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SEMIGROUPS OF m-POINT MOTIONS OF THE ARRATIA FLOW, AND BINARY FORESTS

We present a core for the generator of the semigroup of an m-point motion of the Arratia flow. We represent an action of the semigroup on functions from this core in terms of binary forests.

1. Introduction

The subject of our investigation is a stochastic flow of coalescing particles. One of the approaches to the study of flows with singular interaction is to construct and analyze its discrete-time approximation. In the article [1] Nishchenko developed an approximation scheme, which is driven by a sequence of smooth Gaussian processes, and obtained conditions under which the discrete time flow approximates Harris flow of Brownian particles [2]. The approximations were constructed via recurrence equation

$$x_0^n(u) = u,$$
$$x_{k+1}^n(u) = x_k^n(u) + \frac{1}{\sqrt{n}}\xi_{k+1}^n(x_k^n(u)),$$

where $\{\xi_k^n(\cdot), k \le n\}$ are independent stationary Gaussian processes with zero mean and a covariance function Γ_n. Using these iterations one can build the random processes $\{x_n(u,t), t \in [0,1]\}$ as polygonal lines with edges $\left(\frac{k}{n}, x_k^n(u)\right), k = 0, \dots, n$. Under certain conditions on the convergence of the covariance functions Γ_n to the indicator of zero, m-point motions $\{x_n(u_1,\cdot),\dots,x_n(u_m,\cdot)\}_{n\ge1}$ converge weakly to m-point motions of the Arratia flow $X(u,\cdot) = (x(u_1,\cdot),\dots,x(u_m,\cdot))$ and the random function $x_n^n(\cdot)$ converges weakly in $D([0,1])$ to the value of the Arratia flow $x(\cdot,1)$. Note that estimations that guarantee weak convergence of the discrete-time flows depend on the convergence rate of the covariance functions Γ_n [1].

It should be pointed out that there is a disordering in the approximation scheme, namely, trajectories of the particles intersect one another. Since in the limiting flow the order between particles does not change, it is natural to suspect that in the approximation scheme "amount" of disordering tends to zero. An interesting question in this area is an investigation of the rate of convergence of this "amount". For example, one of the

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functionals which detect disordering in the discrete time system of particles is

$$\Phi_n = \int_0^1 1\!\!\mathrm{I}_{\{x_n(u_2,t)-x_n(u_1,t)<0\}} dt$$

with $u_1 < u_2$. In the paper [3], we obtained that the rate of decrease to zero of Φ_n is governed by convergence of the covariance functions Γ_n. This brings up the question: "does the rate of convergence of an "amount" of disordering depends on the rate of weak convergence of $\{x_n(u_1,\cdot),\ldots,x_n(u_m,\cdot)\}_{n\geq 1}$ to an m-point motion of the Arratia flow?".

The investigation of weak convergence of a discrete-time m-point motion necessitated us to study functionals of the limiting flow's trajectories. In this paper we consider the semigroup of an m-point motion of the Arratia flow and present a system of boundary value problems for functions

$$Q_{m,t}f(u) = \mathbb{E}f(X(u,t)),$$

where $f \in C_0(\mathbb{R}^m)$. We rewrite obtained integral representation for $Q_{m,t}f$ in terms of binary forests that correspond to the order of trajectories' coalescence.

2. Generator for semigroup of m-point motion of the Arratia flow

In this section we consider semigroups of an m-point motions of the Arratia flow and present a core for its generator. At the informal level, the Arratia flow can be described as a collection of Brownian particles that start from points at the real line and move independently until they meet. Upon their meeting, two particles coalesce. Coalescing stochastic flow was investigated in [2]. We give here a definition of the Arratia flow as it was given in [4, 5]

Definition 1. The Arratia flow is a family $(x(u,t)),\ u \in \mathbb{R},\ t \geq 0$ of continuous martingales adapted to a common filtration (F_t), satisfying the following conditions:
(i) For each u $x(u,\cdot)$ is an F_t-Brownian motion starting at u.
(ii) For each u, v the joint covariation of $(x(u,\cdot))$ and $(x(v,\cdot))$ is given by

$$d\langle x(u,\cdot), x(v,\cdot)\rangle(t) = 1\!\!\mathrm{I}_{\{x(u,t)=x(v,t)\}} dt,$$

where $\langle\cdot,\cdot\rangle$ is quadratic covariation.
(iii) $(x(\cdot,t))$ is monotone in u for each t.

In the paper [6] Y. Le. Jan and O. Raimond constructed a system of n coalescing particles on a manifold in terms of transition semigroups. Precisely, let $(P_t^{(n)}, n \geq 1)$ be a compatible family of Feller semigroups on a locally compact separable metric space M, i.e. for all $k \leq n$

$$P_t^{(k)} f(x_1,\ldots,x_k) = P_t^{(n)} g(y_1,\ldots,y_n),$$

where f and g are any continuous functions such that

$$g(y_1,\ldots,y_n) = f(y_{i_1},\ldots,y_{i_k}),$$

with $\{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}$ and $(x_1,\ldots,x_k) = (y_{i_1},\ldots,y_{i_k})$.

