (26), and (27), we obtain

$$\int\limits_0^t(a(\gamma(n)\eta_n(r))-v)dr=\frac{2\varphi^2(n)}{\gamma^2(n)}\int\limits_0^{\gamma(n)\eta_n(t)}\int\limits_0^r\theta_1(s)dsdr-\frac{2\varphi^2(n)}{\gamma^2(n)}\int\limits_0^{\gamma(n)x_0}\int\limits_0^r\theta_1(s)dsdr-$$
$$-\frac{2\varphi^2(n)}{\gamma(n)}\int\limits_0^t\Big(\int\limits_0^{\gamma(n)\eta_n(r)}\theta_1(s)ds\Big)a(\gamma(n)\eta_n(r))dr-$$
$$-\frac{2\varphi(n)}{\gamma(n)\sqrt{n}}\int\limits_0^t\int\Big(\int\limits_0^{\gamma(n)\eta_n(r)}\theta_1(s)ds\Big)f(u,\gamma(n)\eta_n(r))\tilde{\nu}_n(du,dr)-$$
$$-\frac{2\varphi^2(n)n}{\gamma^2(n)}\int\limits_0^t\int\rho_n(u,r)\Pi(du)dr-\frac{2\varphi^2(n)}{\gamma^2(n)}\int\limits_0^t\int\tilde{\rho}_n(u,r)\tilde{\nu}_n(du,dr)=$$
$$=I_1(t)-I_2-I_3(t)-I_4(t)-I_5(t)-I_6(t). \tag{29}$$

Let us estimate the absolute values of terms on the right-hand side of equality (29). From (23) and the fact that the function $\theta_1(x)$ is continuous and periodic with period 1, it follows that there exists the constant $\lambda>1$ such as

$$\sup_r\int\limits_0^r\theta_1(x)dx\le\frac{\lambda}{2}. \tag{30}$$

Using (30), we have that, almost surely,

$$|I_1(t)|=\frac{2\varphi^2(n)}{\gamma^2(n)}\Big|\int\limits_0^{\gamma(n)\eta_n(t)}\int\limits_0^r\theta_1(s)dsdr\Big|\le\frac{\varphi^2(n)\lambda}{\gamma(n)}\sup_{t\in[0,1]}|\eta_n(t)|, \tag{31}$$

$$|I_2|=\frac{2\varphi^2(n)}{\gamma^2(n)}\Big|\int\limits_0^{\gamma(n)x_0}\int\limits_0^r\theta_1(s)dsdr\Big|\le\frac{\varphi^2(n)\lambda}{\gamma(n)}|x_0|. \tag{32}$$

From (30) and condition 2.1, it follows that, almost surely,

$$|I_3(t)| = \frac{2\varphi^2(n)}{\gamma(n)}\Big|\int_0^t\Big(\int_0^{\gamma(n)\eta_n(r)}\theta_1(s)ds\Big)a(\gamma(n)\eta_n(r))dr\Big| \le \frac{\varphi^2(n)\lambda^2}{\gamma(n)}. \qquad (33)$$

We will use the notation

$$B_c = \{\omega : \sup_{t\in[0,1]}|\eta_n(t)| > c\}$$

for $c > \lambda + |x_0|$.

Using Lemma 2.1 and the fact that $\sup_x |a(x)| \le \lambda$, we have

$$P(B_c) \le P\Big(\sup_{t\in[0,1]}\frac{1}{\sqrt{n}\varphi(n)}\Big|\int_0^t\int f(u,\gamma(n)\eta_n(s))\tilde{\nu}_n(du,ds)\Big| \ge c-\lambda-|x_0|\Big) \le$$

$$\le 2\exp\Big\{-\frac{\varphi^2(n)(c-\lambda-|x_0|)^2}{6\lambda}\Big\}. \qquad (34)$$

We denote

$$D_\delta = \Big\{\omega : \sup_{t\in[0,1]}\Big|\frac{2\varphi(n)}{\gamma(n)\sqrt{n}}\int_0^t\int\Big(\int_0^{\gamma(n)\eta_n(r)}\theta_1(s)ds\Big)f(u,\gamma(n)\eta_n(r))\tilde{\nu}_n(du,dr)\Big| > \delta\Big\},$$

where $\delta > 0$.

Using Lemma 2.1, conditions 1.2 and 1.3 and inequality (30), we obtain

$$P(D_\delta) \le 2\exp\Big\{-\frac{\gamma^2(n)\delta^2}{6\lambda^2\varphi^2(n)}\Big\}. \qquad (35)$$

Let

$$G_\delta = \Big\{\omega : \sup_{t\in[0,1]}\Big|\frac{2\varphi^2(n)}{\gamma^2(n)}\int_0^t\int\tilde{\rho}_n(u,r)\tilde{\nu}_n(du,dr)\Big| > \delta\Big\}.$$

In view of Lemma 2.1, conditions 1.2 and 1.3, and (28), we obtain

$$P(G_\delta) \le 2\exp\Big\{-\frac{\gamma^2(n)\delta^2c^2}{6\lambda^2\varphi^2(n)}\Big\} \qquad (36)$$

for all $c > 0$ and sufficiently large n.

