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DISTRIBUTION OF SOME FUNCTIONALS FOR A LÉVY PROCESS WITH MATRIX-EXPONENTIAL JUMPS OF THE SAME SIGN

This paper provides a framework for investigations in fluctuation theory for Lévy processes with matrix-exponential jumps. We present a matrix form of the components of the infinitely divisible factorization. Using this representation we establish generalizations of some results known for compound Poisson processes with exponential jumps in one direction and generally distributed jumps in the other direction.

Lévy processes have many applications in practice as a base model in risk theory, queuing and financial mathematics. Many problems can be connected to the fluctuation theory, in which the factorization method plays a crucial role (see, for instance, [1]).

The most studied class of Lévy processes is the class of semi-continuous processes (with Lévy measure supported on a half-axis). One of the factorization components for the semi-continuous processes is entirely defined by the real root of the cumulant equation (or more specifically, by the right-inverse of the cumulant function).

This result can be generalized for Lévy processes with matrix-exponential upward (or downward) jumps (see [2, 3] and references therein). For such processes one of the factorization components is a rational function with finite number of poles, which are the (possibly complex) roots of the cumulant equation. Using the properties of matrix-exponential distribution we can invert the component to find the distribution of corresponding killed extremum. The convolution of the distribution of this extremum and the integral transform of the Lévy measure defines the moment generating function of other extremum.

We use the relations for the factorization components to obtain in closed form the moment generating function of occupation time of a half-line (for semi-continuous case we refer to [4] and for some other cases to [5]). Further generalization could be done for meromorphic Lévy processes with the main difference that the factorization components have infinitely many poles (see [6]).

1. Matrix-exponential distribution

The class of matrix-exponential (ME) distributions is the generalization of exponential distribution and it comprises the phase-type distributions. The ME class can be defined as a class of distributions with a rational moment generating function (see [7]). The properties of ME distributions allow us to find in the closed form some generalizations of the results known for Lévy processes with exponential jumps.

A nonnegative random variable has a ME(d) distribution ($d \geq 1$), if its cumulative distribution function is as follows

$$F(x) = \begin{cases} 1 + \boldsymbol{\beta} e^{\mathbf{R}x} \mathbf{R}^{-1} \mathbf{t} & x > 0; \\ 0 & x \leq 0, \end{cases}$$

where $\boldsymbol{\beta}$ is a $1 \times d$ vector, $\mathbf{R}$ is a non singular $d \times d$ matrix, $\mathbf{t}$ is a $d \times 1$ vector, and each possibly have complex entries. The triple $(\boldsymbol{\beta}, \mathbf{R}, \mathbf{t})$ is called a representation of the ME distribution. Note that, the same distribution may have several representations. If the cumulative function of a ME distribution can be represented as $F(x) = 1 - \boldsymbol{\alpha} e^{\mathbf{T}x}\mathbf{e}, x > 0$, where $\boldsymbol{\alpha}$ is a probability vector and $\mathbf{T}$ is the intensity matrix of a Markov chain, $\mathbf{e} = (0, \ldots, 0, 1)^{\top}$, then the ME distribution is called phase-type distribution. For details and general results on matrix-exponential distributions we refer to [8].

For $x > 0$ a ME distribution has density $f(x) = \boldsymbol{\beta} e^{\mathbf{R}x}\mathbf{t}$, which can be rewritten as (see [2] and [3]):

$$f(x) = \sum_{i=1}^{m} P_i(x) e^{-r_i x},$$

where $P_i(x)$ are polynomials of degree k_i, $\Re[r_m] \geq \ldots \geq \Re[r_2] > r_1 > 0$ and $\sum_{i=1}^{m} k_i + m = d$.

If $p_0 = 1 + \boldsymbol{\beta}\mathbf{R}^{-1}\mathbf{t} \neq 0$, then the ME distribution has nonzero mass at zero, and the moment generating function has the form

$$\int_0^{\infty} e^{rx} dF(x) = p_0 - \boldsymbol{\beta}(r\mathbf{I} + \mathbf{R})^{-1}\mathbf{t}, \quad \Re[r] = 0.$$

To find a representation of the distribution with known moment generating function we can follow the approach given in [7]. Denote the vectors $\boldsymbol{\rho} = (\rho_d, \ldots, \rho_1)$, $(\boldsymbol{\rho}, 1) = (\rho_d, \ldots, \rho_1, 1)$, $\mathbf{h}_d(r) = (1, r, \ldots, r^d)^{\top}$. If the Laplace transform of $f(x)$ has the form

$$(1) \qquad \int_0^{\infty} e^{-rx} f(x)\, dx = \frac{\beta_1 r^{d-1} + \beta_2 r^{d-2} + \ldots + \beta_d}{r^d + \rho_1 r^{d-1} + \ldots + \rho_{d-1} r + \rho_d} = \frac{\boldsymbol{\beta}\mathbf{h}_{d-1}(r)}{(\boldsymbol{\rho}, 1)\mathbf{h}_d(r)},$$

then the corresponding density is $f(x) = \boldsymbol{\beta} e^{\mathbf{R}x}\mathbf{e}$, $x > 0$, where $\boldsymbol{\beta} = (\beta_d, \ldots, \beta_1)$, $\mathbf{e} = (0, \ldots, 0, 1)^{\top}$, and $\mathbf{R} = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ -\rho_d & -\rho_{d-1} & \ldots & -\rho_1 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{I} \\ -\boldsymbol{\rho} \end{pmatrix}$.

In the case when a ME distribution is defined on negative half-axis, we can follow the similar reasoning. If

$$(2) \qquad \int_{-\infty}^{0} e^{rx} f(x)\, dx = \frac{\boldsymbol{\beta}\mathbf{h}_{d-1}(r)}{(\boldsymbol{\rho}, 1)\mathbf{h}_d(r)},$$

then $f(x) = \boldsymbol{\beta} e^{\mathbf{R}x}\mathbf{e}$, $x < 0$, where $\boldsymbol{\beta} = (\beta_d, \ldots, \beta_1)$, $\mathbf{R} = \begin{pmatrix} 0 & -\mathbf{I} \\ \boldsymbol{\rho} \end{pmatrix}$. If a ME distribution has support on the entire real line, then it is called bilateral matrix-exponential distribution (see [9]).