Denote by $P^{(n)}_{(x_1,\ldots,x_n)}$ the law of the Markov process $X_t^{(n)}$ associated with $P_t^{(n)}$ starting from $(x_1,\ldots,x_n)$. This Markov process will be called the n-point motion of this family of semigroups. It is defined on the set of càdlàg paths on M^n. Let $\delta_n = \{x \in M^n, \exists i \neq j, x_i = x_j\}$ and $T_{\delta_n} = \inf\{t \geq 0, X_t^{(n)} \in \delta_n\}$,

Theorem 1 (Theorem 4.1, [6])**.** *There exists a unique compatible family $(P_t^{(n),c}, n \geq 1)$ of Markovian semigroups on M such that if $X^{(n),c}$ is the associated n-point motion and $T^c_{\delta_n} = \inf\{t \geq 0, X_t^{(n),c} \in \delta_n\}$, then*

(i) $(X_t^{(n),c}, t \leq T^c_{\delta_n})$ is equal in law to $(X_t^{(n)}, t \leq T_{\delta_n})$,

(ii) for $t \geq T^c_{\delta_n}$, $X^{(n),c}_t \in \delta_n$.
Moreover, this family is constituted of Feller semigroups if condition (c) is satisfied:
(c) For all $t > 0, \varepsilon > 0$ *and* $x \in M$

$$\lim_{y \to x} \mathbb{P}^{(2)}_{(x,y)}(\{T_{\delta_2} > t\} \cap \{d(X_t, Y_t) > \varepsilon\}) = 0,$$

where $(X_t, Y_t) = X^{(2)}_t$. *And for some* x *and* y *in* M $\mathbb{P}^{(2)}_{(x,y)}[T_{\delta_2} < \infty] > 0$.
In this case the family $(P^{(n),c}_t, n \geq 1)$ *satisfies*

$$P^{(2),c}_t f^{\otimes 2}(x,x) = P_t f^2(x)$$

and is associated with a coalescence flow.

Applying this theorem with $M = \mathbb{R}$ and a family $(P^{\otimes n}_t, n \geq 1)$ of Feller semigroups, where P_t is the semigroup of a Brownian motion, Y. Le Jan and O. Raimond (example 4.4.1, [6]) obtained a family $(Q_{n,t}, n \geq 1)$ of Feller semigroups such that the flow associated with $(Q_{n,t}, n \geq 1)$ is an n-point motion of the Arratia flow.

For $u = (u_1, ..., u_m) \in \mathbb{R}^m$ we will denote by $X(u,t) = (x(u_1,t), \ldots, x(u_m,t))$ the m-point of the Arratia flow which starts from the point $u_1 < u_2 < \ldots < u_m$.

Let us denote $\Delta_m = \{u \in \mathbb{R}^m : u_1 \leq u_2 \leq \ldots \leq u_m\}$,

$$C^2_0(\Delta_m) = \{f \in C^2(\Delta_m) : \ f(x) \to 0, \ \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \to 0, \text{ as } ||x|| \to \infty\}$$

Recall the definition of a core of a closed linear operator A with domain $\mathcal{D}(A)$ [7]. A subspace D of $\mathcal{D}(A)$ is said to be a *core* for A if the closure of the restriction of A to D is equal to A. A subspace $D \subset C_0$ is said to be *invariant* under semigroup (T_t) if $T_t D \subset D$ for all $t \geq 0$. It is known ([7]) that the generator of a Feller semigroup is closed. To show that some set D is a core for A we will use next proposition:

Proposition 1 (Proposition 19.9, [7]). *If* $(A, \mathcal{D})$ *is the generator of a Feller semigroup, then any dense, invariant subspace* $D \subset \mathcal{D}$ *is a core for* A

In the next theorem we present a core for the generator $\mathcal{A}$ of the semigroup $(Q_{m,t})$ of m-point motion of the Arratia flow.

Theorem 2. *The set of functions*

$$D_m = \{f \in C^2_0(\Delta_m) : \frac{\partial^2 f}{\partial x_i \partial x_j} \in C_0(\Delta_m), \quad \frac{\partial^2 f}{\partial x_i \partial x_j} 1\!\!\mathrm{I}_{\{x_i = x_j\}}(x) = 0, \, i \neq j\}$$

is a core for a generator $\mathcal{A}$ *of the semigroup* $Q_{m,t}$ *and for any* $f \in D_m$

$$\mathcal{A}f(u) = \frac{1}{2}\Delta f(u), \ u \in \Delta_m.$$

Proof. Let D be a domain of the generator $\mathcal{A}$ of the semigroup $Q_{m,t}$. First, we show that $D_m \subset D$, i.e. for any function $f \in D_m$

$$\frac{Q_{m,t} f - f}{t} \to \mathcal{A}f \text{ in } C_0(\Delta_m) \text{ as } t \to 0.$$