It follows from (25) that, almost surely,

$$|I_5(t)| = \frac{2\varphi^2(n)n}{\gamma^2(n)}\int_0^t\int\rho_n(u,r)\Pi(du)dr \le \frac{\delta}{\lambda}\int_0^t\int f^2(u,\gamma(n)\eta_n(r))\Pi(du) \le \delta, \qquad (37)$$

for all $\delta > 0$ and sufficiently large n.

We now denote

$$H_1^\delta = \{\omega : \sup_{t\in[0,1]}|I_1(t)| > \delta\},\ H_2^\delta = \{\omega : \sup_{t\in[0,1]}|I_2| > \delta\},\ H_3^\delta = \{\omega : \sup_{t\in[0,1]}|I_3(t)| > \delta\},$$

$$H_4^\delta = \{\omega : \sup_{t\in[0,1]}|I_5(t)| > \delta\},\ A_\varepsilon = \Big\{\omega : \sup_{t\in[0,1]}\Big|\int_0^t(a(\gamma(n)\eta_n(s)) - v)ds\Big| > \varepsilon\Big\}.$$

From (29), estimates (31)-(37), and condition 1.3, it follows that

$$P(A_\varepsilon) \le P(A_\varepsilon \cap \overline{B_c}) + P(B_c) \le$$

$$\le P((H_1^{\varepsilon/6} \cup H_2^{\varepsilon/6} \cup H_3^{\varepsilon/6} \cup H_4^{\varepsilon/6} \cup D_{\varepsilon/6} \cup G_{\varepsilon/6}) \cap \overline{B_c}) + P(B_c) \le$$

$$\leq P\left(\frac{\varphi^2(n)\lambda c}{\gamma(n)} \geq \frac{\varepsilon}{6}\right)+P\left(\frac{\varphi^2(n)\lambda|x_0|}{\gamma(n)} \geq \frac{\varepsilon}{6}\right)+P\left(\frac{\varphi^2(n)\lambda^2}{\gamma(n)} \geq \frac{\varepsilon}{6}\right)+$$

$$+2\exp\left\{-\frac{\gamma^2(n)\varepsilon^2}{216\lambda^2\varphi^2(n)}\right\}+2\exp\left\{-\frac{\gamma^2(n)\varepsilon^2c^2}{216\lambda^2\varphi^2(n)}\right\}+2\exp\left\{-\frac{\varphi^2(n)(c-\lambda-|x_0|)^2}{6\lambda}\right\}=$$

$$=2\exp\left\{-\frac{\gamma^2(n)\varepsilon^2}{216\lambda^2\varphi^2(n)}\right\}+2\exp\left\{-\frac{\gamma^2(n)\varepsilon^2c^2}{216\lambda^2\varphi^2(n)}\right\}+2\exp\left\{-\frac{\varphi^2(n)(c-\lambda-|x_0|)^2}{6\lambda}\right\}\leq$$

$$\leq 6\exp\left\{-\frac{\varphi^2(n)(c-\lambda-|x_0|)^2}{6\lambda}\right\} \tag{38}$$

for all $c > \lambda + |x_0|$ and sufficiently large n.

Using (38), we obtain

$$\lim_{n\to\infty}\frac{1}{\varphi^2(n)}\ln P(A_\varepsilon) \leq -\frac{(c-\lambda-|x_0|)^2}{6\lambda} \tag{39}$$

for all $c > \lambda + |x_0|$.

Inequality (39) holds for all $c > \lambda + |x_0|$. Hence, passing to the limit $c \to +\infty$, we get (21).

To prove (22), we apply the Itô formula to the function $H_2(\gamma(n)\eta_n(t))$, where

$$H_2(x) = \int_0^x\int_0^r \theta_2(s)dsdr, \quad \theta_2(x) = 1 - \frac{q}{\tilde{f}(x)}.$$

□

As an illustration of Theorem 4.1, we give an example. Let

$$\eta_n(t) = \int_0^t \frac{ds}{5+\sin(2\pi n^{2/3}\eta_n(s))} + \frac{1}{n^{3/4}}\int_0^t\int \frac{l(u)}{\sqrt{5+\cos(2\pi n^{2/3}\eta_n(s))}}\tilde{\nu}_n(du,ds),$$

where the function $l(u)$ is such that $\int l^2(u)\Pi(du) = 1$ and $l^2(u) \leq \lambda$. The conditions of Theorem 4.1 hold with $\varphi(n) = n^{1/4}$, $\gamma(n) = n^{2/3}$. In that case, the normalizing function $\psi(n) = n^{1/2}$, the constants $q = 1/5$, $v = \frac{1}{2\sqrt{6}}$ and the rate functional

$$S_5(x) = \begin{cases} \frac{5}{2}\int_0^1 \left(\dot{x}(t) - \frac{1}{2\sqrt{6}}\right)^2 dt, & \text{if } x(\cdot) \in AC_0[0,1], \\ +\infty, & \text{if } x(\cdot) \notin AC_0[0,1]. \end{cases}$$

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74, R. Luxemburgh Str., Donetsk 83114, Ukraine
E-mail address: omboldovskaya@mail.ru