2. Extrema and overshoot

Let us suppose that $X_t, t \geq 0$ is a Lévy process with cumulant function

$$k(r) = a'r + \frac{\sigma^2}{2}r^2 + \int_{-\infty}^{\infty}\left(e^{rx} - 1 - rxI_{\{|x| \leq 1\}}\right)\Pi(dx),$$

where a' is a real constant, $\sigma > 0$, and Π is a non negative measure, defined on $R\backslash\{0\}$: $\int_R \min\{x^2, 1\}\Pi(dx) < \infty$.

Throughout we impose the restriction that $\int_{-1}^{1} |x|\Pi(dx) < \infty$, then the cumulant function can be represented as follows

$$(3) \qquad k(r) = ar + \frac{\sigma^2}{2}r^2 + \int_{-\infty}^{\infty}(e^{rx} - 1)\Pi(dx),$$

where $a = a' - \int_{-1}^{1} |x| \Pi(dx)$.

Denote by θ_s an exponential random variable with parameter $s > 0$: $\mathsf{P}\{\theta_s > t\} = e^{-st}, t > 0$, independent of process X_t, and by definition $\theta_0 = \infty$. Then X_{θ_s} is called a Lévy process killed at rate s (see [10]). For the moment generating function of X_{θ_s}

$$\mathsf{E}e^{rX_{\theta_s}} = \frac{s}{s - k(r)}$$

the identity of infinitely divisible factorization takes place

$$\mathsf{E}e^{rX_{\theta_s}} = \mathsf{E}e^{rX_{\theta_s}^+}\mathsf{E}e^{rX_{\theta_s}^-}, \Re[r] = 0,$$

where $X_{\theta_s}^+ = \sup_{0 \leq t \leq \theta_s} X_t$, $X_{\theta_s}^- = \inf_{0 \leq t \leq \theta_s} X_t$ are the supremum and infimum of the process, respectively, killed at rate s.

In general case the closed formulae for factorization components are not known, so we should impose additional restrictions on the parameters of the process. Following [3], we consider Lévy processes that have finite intensity negative (or positive) jumps with ME distribution, arbitrary positive (negative) jumps, and possibly drift and gaussian component.

That is, we assume that $\Pi(dx) = \lambda_- \sum_{i=1}^{m_-} P_i^-(x) e^{b_i x} dx$, $x < 0$ (or $\Pi(dx) = \lambda_+ \sum_{i=1}^{m_+} P_i^+(x) e^{-c_i x} dx$, $x > 0$) where $\lambda_\pm = \int_{R^\pm} \Pi(dx) < \infty$, $P_i^\pm(x)$ are the polynomials of degree $k_i^\pm$, $\sum_{i=1}^{m_\pm} k_i^\pm + m_\pm = d_\pm$, $\Re[b_{m_-}] \geq \ldots \geq \Re[b_2] > b_1 > 0$ and $\Re[c_{m_+}] \geq \ldots \geq \Re[c_2] > c_1 > 0$. Also we split two cases (for details, see [3]):

$(NS)_\pm$: $\sigma > 0$ or $\sigma = 0, \pm a > 0$,
$(S)_\pm$: $\sigma = 0, \mp a \geq 0$,

where sign '+' corresponds to the case when positive jumps have ME distribution while sign '−' corresponds to the case when negative jumps have ME distribution.

Due to [2, 3], in any of the cases $(NS)_\pm$ or $(S)_\pm$ the moment generating function $\mathsf{E}e^{rX_{\theta_s}^\pm}$ is a rational function and $X_{\theta_s}^\pm$ has a ME distribution. In addition, the cumulant equation

$$k(r) = s$$

has the roots $\{\pm r_i^\pm(s)\}_{i=1}^{N_\pm}$ in half-plane $\pm\Re[r] > 0$, where $N_\pm = \begin{cases} d_\pm + 1 & (NS)_\pm, \\ d_\pm & (S)_\pm, \end{cases}$

$r_1^+(s)$ is the unique root on $[0, c_1]$ ($-r_1^-(s)$ is the unique root on $[-b_1, 0]$). These roots entirely define the distribution of corresponding extrema.

Write

$$\beta_k^- = \sum_{1 \leq i_1 < \ldots < i_k \leq N_-} b_{i_1} \ldots b_{i_k}, \beta_k^+ = \sum_{1 \leq i_1 < \ldots < i_k \leq N_+} c_{i_1} \ldots c_{i_k},$$

$$\rho_k^\pm(s) = \sum_{1 \leq i_1 < \ldots < i_k \leq N_\pm} r_{i_1}^\pm(s) \ldots r_{i_k}^\pm(s),$$

then the distribution of $X_{\theta_s}^\pm$ can be represented by the parameters:

$$\boldsymbol{\beta}_\pm = \left(\beta_{d_\pm}^\pm, \ldots, \beta_1^\pm\right), \boldsymbol{\rho}_\pm(s) = \left(\rho_{N_\pm}^\pm(s), \ldots, \rho_1^\pm(s)\right), \mathbf{R}_\pm(s) = \begin{pmatrix} 0 & \pm\mathbf{I} \\ \mp\boldsymbol{\rho}_\pm(s) \end{pmatrix}.$$

Under additional conditions the moment generating function of $X_{\theta_s}^\mp$ we can determine in terms of integral transforms of the Lévy measure:

$$\overline{\Pi}^+(x) = \int_x^\infty \Pi(dx), x > 0; \overline{\Pi}^-(x) = \int_{-\infty}^x \Pi(dx), x < 0; \tilde{\Pi}^\pm(r) = \int_{R^\pm} e^{rx}\overline{\Pi}^\pm(x) dx.$$

The following statement is essentially based on the results given in [2] and [3].