By the Ito formula we get

$$f(X(u,t)) = f(u) + \sum_{i=1}^m \int_0^t f'_i(X(u,s))\, dx(u_i,s) +$$

$$+ \frac{1}{2} \sum_{i,j=1}^m \int_0^t f''_{ij}(X(u,s))\, d\langle x(u_i,\cdot), x(u_j,\cdot)\rangle(s) =$$

$$= f(u) + \sum_{i=1}^{m} \int_0^t f_i'(X(u,s))\, dx(u_i,s) + \frac{1}{2}\sum_{i=1}^{m}\int_0^t f_{ii}''(X(u,s))\, ds,$$

where we use condition (ii) from the definition 1. Taking the expectation we obtain:

$$\frac{1}{t}\mathbb{E}[f(X(u,t)) - f(u)] = \frac{1}{2t}\sum_{i=1}^{m}\mathbb{E}\int_0^t f_{ii}''(X(u,s))\, ds =$$

$$= \frac{1}{2t}\sum_{i=1}^{m}\int_0^t Q_{m,s} f_{ii}''(u)\, ds.$$

For any $g \in C_0(\Delta_m)$ using strong continuity of the Feller semigroup $(Q_{m,t})$

$$\Big\| \frac{1}{t}\int_0^t Q_{m,s} g\, ds - g \Big\| \le \frac{1}{t}\int_0^t \|g - Q_{m,s} g\|\, ds \to 0 \text{ as } t \to 0.$$

By assumption, $f_{ii}'' \in C_0(\Delta_m)$, so by the same argument,

$$\frac{1}{t}\sum_{i=0}^{m}\int_0^t Q_{m,s} f_{ii}''(u)\, ds \to \frac{1}{2}\Delta f(u) \text{ as } t \to 0$$

uniformly in $u \in \Delta_m$.

To prove the invariance of D_m under $Q_{m,t}$

(1) $$Q_{m,t} D_m \subset D_m$$

we will use mathematical induction. The next lemma gives the base of induction.

Lemma 1. *For $m = 2$ relation (1) holds.*

Proof. Let f be any function of class D_2. Note that 2-point motion of the Arratia flow can be represented via two independent Brownian motions $\{w(u_1,t)\}_{t\ge 0}, \{w(u_2,t)\}_{t\ge 0}$, $w(u_i,0) = u_i$:

$$x(u_2,t) = w(u_2,t),$$
$$x(u_1,t) = w(u_1,t)1\!\!\mathrm{I}_{\{t<\tau\}} + w(u_2,t)1\!\!\mathrm{I}_{\{t\ge\tau\}},$$

where $\tau = \inf\{t : w(u_1,t) = w(u_2,t)\}$.

We will use the transition density for non-intersecting Brownian motions obtained by Karlin and McGregor [8]:

$$\mathbb{P}\{w^1(t) \in dy_1, \ldots, w^m(t) \in dy_m,\ w^1(s) < \ldots < w^m(s),\ 0 \le s \le t\} =$$
$$= \det\big(p_t(u_i, y_j)\big)_{\substack{i=1,\ldots,m\\ j=1,\ldots,m}} dy_1 \ldots dy_m,$$

where $(w^1(t),\ldots,w^m(t))_{t\ge 0}$ is a Brownian motion that starts at $(x_1,\ldots,x_m), x_1 < x_2 < \ldots < x_m$, and $p_t(u,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(u-y)^2}{2t}}$.

So we get

$$Q_{2,t} f(u_1,u_2) = \mathbb{E} f(w(u_1,t), w(u_2,t))1\!\!\mathrm{I}_{\{t<\tau\}} + \mathbb{E} f(w(u_2,t), w(u_2,t))(1 - 1\!\!\mathrm{I}_{\{t<\tau\}}) =$$

$$= \iint_{y_1 \le y_2} (f(y_1,y_2) - f(y_2,y_2)) \begin{vmatrix} p_t(u_1,y_1) & p_t(u_1,y_2) \\ p_t(u_2,y_1) & p_t(u_2,y_2) \end{vmatrix} dy_2\, dy_1 +$$

$$+ \int_{\mathbb{R}} f(y_1,y_2) p_t(u_2,y_2)\, dy_2,$$

The second summand in the obtained expression belongs to D_2 since it does not depend on u_1. To check that the first summand belongs to D_2, we note, that under assumption of f, the function

$$g(u_1,u_2,y_1) = \int_{y_1}^{+\infty} (f(y_1,y_2) - f(y_2,y_2)) \begin{vmatrix} p_t(u_1,y_1) & p_t(u_1,y_2) \\ p_t(u_2,y_1) & p_t(u_2,y_2) \end{vmatrix} dy_2$$

is differentiable with respect to u_1, u_2 and

$$\frac{\partial^2}{\partial u_1 \partial u_2} \int_{\mathbb{R}} g(u_1, u_2, y_1)\, dy_1 = \int_{\mathbb{R}} \frac{\partial^2}{\partial u_1 \partial u_2} g(u_1, u_2, y_1)\, dy_1 =$$

$$= \int_{\mathbb{R}} \int_{y_1}^{+\infty} (f(y_1, y_2) - f(y_2, y_2)) \begin{vmatrix} p_t(u_1, y_1) \frac{y_1 - u_1}{t} & p_t(u_1, y_2) \frac{y_2 - u_1}{t} \\ p_t(u_2, y_1) \frac{y_1 - u_2}{t} & p_t(u_2, y_2) \frac{y_2 - u_2}{t} \end{vmatrix} dy_2\, dy_1,$$

where we use

$$\frac{\partial}{\partial u} \begin{vmatrix} g(u) & g(u) \\ g(v) & g(v) \end{vmatrix} = \begin{vmatrix} g'(u) & g'(u) \\ g(v) & g(v) \end{vmatrix}.$$