Theorem 2.1. *If for Lévy process X_t one of the cases $(NS)_-$ or $(S)_-$ holds, then*

$$(4) \qquad P'_-(s,x) = \frac{\partial}{\partial x} \mathsf{P}\left\{X^-_{\theta_s} < x\right\} = \mathbf{q}_-(s)\, e^{\mathbf{R}_-(s)x}\mathbf{e}, \, x < 0,$$

where

$$(5) \qquad \mathbf{q}_-(s) = \begin{cases} \frac{\rho^-_{d_-+1}(s)}{\beta^-_{d_-}}(\boldsymbol{\beta}_-, 1) & (NS)_-\,, \\[2ex] \frac{\rho^-_{d_-}(s)}{\beta^-_{d_-}}(\boldsymbol{\beta}_- - \boldsymbol{\rho}_-(s)) & (S)_-\,. \end{cases}$$

Moreover, in case $(S)_-$: $p_-(s) = \mathsf{P}\left\{X^-_{\theta_s} = 0\right\} \neq 0$ and $p_-(s) = \frac{\rho^-_{d_-}(s)}{\beta^-_{d_-}} = \left(\prod_{i=1}^{d_-} \frac{r^-_i(s)}{b_i}\right)$.

If additionally $\mathsf{D}X_1 < \infty$, $\mathsf{E}X^+_{\theta_s} < \infty$, the moment generating function of $X^+_{\theta_s}$ could be represented as

$$(6)$$
$$\mathsf{E}e^{rX^+_{\theta_s}} = \left(1 - \frac{r}{s}\left(A^-_*(s) + \mathsf{E}e^{rX^-_{\theta_s}}\tilde{\Pi}^+(r) - \mathbf{q}_-(s)\,(r\mathbf{I} + \mathbf{R}_-(s))^{-1}\tilde{\Pi}^+(-\mathbf{R}_-(s))\mathbf{e}\right)\right)^{-1},$$

$$A^-_*(s) = \begin{cases} \frac{\sigma^2}{2}\frac{\partial}{\partial y}\mathsf{P}\left\{X^-_{\theta_s} < y\right\}\Big|_{y=0} & \sigma > 0, \\[2ex] \mathsf{P}\left\{X^-_{\theta_s} = 0\right\}\max\{0, a\} & \sigma = 0, \end{cases} = \begin{cases} \frac{\sigma^2}{2}\frac{\rho^-_{d_-+1}(s)}{\beta^-_{d_-}} & (NS)_-\,, \\[2ex] a\frac{\rho^-_{d_-}(s)}{\beta^-_{d_-}} & (S)_-\,. \end{cases}$$

Denote the first passage time by $\tau^+_x = \inf\{t > 0 : X_t > x\}$, then the distribution of discounted overshoot $X_{\tau^+_x} - x$, $x > 0$, is defined by

$$(7) \quad \mathsf{E}\left[e^{-s\tau^+_x}, X_{\tau^+_x} - x \in dv, \tau^+_x < \infty\right] = s^{-1}A^-_*(s)\frac{\partial}{\partial x}\mathsf{P}\left\{X^+_{\theta_s} < x\right\}\delta(v)\,dv +$$

$$s^{-1}\int_0^x\left(p_-(s)\,\Pi(dv+y) + \int_y^\infty \Pi(dv+z)\,\mathbf{q}_-(s)\,e^{\mathbf{R}_-(s)(y-z)}\mathbf{e}\,dz\right)\mathsf{P}\left\{X^+_{\theta_s} \in x - dy\right\},$$

where $\delta(v)$ is the Dirac delta function.

Proof. If one of the cases $(NS)_-$ or $(S)_-$ holds, then according to [3] the moment generating function of $X^-_{\theta_s}$ can be defined by the relation

$$(8) \qquad \mathsf{E}e^{rX^-_{\theta_s}} = \frac{\prod_{i=1}^{N_-} r^-_i(s)}{\prod_{i=1}^{d_-} b_i}\frac{\prod_{i=1}^{d_-}(r + b_i)}{\prod_{i=1}^{N_-}(r + r^-_i(s))}.$$

Using notation given above we can rewrite this relation as
$$(9)$$
$$\mathsf{E}e^{rX^-_{\theta_s}} = \frac{\rho^-_{N_-}(s)}{\beta^-_{d_-}}\frac{(\boldsymbol{\beta}_-, 1)\,\mathbf{h}_{d_-}(r)}{(\boldsymbol{\rho}_-(s), 1)\,\mathbf{h}_{N_-}(r)} = \begin{cases} \frac{\rho^-_{d_-+1}(s)}{\beta^-_{d_-}}\frac{(\boldsymbol{\beta}_-, 1)\mathbf{h}_{d_-}(r)}{(\boldsymbol{\rho}_-(s), 1)\mathbf{h}_{d_-+1}(r)} & (NS)_-\,, \\[2ex] \frac{\rho^-_{d_-}(s)}{\beta^-_{d_-}}\left(1 + \frac{(\boldsymbol{\beta}_- - \boldsymbol{\rho}_-(s))\mathbf{h}_{d_--1}(r)}{(\boldsymbol{\rho}_-(s), 1)\mathbf{h}_{d_-}(r)}\right) & (S)_-\,. \end{cases}$$

which allows for inversion in r, so we get (4) and (5).

Under conditions of the theorem (see [5, Corollary 2.2]):

$$(10) \qquad \mathsf{E}e^{rX^+_{\theta_s}} = \left(1 - s^{-1}r\left(A^-_*(s) + \int_0^\infty e^{rx}\int_{-\infty}^0 \overline{\Pi}^+(x - y)\,dP_-(s, y)\right)\right)^{-1}.$$

Using (4) and (5) we get

$$
\int_0^\infty e^{rx} \int_{-\infty}^0 \overline{\Pi}^+ (x - y)\, dP_- (s, y) =
$$

$$
= p_- (s) \tilde{\Pi}^+ (r) + \mathbf{q}_- (s) (r\mathbf{I} + \mathbf{R}_- (s))^{-1} \left(\tilde{\Pi}^+ (r) - \tilde{\Pi}^+ (-\mathbf{R}_- (s)) \right) \mathbf{e}_- =
$$

$$
= \mathsf{E} e^{r X_{\theta_s}^-} \tilde{\Pi}^+ (r) - \mathbf{q}_- (s) (r\mathbf{I} + \mathbf{R}_- (s))^{-1} \tilde{\Pi}^+ (-\mathbf{R}_- (s)) \mathbf{e}_-,
$$

Substituting the last relation in (10) yields (6).

Using formula (4), relation (7) can be deduce by integration of the Gerber-Shiu measure (see [6]):

$$
\mathsf{E} \left[e^{-s\tau_x^+}, x - X_{\tau_x^+ - 0}^+ \in dy, x - X_{\tau_x^+ - 0} \in dz, X_{\tau_x^+} - x \in dv, \tau_x^+ < \infty \right] =
$$

$$
= s^{-1} \mathsf{P} \left\{ X_{\theta_s}^+ \in x - dy \right\} \mathsf{P} \left\{ -X_{\theta_s}^- \in dz - v \right\} \Pi (dv + z),\; v, z > 0, 0 \le y \le \min \{x, z\},
$$

with respect to y and z and taking into account that $\mathsf{E} \left[e^{-s\tau_x^+}, x - X_{\tau_x^+ - 0} = 0, \tau_x^+ < \infty \right] = s^{-1} A_*^- (s) \frac{\partial}{\partial x} \mathsf{P} \left\{ X_{\theta_s}^+ < x \right\}$ (see [5, (2.55)]). $\qquad \square$

Corollary 2.1. *If one of the cases* $(NS)_+$ *or* $(S)_+$ *holds, then*

$$
(11) \qquad\qquad P_+' (s, x) = \frac{\partial}{\partial x} \mathsf{P} \left\{ X_{\theta_s}^+ < x \right\} = \mathbf{q}_+ (s) e^{\mathbf{R}_+ (s) x} \mathbf{e},\, x > 0,
$$

where

$$
(12) \qquad\qquad \mathbf{q}_+ (s) = \begin{cases} \dfrac{\rho_{d_+ + 1}^+ (s)}{\beta_{d_+}^+} (\boldsymbol{\beta}_+, 1) & (NS)_+, \\[2ex] \dfrac{\rho_{d_+}^+ (s)}{\beta_{d_+}^+} (\boldsymbol{\beta}_+ - \boldsymbol{\rho}_+ (s)) & (S)_+. \end{cases}
$$

Moreover, in the case $(S)_+$: $p_+ (s) = \mathsf{P} \left\{ X_{\theta_s}^+ = 0 \right\} \ne 0$ *and* $p_+ (s) = \rho_{d_+}^+ (s) / \beta_{d_+}^+$.

If additionally, $\mathsf{D} X_1 < \infty, \mathsf{E} X_{\theta_s}^- < \infty$, *then the moment generating function of* $X_{\theta_s}^-$ *admits the representation*
$$
(13)
$$
$$
\mathsf{E} e^{r X_{\theta_s}^-} = \left(1 + \frac{r}{s} \left(A_*^+ (s) + \mathsf{E} e^{r X_{\theta_s}^+} \tilde{\Pi}^- (r) + \mathbf{q}_- (s) (r\mathbf{I} + \mathbf{R}_+ (s))^{-1} \tilde{\Pi}^- (-\mathbf{R}_+ (s)) \mathbf{e} \right) \right)^{-1},
$$

where

$$
A_*^+ (s) = \begin{cases} \dfrac{\sigma^2}{2} \dfrac{\rho_{d_+ + 1}^+ (s)}{\beta_{d_+}^+} & (NS)_+, \\[2ex] a \dfrac{\rho_{d_+}^+ (s)}{\beta_{d_+}^+} & (S)_+. \end{cases}
$$

Proof. To prove relations (11) – (13) we can use (4) – (6) and the fact that if for the dual process $Y_t = -X_t$ one of the cases $(NS)_-$ or $(S)_-$ holds, then for X_t we have the case $(NS)_+$ or $(S)_+$ correspondingly, and $\mathsf{E} e^{r Y_{\theta_s}^+} = \mathsf{E} e^{-r X_{\theta_s}^-}$. $\qquad \square$

If we have cases $(NS)_-$ and $(NS)_+$ $((S)_-$ and $(S)_+)$ at the same time, then the Lévy process X_t has the gaussian part with possibly drift (with zero drift and without gaussian part, correspondingly) and the jump part is a compound Poisson process with bilateral matrix-exponential distributed jumps.

Corollary 2.2. *If we have the cases* $(NS)_-$ *and* $(NS)_+$ *at the same time, then*

$$
(14) \qquad\qquad \frac{\partial}{\partial x} \mathsf{P} \left\{ X_{\theta_s}^\pm < x \right\} = \mathbf{q}_\pm (s) e^{\mathbf{R}_\pm (s) x} \mathbf{e}, \pm x > 0,
$$

where $\mathbf{q}_\pm(s) = \frac{\rho^\pm_{d_\pm+1}(s)}{\beta^\pm_{d_\pm}}(\boldsymbol{\beta}_\pm, 1)$.

If we have $(S)_-$ and $(S)_+$ simultaneously, then $\mathsf{P}\left\{X^\pm_{\theta_s} = 0\right\} = \rho^\pm_{d_\pm}(s)/\beta^\pm_{d_\pm}$ and

$$(15) \qquad \frac{\partial}{\partial x}\mathsf{P}\left\{X^\pm_{\theta_s} < x\right\} = \mathbf{q}_\pm(s)\,e^{\mathbf{R}_\pm(s)x}\mathbf{e}, \pm x > 0,$$

where $\mathbf{q}_\pm(s) = \frac{\rho^\pm_{d_\pm}(s)}{\beta^\pm_{d_\pm}}(\boldsymbol{\beta}_\pm - \boldsymbol{\rho}_\pm(s))$.