From this expression and properties of Gaussian distribution one can see that

$$\frac{\partial^2}{\partial u_1 \partial u_2} \int_{\mathbb{R}} g(u_1, u_2, y_1)\, dy_1 \in C_0(\Delta_2)\ .$$

Notice that $\begin{vmatrix} p_t(u_1, y_1) \frac{y_1 - u_1}{t} & p_t(u_1, y_2) \frac{y_2 - u_1}{t} \\ p_t(u_2, y_1) \frac{y_1 - u_2}{t} & p_t(u_2, y_2) \frac{y_2 - u_2}{t} \end{vmatrix} = 0$ when $u_1 = u_2$, so we get

$$\frac{\partial^2}{\partial u_1 \partial u_2} \int_{\mathbb{R}} g(u_1, u_2, y_1)\, dy_1 \Big|_{u_2 = u_1} = 0.$$

Thus we obtain that $Q_{2,t} f \in D_2$ whenever $f \in D_2$. The lemma 1 is proved. □

Suppose now that property (1) holds for $m - 1$ and prove it for m. For any function $f \in D_m$

$$\begin{aligned} Q_{m,t} f(u) &= \mathbb{E} f(X(u,t)) 1\!\!1_{\{\tau > t\}} + \mathbb{E} f(X(u,t)) 1\!\!1_{\{\tau \le t\}} = \\ &= \mathbb{E} f(w(u_1, t), \dots, w(u_m, t)) 1\!\!1_{\{\tau > t\}} + \mathbb{E} f(X(u,t)) 1\!\!1_{\{\tau \le t\}}, \end{aligned} \tag{2}$$

where $\{w(u_1, t)\}_{t \ge 0}, \dots, \{w(u_m, t)\}_{t \ge 0}$ are independent Brownian motions such that $w(u_i, 0) = u_i$, and

$$\tau = \inf\{t : X(u,t) \in \partial \Delta_m\} \overset{d}{=} \inf\{t : (w(u_1, t), \dots, w(u_m, t)) \in \partial \Delta_m\}.$$

Using the Karlin–McGregor formula for the transition density for non-intersecting Brownian motions we get:

$$\mathbb{E} f(w(u_1, t), \dots, w(u_m, t)) 1\!\!1_{\{t < \tau\}} = \int \dots \int_{y_1 < \dots < y_m} f(y) \det\bigl(p_t(u_i, y_j)\bigr)_{\substack{i=1,\dots,m \\ j=1,\dots,m}} dy.$$

Using properties of Gaussian distribution, as in case $m = 2$, one can check that under the assumption on f, the last integral as function of $u_1, \dots, u_m$ belongs to the class D_m. It is more difficult to check that the second summand in (2) is in D_m. Using the strong Markov property for the m-point motion of the Arratia flow we can write:

$$\mathbb{E} f(X(u,t)) 1\!\!1_{\{\tau \le t\}} = \mathbb{E}\, \mathbb{E}\bigl(f(X(u,t)) 1\!\!1_{\{\tau \le t\}} \big| \mathcal{F}_\tau^{u_1, \dots, u_m}\bigr) =$$

$$= \mathbb{E}\, 1\!\!1_{\{\tau \le t\}}\, Q_{m, t-\tau} f(X(u, \tau)) = \mathbb{E}\, 1\!\!1_{\{\tau \le t\}}\, Q_{m, t-\tau} f(w(u_1, \tau), \dots, w(u_m, \tau)),$$

where $\mathcal{F}_t^{u_1, \dots, u_m} = \sigma\{x(u_1, s), \dots, x(u_m, s), s \le t\}$.

We will now prove that the obtained expression is a solution to some boundary value problem and then establish properties of this solution depending on boundary condition.

Denote

$$S_i^m = \{u \in \partial \Delta_m : u_i = u_{i+1}\}. \tag{3}$$

Define functions $\pi_i : \Delta_m \to \Delta_{m-1}$ by the rule:

$$\pi_i(u_1, u_2, \dots, u_m) = (u_1, \dots, u_i, u_{i+2}, \dots, u_m) \tag{4}$$

and functions $\pi_i^{-1} : \Delta_{m-1} \to \Delta_m$ by the rule

$$\pi_i^{-1}(u_1, \dots, u_{m-1}) = (u_1, \dots, u_i, u_i, u_{i+1}, \dots, u_{m-1}). \tag{5}$$

For any function $\varphi \in C^2(\partial\Delta_m)$ let φ_i be a restriction of φ on S_i^m, i.e. $\varphi_i = \varphi|_{S_i^m}$. Define a set of functions on $\partial\Delta_m$:

$$D(\partial\Delta_m) = \{\varphi \in C^2(\partial\Delta_m) : \varphi_i \circ \pi_i^{-1} \in D_m,\ i = 1, \dots, m\}.$$

We will denote by $\overset{\circ}{A}$ the interior of a set A.

Lemma 2. *Let F be a solution to a boundary value problem:*

$$\frac{\partial}{\partial s} F(u,s) = -\frac{1}{2}\Delta F(u,s) \text{ in } \overset{\circ}{\Delta}_m \times [0,t) \tag{6}$$

$$\lim_{s\to t} F(u,s) = 0 \tag{7}$$

$$F(u,s) = \varphi(u),\ u \in \partial\Delta_m, \tag{8}$$

$$F \in C^2(\overset{\circ}{\Delta}_m \times (0,t)).$$

Let $\{w_s(u_i,t), t \geq s\}_{i=1}^m$ be independent Brownian motions, $w_s(u_i,s) = u_i, u \in \Delta_m$. Denote

$$\tau = \inf\{t \geq s : (w_s(u_1,t), \dots, w_s(u_m,t)) \in \partial\Delta_m\}.$$

Then for $\varphi \in C_0^2(\partial\Delta_m)$ and $t > s$

$$f(u) = \mathbb{E}\, 1\!\!1_{\{t\geq\tau\}} \varphi(w_0(u_1,\tau), \dots, w_0(u_m,\tau)) = F(u,0).$$

Proof. It is easy to see that the Green function G to the problem (6)-(8) can be obtained by the method of images [9] and that $G \in C^\infty(\Delta_m)$. Since $F(u,s) = \varphi(u),\ u \in \partial\Delta_m$ and $\varphi \in C_0^2(\partial\Delta_m)$, solution to (6)–(8) belongs to $C^2(\Delta_m)$. Applying the Ito formula to $F(w(s),s)$ and using Doob's optional sampling theorem with bounded stopping time $t \wedge \tau$

$$\begin{aligned} F(w_s(u_1, t\wedge\tau), \dots, w_s(u_m, t\wedge\tau), t\wedge\tau) = F(u,s) + \\ + \sum_{i=1}^m \int_s^{t\wedge\tau} F_i'(w_s(u_1, r\wedge\tau), \dots, w_s(u_m, r\wedge\tau), r\wedge\tau) dw(u_i,r) + \\ + \int_s^{t\wedge\tau} F_{m+1}'(w_s(u_1,r), \dots, w_s(u_m,r), r) dr + \\ + \frac{1}{2}\sum_{i=1}^m \int_s^{t\wedge\tau} F_{ii}''(w_s(u_1,r), \dots, w_s(u_m,r), r) dr. \end{aligned}$$

Since F satisfies (6) we get

$$\begin{aligned} F(w_s(u_1, t\wedge\tau), \dots, w_s(u_m, t\wedge\tau), t\wedge\tau) = F(u,s) + \\ + \sum_{i=1}^m \int_s^{t\wedge\tau} F_i'(w_s(u_1,r), \dots, w_s(u_m,r), r) dw(u_i,r). \end{aligned}$$

Taking the expectation:

$$F(u,s) = \mathbb{E}\, F(w_s(u_1, t\wedge\tau), \dots, w_s(u_m, t\wedge\tau), t\wedge\tau).$$

Conditions (7), (8) imply that

$$\begin{aligned} F(u,0) = \mathbb{E}\, F(w_0(u_1, t\wedge\tau), \dots, w_0(u_m, t\wedge\tau), t\wedge\tau) 1\!\!1_{\{t<\tau\}} + \\ + \mathbb{E}\, F(w_0(u_1, t\wedge\tau), \dots, w_0(u_m, t\wedge\tau), t\wedge\tau) 1\!\!1_{\{t\geq\tau\}} = \\ = \mathbb{E}\, \varphi(w_0(u_1,\tau), \dots, w_0(u_m,\tau)) 1\!\!1_{\{t\geq\tau\}}. \end{aligned}$$

□

The next lemma gives some properties of a solution to the boundary value problem (6)–(8).

Lemma 3. *Let φ be some function of class $D(\partial\Delta_m)$ and let F be a solution to the problem (6)–(8). Then $F(\cdot,0)$ belongs to the set D_m.*

Proof. The Green function for the boundary value problem (6)–(8) is the Karlin–McGregor determinant

$$G(x,y,s,r)=\det\big(p_{r-s}(x_i,y_j)\big)_{\substack{i=1,\dots,m\\ j=1,\dots,m}}.$$

Then the solution to the problem (6)–(8) has the form [9]

$$F(u,s)=-\sum_{i=1}^{m-1}\int_s^t\int_{K_i}\frac{1}{2}\varphi(y)\frac{\partial}{\partial N_y}G(u,y,s,r)\,dS_y\,dr,$$

where $\frac{\partial}{\partial N}$ is the operator of differentiation along the outward normal to $K_i=\{u\in\partial\Delta_m : u_i=u_{i+1}, u_j<u_{j+1}, j\neq i\}$ and S_y is a surface measure. Since for $y\in K_i$, $\frac{\partial}{\partial N_y}=\Big(\frac{\partial}{\partial y_{i+1}}-\frac{\partial}{\partial y_i}\Big)$, we get

$$F(u,s)=\frac{1}{2}\sum_{i=1}^{m-1}\int_s^t\int_{K_i}\varphi(y)\Big(\frac{\partial}{\partial y_{i+1}}-\frac{\partial}{\partial y_i}\Big)\frac{1}{\sqrt{2}}G(u,y,s,r)\,dS_y\,dr=$$

$$=\frac{1}{2}\sum_{i=1}^{m-1}\int_s^t\int_{\Delta_{m-1}}\varphi(\pi_i^{-1}v)\Big(\frac{\partial}{\partial y_{i+1}}-\frac{\partial}{\partial y_i}\Big)G(u,y,s,r)\Big|_{\substack{y_1=v_1,\dots,y_i=v_i\\ y_{i+1}=v_i,\dots,y_m=v_{m-1}}}dv\,dr$$