Note that, if $\sigma = 0, a \geq 0$, and $\Pi(dx) = \lambda_- b_1 e^{b_1 x} dx$ for $x < 0$, then the process X_t is called almost lower semi-continuous (for details see [5]) and we have the case $(S)_-$ with $d_- = 1$, hence $N_- = 1$ and the cumulant equation has a unique negative real root $-r_1^-(s) > -b_1$. Hence, by (4) the density of infimum is $P'_-(s,x) = \frac{r_1^-(s)}{b_1}\left(b_1 - r_1^-(s)\right)e^{r_1^-(s)x}, x > 0$ (cf. [5, (3.110)]).

To find the distribution of absolute supremum $X^+ = \sup_{0 \leq t < \infty} X_t$ or infimum $X^- = \inf_{0 \leq t < \infty} X_t$ we should take into consideration the sign of $\mathsf{E}X_1$. If $\mu = \mathsf{E}X_1 < 0\ (\mu > 0)$, then the distribution of $X^+\ (X^-)$ is non degenerate and it is defined in terms of the roots of the cumulant equation for $s = 0$ (see, for instance, [5]).

If we have one of the cases $(NS)_\pm$ or $(S)_\pm$, then from [2] it can be seen that for $\pm\mu < 0$: $r_i^\pm(s) \underset{s\to 0}{\longrightarrow} r_i^\pm, \Re\left[r_i^\pm\right] > 0, i = 1,\dots,N_\pm$, and for $\mp\mu < 0$: $r_i^\pm(s) \underset{s\to 0}{\longrightarrow} r_i^\pm, \Re\left[r_i^\pm\right] > 0$, $i = 2,\dots,N_\pm$, $r_1^\pm(s) \underset{s\to 0}{\longrightarrow} 0$, $s^{-1}r_1^\pm(s) \underset{s\to 0}{\longrightarrow} |\mu|$. Thus we obtain the next corollary of Theorem 2.1.

Corollary 2.3. *Let one of the cases* $(NS)_-$ *or* $(S)_-$ *hold and* $\mu = \mathsf{E}X_1 < 0$, *then*

$$(16) \qquad \lim_{s\to 0} s^{-1}P'_-(s,x) = \mathbf{q}'_- e^{\mathbf{R}_-(0)x}\mathbf{e}, x < 0,$$

$$(17) \qquad \mathbf{q}'_- = \begin{cases} \dfrac{\prod_{i=2}^{d_-+1} r_i^-}{|\mu|\beta^-_{d_-}}(\boldsymbol{\beta}_-, 1) & (NS)_-\,, \\[4mm] \dfrac{\prod_{i=2}^{d_-} r_i^-}{|\mu|\beta^-_{d_-}}(\boldsymbol{\beta}_- - \boldsymbol{\rho}_-(0)) & (S)_-\,. \end{cases}$$

Moreover, in case $(S)_-$: $p'_- = \lim_{s\to 0} s^{-1}p_-(s) = \left(\prod_{i=2}^{d_-} r_i^-\right)/\left(|\mu|\prod_{i=1}^{d_-} b_i\right)$.

If additionally $\mathsf{D}X_1 < \infty$, *then the moment generating function of* X^+ *has the form*

$$(18) \quad \mathsf{E}e^{rX^+} =$$

$$= \left(1 - r\left(A'_- + p'_-\tilde{\Pi}^+(r) + \mathbf{q}'_-\left(r\mathbf{I} + \mathbf{R}_-(0)\right)^{-1}\left(\tilde{\Pi}^+(r)\mathbf{I} - \tilde{\Pi}^+(-\mathbf{R}_-(0))\right)\mathbf{e}\right)\right)^{-1},$$

$$A'_- = \lim_{s\to 0} s^{-1}A^-_*(s) = \begin{cases} \dfrac{\sigma^2}{2}\dfrac{\prod_{i=2}^{d_-+1} r_i^-}{|\mu|\beta^-_{d_-}} & (NS)_-\,, \\[4mm] a\dfrac{\prod_{i=2}^{d_-} r_i^-}{|\mu|\beta^-_{d_-}} & (S)_-\,. \end{cases}$$

The distribution of the overjump is defined by the relation

$$(19) \quad \mathsf{P}\left\{X_{\tau^+_x} - x \in dv\right\} = A'_-\frac{\partial}{\partial x}\mathsf{P}\left\{X^+ < x\right\}\delta(v)\,dv +$$

$$\int_0^x \left(p'_-\Pi(dv+y) + \int_y^\infty \Pi(dv+z)\mathbf{q}'_- e^{\mathbf{R}_-(0)(y-z)}\mathbf{e}\,dz\right)\mathsf{P}\left\{X^+ \in x - dy\right\}.$$

3. Occupation time and ladder process

Denote the moment generating function for the time that the process X_t spends in the interval $(x, +\infty)$ until θ_s by

$$D_x(s, u) = \mathsf{E}e^{-u \int_0^{\theta_s} I\{X_t > x\} dt}.$$

Combining the results of the previous section with the relations for $D_x(s, u)$ given in [5] yields the following statement.