Denote

$$G^{(i)}(u,v,s,r)=\Big(\frac{\partial}{\partial y_{i+1}}-\frac{\partial}{\partial y_i}\Big)G(u,y,s,r)\Big|_{\substack{y_1=v_1,\dots,y_i=v_i\\ y_{i+1}=v_i,\dots,y_m=v_{m-1}}},$$

where $u=(u_1,...,u_m)\in\Delta_m$ and $v=(v_1,...,v_{m-1})\in\Delta_{m-1}$. One can obtain the explicit formula for $G^{(i)}$:

$$G^{(i)}(u,v,s,r)=2\begin{vmatrix} p_{r-s}(u_1,v_1) & \dots & p_{r-s}(u_m,v_1)\\ & \vdots & \\ p_{r-s}(u_1,v_i) & \dots & p_{r-s}(u_m,v_i)\\ p_{r-s}(u_1,v_i)\frac{u_1-v_i}{r-s} & \dots & p_{r-s}(u_m,v_i)\frac{u_m-v_i}{r-s}\\ p_{r-s}(u_1,v_{i+1}) & \dots & p_{r-s}(u_m,v_{i+1})\\ & \vdots & \\ p_{r-s}(u_1,v_{m-1}) & \dots & p_{r-s}(u_m,v_{m-1})\end{vmatrix}$$

Representation for a solution to boundary value problem via functions $G^{(i)}$ allows to check that $F(u,0)\in D(\Delta_m)$. Condition $\varphi(\pi_i^{-1}v)\in C_0(\Delta_m)$ and smoothness of Gaussian distribution make it possible to differentiate the integral in the representation for the solution and $F(\cdot,s)\in C_0^2(\Delta_m)$. Property $\frac{\partial^2 F}{\partial u_k\partial u_j}1\!\!\mathrm{I}_{\{u_k=u_j\}}=0$ follows from the same property for functions $G^{(i)}$. □

Lemma 3 completes the proof of invariance of the set D_m under semigroup $Q_{m,t}$ if we note that

$$\mathbb{E}1\!\!\mathrm{I}_{\{\tau\le t\}}Q_{m,t-\tau}f(w(u_1,\tau),\dots,w(u_m,\tau))=\mathbb{E}1\!\!\mathrm{I}_{\{\tau\le t\}}\varphi(w(u_1,\tau),\dots,w(u_m,\tau)),$$

where φ is defined on $\partial\Delta_m$ and by inductive hypothesis $\varphi\in D(\partial\Delta_m)$.

The proof of density of the set D_m in $C_0(\Delta_m)$ is easy and omitted. □

3. The action of the semigroup $Q_{m,t}$ in terms of binary forests

In the previous section we described the core D_m for the generator of the semigroup $Q_{m,t}$, and obtained the action of the generator on functions from D_m. The function $Q_{m,t}f(u), f \in D_m$ satisfies Kolmogorov forward equation (Theorem 19.6 [7]):

$$\frac{\partial}{\partial t} Q_{m,t} f(u) = \mathcal{A} Q_{m,t} f(u), \ u \in \Delta_m, \ t > 0.$$

Taking into consideration the action of $Q_{m,t}$ on f at points of boundary of the set Δ_m we get

$$\begin{aligned} Q_{m,t}f(u) &= \mathbb{E} f(x(u_1,t),\ldots,x(u_i,t),x(u_i,t),x(u_{i+2},t),\ldots,x(u_m,t)) = \\ &= \mathbb{E} f \circ \pi_i^{-1}(x(u_1,t),\ldots,x(u_i,t),x(u_{i+2},t),\ldots,x(u_m,t)) = \\ &= Q_{mp-1,t} f \circ \pi_i^{-1}(\pi_i u), \end{aligned}$$

for $u \in K_i^m = \{u \in \partial\Delta_m : u_i = u_{i+1}, u_j < u_{j+1}, i \neq j\}$ (recall that π_i, π_i^{-1} were defined in the previous section in (4), (5)). Therefore, we obtain an iterative scheme of boundary value problems:

$$\frac{\partial}{\partial t} Q_{m,t} f(u) = \frac{1}{2} \Delta Q_{m,t} f(u), u \in \Delta_m, t \geq 0, \tag{9}$$

$$Q_{m,0} f(u) = f(u) \tag{10}$$

$$Q_{m,t} f(u) = (Q_{m-1,t} f \circ \pi_i^{-1})(\pi_i u), u \in K_i^m, Q_{m,t} f(\cdot) \in D_m. \tag{11}$$

A solution to the boundary value problem (9)–(11) we will represent as a sum in which each summand is indexed by a binary forest.