Theorem 3.1. *If one of the cases* $(NS)_-$ *or* $(S)_-$ *($a > 0$) holds, then*

$$D_0(s, u) = \frac{s}{s+u} \frac{\rho_{N_-}^-(s+u)}{\rho_{N_-}^-(s)},$$

$$(20) \quad D_x(s, u) = \frac{s}{s+u} \times$$
$$\times \left(1 - \frac{\rho_{N_-}^-(s+u)}{\rho_{N_-}^-(s)} \left(\boldsymbol{\rho}_-(s) - \boldsymbol{\rho}_-(s+u)\right) \mathbf{R}_-^{-1}(s+u) e^{\mathbf{R}_-(s+u)x} \mathbf{e}\right), x < 0.$$

If one of the cases $(NS)_+$ *or* $(S)_+$ *($a < 0$) holds, then*

$$D_0(s, u) = \frac{\rho_{N_+}^+(s)}{\rho_{N_+}^+(s+u)},$$

$$(21) \qquad D_x(s, u) = 1 + \frac{\rho_{N_+}^+(s)}{\rho_{N_+}^+(s+u)} \left(\boldsymbol{\rho}_+(s+u) - \boldsymbol{\rho}_+(s)\right) \mathbf{R}_+^{-1}(s) e^{\mathbf{R}_+(s)x} \mathbf{e}, x > 0.$$

Proof. Following [5, Theorem 2.6], for non step-wise processes the next relations are true

$$\int_{-0}^{+\infty} e^{rx} D_x'(s, u) dx + D_0(s, u) = \frac{\mathsf{E}e^{rX_{\theta_s}^+}}{\mathsf{E}e^{rX_{\theta_{s+u}}^+}},$$

$$(22) \qquad \int_{-\infty}^{+0} e^{rx} D_x'(s, u) dx - D_0(s, u) = -\frac{s}{s+u} \frac{\mathsf{E}e^{rX_{\theta_{s+u}}^-}}{\mathsf{E}e^{rX_{\theta_s}^-}}.$$

If one of the cases $(NS)_-$ or $(S)_-$ ($a > 0$) holds, then recalling Theorem 2.1 it follows that

$$\frac{\mathsf{E}e^{rX_{\theta_{s+u}}^-}}{\mathsf{E}e^{rX_{\theta_s}^-}} = \frac{\rho_{N_-}^-(s+u)}{\rho_{N_-}^-(s)} \frac{\left(\boldsymbol{\rho}_-(s), 1\right) \mathbf{h}_{N_-}(r)}{\left(\boldsymbol{\rho}_-(s+u), 1\right) \mathbf{h}_{N_-}(r)} =$$

$$= \frac{\rho_{N_-}^-(s+u)}{\rho_{N_-}^-(s)} \left(1 - \frac{\left(\boldsymbol{\rho}_-(s+u) - \boldsymbol{\rho}_-(s)\right) \mathbf{h}_{N_- - 1}(r)}{\left(\boldsymbol{\rho}_-(s+u), 1\right) \mathbf{h}_{N_-}(r)}\right).$$

Taking account of formula (22) this gives us the following

$$D_x'(s, u) = \frac{s}{s+u} \frac{\rho_{N_-}^-(s+u)}{\rho_{N_-}^-(s)} \left(\boldsymbol{\rho}_-(s+u) - \boldsymbol{\rho}_-(s)\right) e^{\mathbf{R}_-(s+u)x} \mathbf{e}, \ x < 0,$$

and combining with $\lim_{x \to -\infty} D_x(s, u) = \frac{s}{s+u}$, we receive (20).

Similarly, in case $(NS)_+$ or $(S)_+$ ($a < 0$):

$$\frac{\mathsf{E}e^{-rX_{\theta_s}^+}}{\mathsf{E}e^{-rX_{\theta_{s+u}}^+}} = \frac{\rho_{N_+}^+(s)}{\rho_{N_+}^+(s+u)} \left(1 + \frac{\left(\boldsymbol{\rho}_+(s+u) - \boldsymbol{\rho}_+(s)\right) \mathbf{h}_{N_+ - 1}(r)}{\left(\boldsymbol{\rho}_+(s), 1\right) \mathbf{h}_{N_+}(r)}\right).$$

Hence,

$$D'_x(s,u) = \frac{\rho^+_{N_+}(s)}{\rho^+_{N_+}(s+u)}\left(\boldsymbol{\rho}_+(s+u) - \boldsymbol{\rho}_+(s)\right)e^{\mathbf{R}_+(s)x}\mathbf{e}, \; x > 0,$$

and taking into account that $\lim_{x\to+\infty} D_x(s,u) = 1$ we deduce (21). $\qquad\square$

This statement generalize the representation of $D_x(s,u)$ known for almost semi-continuous processes given in [5].

Note that, for a non step-wise Lévy process $\mathsf{P}\{X_t = 0\} = 0$, then by [10, VI, Lemma 15] for any $t \geq 0$ the time it spends in $[0,\infty)$ $Q_0(t) = \int_0^t I\{X_v \geq 0\}\,dv$ and the instant of its last supremum $g_t = \sup\{v < t : X_v = X_v^+\}$ have the same law. Moreover, by [5, Theorem 2.9] $Q_0(t)$ and the time the maximum is achieved $T_t = \inf\{v > 0 : X_v = X_v^+\}$ also have the same law. Hence, the results of Theorem 3.1 define the moment generating functions of g_{θ_s} and T_{θ_s}.

Let $L(t)$ be the local time in $[0,t]$ that $X_t^+ - X_t$ spends at zero and $L^{-1}(t) = \inf\{v > 0 : L(v) > t\}$ is the inverse local time (for details, see [10, VI]). Denote by $\kappa(s,r)$ the Laplace exponent of the so called ladder process $\{L^{-1}, X_{L^{-1}}\}$:

$$e^{-\kappa(s,r)} = \mathsf{E}\left[e^{-sL^{-1}(1)-rX_{L^{-1}(1)}}, 1 < L_\infty\right].$$

According to [10, VI, (1)]:

$$\mathsf{E}e^{-rX_{\theta_s}^+ - ug_{\theta_s}} = \frac{\kappa(s,0)}{\kappa(s+u,r)}.$$

Assuming that the normalization constant of the local time is 1, we can deduce that $\kappa(s,0) = \mathsf{E}e^{-(1-s)g_{\theta_s}}$. Taking into account that for non step-wise processes $Q_0(\theta_s)$ and g_{θ_s} have the same distribution we can write that

$$(23) \qquad \kappa(s,r) = \frac{D_0(s,1-s)}{\mathsf{E}e^{-rX_{\theta_s}^+}}.$$

Hence, using Theorem 2.1 and Theorem 3.1, we can deduce the following statement.

Corollary 3.1. *If one of the cases* $(NS)_-$ *or* $(S)_-$ $(a > 0)$ *holds, then*

$$\kappa(s,-r) = \frac{\rho^-_{N_-}(1)}{\rho^-_{N_-}(s)}\times$$

$$\times\left(s - r\left(A_*^-(s) + \mathsf{E}e^{rX_{\theta_s}^-}\tilde{\Pi}^+(r) - \mathbf{q}_-(s)\left(r\mathbf{I} + \mathbf{R}_-(s)\right)^{-1}\tilde{\Pi}^+(-\mathbf{R}_-(s))\mathbf{e}\right)\right).$$

If X_t *is a compound Poisson process with negative drift* $a < 0$, *without gaussian part* $(\sigma = 0)$, *and with bilateral ME distributed jumps, then*

$$\kappa(s,r) = \frac{\rho^-_{d_-+1}(1)}{\rho^-_{d_-+1}(s)}\frac{\beta^+_{d_+}}{\rho^+_{d_+}(s)}\left(1 + \frac{\left(\boldsymbol{\rho}_+(s) - \boldsymbol{\beta}_+\right)\mathbf{h}_{d_+-1}(r)}{\left(\boldsymbol{\rho}_+(s),1\right)\mathbf{h}_{d_+}(r)}\right).$$

The next statement applies Theorem 3.1 and Corollary 2.3 to get a representation of the moment generating function of the total sojourn time over a level $D_x(0,u) = \mathsf{E}e^{-u\int_0^\infty I\{X_t > x\}dt}$, which in risk theory defines the time in risk zone (for details, see [5]).

Corollary 3.2. *If one of the cases* $(NS)_-$ *or* $(S)_-$ $(a > 0)$ *holds and* $\mu = \mathsf{E}X_1 < 0$, *then*

$$\mathsf{E}e^{-u\int_0^\infty I\{X_t > 0\}dt} = \frac{|\mu|}{u}\frac{\prod_{i=1}^{N_-} r_i^-(u)}{\prod_{i=2}^{N_-} r_i^-},$$

$$(24) \quad \mathsf{E}e^{-u\int_0^\infty I\{X_t > x\}dt} = \frac{|\mu|\prod_{i=1}^{N_-} r_i^-(u)}{u\prod_{i=2}^{N_-} r_i^-}\left(\boldsymbol{\rho}_-(u) - \boldsymbol{\rho}_-(0)\right)\mathbf{R}_-^{-1}(u)e^{\mathbf{R}_-(u)x}\mathbf{e}, \; x < 0.$$

The integral transform of the moment generating function of the sojourn time over a positive level has the next representation

$$(25) \quad \int_{-0}^{+\infty} e^{rx} D_x'(0,u)\, dx + D_0(0,u) = \frac{\mathsf{E}e^{rX^+}}{\mathsf{E}e^{rX_{\theta_u}^+}} =$$

$$= \frac{1 - \frac{r}{u}\left(A_*^-(u) + \mathsf{E}e^{rX_{\theta_u}^-}\tilde{\Pi}^+(r) - \mathbf{q}_-(u)\left(r\mathbf{I} + \mathbf{R}_-(u)\right)^{-1}\tilde{\Pi}^+(-\mathbf{R}_-(u))\,\mathbf{e}\right)}{1 - r\left(A_-' + p_-'\tilde{\Pi}^+(r) + \mathbf{q}_-'\left(r\mathbf{I} + \mathbf{R}_-(0)\right)^{-1}\left(\tilde{\Pi}^+(r)\mathbf{I} - \tilde{\Pi}^+(-\mathbf{R}_-(0))\right)\mathbf{e}\right)}.$$

If for the process X_t: $\sigma = 0, a \le 0$, negative jumps have a ME distribution and $\mu < 0$, then

$$(26) \quad \mathsf{E}e^{-u\int_0^\infty I\{X_t > x\}dt} = \mathsf{P}\left\{X^+ < x\right\} + \frac{p_+(u)}{u}\left(p_-(u)\int_0^x \overline{\Pi}^+(x-z)\,\mathsf{P}\left\{X^+ \in dz\right\} + \right.$$

$$\left. + \mathbf{q}_-(u)\int_0^\infty \overline{\Pi}^+(y)\int_{\max\{0,x-y\}}^x e^{\mathbf{R}_-(u)(x-y-z)}\mathsf{P}\left\{X^+ \in dz\right\}\mathbf{e}\,dy\right).$$

Proof. Equality (24) follows by taking the limit as $s \to 0$ in (20). Formula (25) is a straightforward consequence of formulas (6), (18) and (22).

If $\sigma = 0, a \le 0$, then $\left\{\tau_0^+, X_{\tau_0^+}\right\}$ has non degenerate joint distribution. Applying prelimit generalization of the Pollaczek-Khinchin formula ([5, Theorem 2.4]), we get

$$\int_{-0}^{+\infty} e^{rx} D_x'(s,u)\, dx + D_{+0}(s,u) = \frac{\mathsf{E}e^{rX_{\theta_s}^+}}{\mathsf{P}\left\{X_{\theta_{s+u}}^+ = 0\right\}}\left(1 - \mathsf{E}\left[e^{-(s+u)\tau_0^+ + rX_{\tau_0^+}}, \tau_0^+ < \infty\right]\right).$$

Whence $D_{+0}(s,u) = \frac{\mathsf{P}\left\{X_{\theta_s}^+ = 0\right\}}{\mathsf{P}\left\{X_{\theta_{s+u}}^+ = 0\right\}}$ and for $x > 0$

$$D_x(s,u) = P_+(s,x) + \int_0^x \mathsf{P}\left\{X_{\theta_{s+u}}^+ > 0, X_{\tau_0^+} > x - z\right\}dP_+(s,z).$$