Let us define a class of forests $T_k^m, k < m$ with m leaves and k roots. Denote by U_k^m a set of vertices:

$$U_k^m = \{u_1^{(k)}, u_2^{(k)}, \ldots, u_k^{(k)}, u_1^{(k+1)}, \ldots, u_{k+1}^{(k+1)}, \ldots, u_1^{(m)}, \ldots, u_m^{(m)}, u^{(j)} \in \Delta_j\}.$$

Edges of a forest will be defined via a set of mappings

$$\mathcal{G}_j = \{\sigma_j : \{1,2,\ldots,j\} \to \{1,2,\ldots,j-1\}, \ \sigma_j \text{ is a surjection } \}.$$

For any mapping $\sigma_j \in \mathcal{G}_j$ there exists a unique pair

$$(l_1, l_2) = (l_1(\sigma_j), l_2(\sigma_j)) \subset \{1,2,\ldots,j\}$$

such that $\sigma_j(l_1) = \sigma_j(l_2), l_1 < l_2$.

For a fixed set of mappings $\{\sigma_m, \sigma_{m-1}, \ldots, \sigma_{k+1}\}$, where $\sigma_j \in \mathcal{G}_j$ define edges

$$R_k^m \equiv R_k^m(\sigma_m, \sigma_{m-1}, \ldots, \sigma_{k+1}) = \Big\{ \Big(u_j^{(i)}, u_{\sigma_i(j)}^{(i-1)} \Big), \ i = k+1, \ldots, m, \ j = 1, \ldots, i \Big\}.$$

Let us define a set of binary forests as

$$T_k^m = \big\{ (U_k^m, R_k^m(\sigma_m, \ldots, \sigma_{k+1})), \ \sigma_j \in \mathcal{G}_j \big\}.$$

We assign time t_j to the set of vertices $\big\{u_1^{(j)}, u_2^{(j)}, \ldots, u_j^{(j)}\big\}$ for every $j \in \{k, \ldots, m\}$.

We say that the edges $\Big(u_j^{(n)}, u_{\sigma_n(j)}^{(n-1)}\Big)$ and $\Big(u_i^{(n)}, u_{\sigma_n(i)}^{(n-1)}\Big)$ intersect if $i < j, \ \sigma_n(i) > \sigma_n(j)$ and denote by $\varepsilon(T)$ the number of intersection of edges in a forest $T \in T_k^m$. We assign a weight to any edge of a forest $T \in T_k^m$, which depends on connected vertices and corresponding moments of time

$$g\big(u_j^{(i)}, u_{\sigma_i(j)}^{(i-1)}, t_i, t_{i-1}\big) = p_{t_i - t_{i-1}}\big(u_j^{(i)}, u_{\sigma_i(j)}^{(i-1)}\big) \sqrt{\frac{u_{l_2}^{(i)} - u_{l_1}^{(i)}}{t_i - t_{i-1}}}$$

for $j = l_1, l_2$, where $l_1 < l_2, \ \sigma_i(l_1) = \sigma_i(l_2)$, and

$$g(u_j^{(i)}, u_{\sigma_i(j)}^{(i-1)}, t_i, t_{i-1}) = p_{t_i - t_{i-1}}\big(u_j^{(i)}, u_{\sigma_i(j)}^{(i-1)}\big)$$

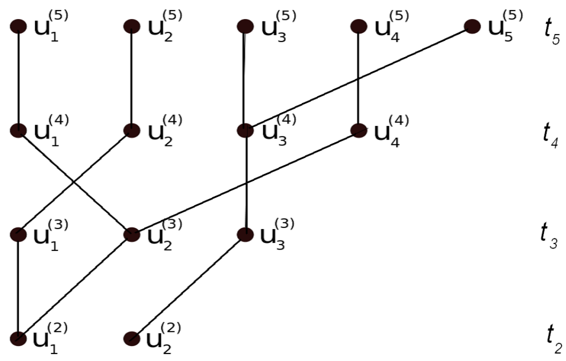

FIGURE 1. An example of $T \in T_2^5$.

for $j \neq l_1, j \neq l_2$, where $p_s(u_1, u_2) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{(u_1-u_2)^2}{2s}}$.

Denote by $|T|$ a product of weights of all edges in a forest $T \in T_k^m$:

$$|T| = |T(u^{(m)}, \dots, u^{(k)}, t_m, \dots, t_k)| = \prod_{i=k+1}^{m} \prod_{j=1}^{i} g\big(u_j^{(i)}, u_{\sigma_i(j)}^{(i-1)}, t_i, t_{i-1}\big).$$

For any forest $T \in T_k^m$ we put into correspondence a set of indexes $(i_{m-1}, i_{m-2}, \dots, i_k)$, where each index i_j is a coordinate of a vector $u^{(j)}$ such that $\sigma_{j+1}(l_1) = \sigma_{j+1}(l_2) = i_j, l_1 \neq l_2$. Define an action of a forest $T \in T_k^m$ on a function $f : \Delta_m \to \mathbb{R}$ by the rule:

$$f_T = f \circ \pi_{m-1}^{-1} \dots \circ \pi_k^{-1}$$

Now we are ready to represent the action of the semigroup of m-point motions in terms of binary forests. Denote by G_m the Green function of the boundary value problem (9)–(11).