Due to [5, Corollary 2.3]:

$$\mathsf{P}\left\{X_{\theta_{s+u}}^+ > 0, X_{\tau_0^+} > z\right\} = \frac{\mathsf{P}\left\{X_{\theta_{s+u}}^+ = 0\right\}}{s+u}\int_{-\infty}^0 \overline{\Pi}^+(z-y)\,dP_-(s+u,y).$$

If negative jumps have the ME distribution, then

$$D_x(s,u) = P_+(s,x) + \frac{p_+(s+u)}{s+u}\left(p_-(s+u)\int_0^x \overline{\Pi}^+(x-z)\,dP_+(s,z) + \right.$$

$$\left. + \mathbf{q}_-(s+u)\int_0^\infty \overline{\Pi}^+(y)\int_{\max\{0,x-y\}}^x e^{\mathbf{R}_-(s+u)(x-y-z)}\mathbf{e}dP_+(s,z)\,dy\right).$$

from here as $s \to 0$ relation (26) follows. $\qquad\square$

Note that, for the step-wise ($a = 0$) almost lower semi-continuous processes formula (26) is reduced to

$$\mathsf{E}e^{-u\int_0^\infty I\{X_t > x\}dt} = \mathsf{P}\left\{X^+ < x\right\} + \frac{1}{u+\lambda}\left(\int_0^x \overline{\Pi}^+(x-z)\,\mathsf{P}\left\{X^+ \in dz\right\} + \right.$$

$$\left. + \left(b_1 - r_1^-(u)\right)\int_0^\infty \overline{\Pi}^+(y)\int_{\max\{0,x-y\}}^x e^{r_1^-(u)(x-y-z)}\mathsf{P}\left\{X^+ \in dz\right\}dy\right).$$

If negative (positive) jumps have hyperexponential distribution, that is, if we have additional condition that $b_{m_-} > \ldots > b_2 > b_1 > 0$ ($c_{m_+} > \ldots > c_2 > c_1 > 0$), then the

roots of the cumulant equation $\left\{-r_i^-(s)\right\}_{i=1}^{N_-}\left(\left\{r_i^+(s)\right\}_{i=1}^{N_+}\right)$ are real and distinct (see [3]), and the matrix exponents in Theorem 3.1 can be simplified.

Corollary 3.3. *If one of the cases* $(NS)_-$ *or* $(S)_-$ *($a > 0$) holds and* $b_{m_-} > \ldots > b_1 > 0$, *then*

$$D_0(s,u) = \frac{s}{s+u}\prod_{i=1}^{N_-}\frac{r_i^-(s+u)}{r_i^-(s)},$$

(27)

$$D_x(s,u) = \frac{s}{s+u}\left(1 + \sum_{k=1}^{N_-}\frac{\prod_{i=1}^{N_-}\left(r_k^-(s+u)/r_i^-(s)-1\right)}{\prod_{i=1,i\neq k}^{N_-}\left(r_k^-(s+u)/r_i^-(s+u)-1\right)}e^{r_k^-(s+u)x}\right), x < 0.$$

For $\mu < 0$

$$D_0(0,u) = \frac{|\mu|}{u}\prod_{i=2}^{N_-}\frac{r_i^-(u)}{r_i^-},$$

(28) $\qquad D_x(0,u) = \frac{|\mu|}{u}\sum_{k=1}^{N_-}\frac{\prod_{i=2}^{N_-}\left(r_k^-(u)/r_i^--1\right)}{\prod_{i=1,i\neq k}^{N_-}\left(r_k^-(u)/r_i^-(u)-1\right)}r_k^-(u)\,e^{r_k^-(u)x}, x < 0.$

If one of the cases $(NS)_+$ *or* $(S)_+$ *($a < 0$) holds and* $c_{m_+} > \ldots > c_1 > 0$, *then*

$$D_0(s,u) = \prod_{i=1}^{N_+}\frac{r_i^+(s)}{r_i^+(s+u)},$$

(29) $\qquad D_x(s,u) = 1 - \sum_{k=1}^{N_+}\frac{\prod_{i=1}^{N_+}\left(1-r_k^+(s)/r_i^+(s+u)\right)}{\prod_{i=1,i\neq k}^{N_+}\left(1-r_k^+(s)/r_i^+(s)\right)}e^{-r_k^+(s)x}, x > 0.$

For $\mu > 0$ *and* $x \geq 0$: $D_x(0,u) = 0$, *in other words* $\mathsf{P}\left\{\int_0^{+\infty}I\left\{X_t > x\right\}dt = +\infty\right\} = 1$.

Proof. If we have one of the cases $(NS)_-$ or $(S)_-$ ($a > 0$) and $b_{m_-} > \ldots > b_2 > b_1 > 0$, then the roots $\left\{-r_i^-(s)\right\}_{i=1}^{N_-}$ are real and distinct, and instead of using formula (20) it is more convenient to substitute relation (8) in (22) and invert with respect to r. Similarly for the case $(NS)_+$ or $(S)_+$ ($a < 0$) and $c_{m_+} > \ldots > c_2 > c_1 > 0$ we can deduce formula (29). To get $D_x(0,u)$ in the corresponding cases apply the limit behavior of the roots of cumulant equation as $s \to 0$. To find the limit as $s \to 0$ in (29) for the case $(NS)_+$ or $(S)_+$ and for $\mu > 0$ we can use the relation

$$\frac{\prod_{i=1}^{N_+}\left(1-r_k^+(s)/r_i^+(s+u)\right)}{\prod_{i=1,i\neq k}^{N_+}\left(1-r_k^+(s)/r_i^+(s)\right)} \xrightarrow[s\to 0]{} \begin{cases} 1 & k=1, \\ 0 & k \neq 1. \end{cases}$$

$\square$

Note that, using the results of [6], Corollary 3.3 could be generalized for the case of the so called meromorphic Lévy processes (the cumulant function is holomorphic except a set of isolated points, the poles of the function), for which $N_\pm = \infty$ in (27)–(29).

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