Theorem 3. *Let f be a function of class D_m. Then*

$$\begin{aligned}
Q_{m,t}f(u) = &\int_{\Delta_m} f(y) G_m(u, y, t, 0)\, dy + \\
&+ \sum_{T \in T_{m-1}^m} (-1)^{\varepsilon(T)} \int_0^t \int_{\Delta_{m-1}} \int_{\Delta_{m-1}} f_T(y) G_{m-1}(u^{(m-1)}, y, t_{m-1}, 0) \cdot \\
&\cdot |T(u, u^{(m)}, t, t_{m-1})|\, du^{(m-1)}\, dy\, dt_{m-1} + \\
&+ \sum_{T \in T_{m-2}^m} \int_0^t \int_0^{t_{m-1}} \int_{\Delta_{m-1}} \int_{\Delta_{m-2}} \int_{\Delta_{m-2}} (-1)^{\varepsilon(T)} f_T(y) G_{m-2}(u^{(m-2)}, y, t_{m-2}, 0) \cdot \\
&\cdot |T(u, u^{(m-1)}, u^{(m-2)}, t, t_{m-1}, t_{m-2})|\, dy\, du^{(m-2)}\, du^{(m-1)}\, dt_{m-2}\, dt_{m-1} + \dots \\
&\dots + \sum_{T \in T_1^m} (-1)^{\varepsilon(T)} \int_0^t \int_0^{t_{m-1}} \dots \int_0^{t_2} \int_{\Delta_{m-1}} \int_{\Delta_{m-2}} \dots \int_{\mathbb{R}} \int_{\mathbb{R}} f_T(y) G_1(u^{(1)}, y, t_1, 0) \cdot \\
&\cdot |T(u, u^{(m-1)}, \dots, u^{(2)}, u^{(1)}, t, t_{m-1}, \dots, t_1)|\, dy\, du^{(1)} \dots du^{(m-1)}\, dt_1 \dots dt_{m-1}.
\end{aligned}$$

Proof. Let us write the solution to boundary value problem via the Green function:

$$Q_{m,t}f(u) = \int_{\Delta_m} f(y) G_m(u, y, t, 0)\, dy +$$

$$+\frac{1}{2}\sum_{i=1}^{m-1}\int_0^t\int_{K_i^m} Q_{m-1,t_{m-1}} f\circ\pi_i^{-1}(\pi_i v)\Big(\frac{\partial}{\partial v_{i+1}}-\frac{\partial}{\partial v_i}\Big)G_m(u,v,t_{m-1},t)\frac{1}{\sqrt{2}}\,dS_v\,dt_{m-1}.$$

Denoting by

$$G_m^{(i)}(u,y,s,t)=\Big(\frac{\partial}{\partial v_{i+1}}-\frac{\partial}{\partial v_i}\Big)G_m(u,v,s,t)\Big|_{\substack{v_1=y_1,\dots,v_i=y_i\\ v_{i+1}=y_i,\dots,v_m=y_{m-1}}}$$

we get

$$Q_{m,t}f(u)=\int_{\Delta_m} f(y)G_m(u,y,t,0)dy+$$

$$+\frac{1}{2}\sum_{i=1}^{m-1}\int_0^t\int_{\Delta_{m-1}} Q_{m-1,t_{m-1}} f\circ\pi_i^{-1}(\pi_i y)G_m^{(i)}(u,y,t_{m-1},t)dydt_{m-1}.$$

Since $G_m(u,v,t,s)=\det\big(p_{t-s}(u_i,v_j)\big)_{\substack{i=1,\dots,m\\ j=1,\dots,m}}$ we have

$$G^{(i)}(u,y,s,t)=2\begin{vmatrix} p_{t-s}(u_1,y_1) & \dots & p_{t-s}(u_m,y_1)\\ & \vdots & \\ p_{t-s}(u_1,y_i) & \dots & p_{t-s}(u_m,y_i)\\ p_{t-s}(u_1,y_i)\frac{u_1-y_i}{t-s} & \dots & p_{t-s}(u_m,y_i)\frac{u_m-y_i}{t-s}\\ p_{r-s}(u_1,v_{i+1}) & \dots & p_{r-s}(u_m,v_{i+1})\\ & \vdots & \\ p_{t-s}(u_1,y_{m-1}) & \dots & p_{t-s}(u_m,y_{m-1})\end{vmatrix}=$$

$$=\sum_{l_1<l_2}(-1)^{l_1+l_2+1}\begin{vmatrix} p_{t-s}(u_{l_1},y_i) & p_{t-s}(u_{l_2},y_i)\\ p_{t-s}(u_{l_1},y_i)\frac{u_{l_1}-y_i}{t-s} & p_{t-s}(u_{l_2},y_i)\frac{u_{l_2}-y_i}{t-s}\end{vmatrix}\det\big(p_{t-s}(u_j,y_k)\big)_{\substack{j\neq l_1,l_2\\ k\neq i}}=$$

$$=\sum_{l_1<l_2}(-1)^{l_1+l_2+1}p_{t-s}(u_{l_1},y_i)p_{t-s}(u_{l_2},y_i)\frac{u_{l_2}-u_{l_1}}{t-s}\det\big(p_{t-s}(u_j,y_k)\big)_{\substack{j\neq l_1,l_2\\ k\neq i}}$$

Let us check that

$$G_m^{(i)}(u,y,t,s)=\sum_{T\in T_{m-1}^m,i_{m-1}=i}|T(u,y,t,s)|, \tag{12}$$

where the last sum is taken over forests $T\subset T_{m-1}^m$, such that $i=\sigma_m(l_1)=\sigma_m(l_2),\ l_1\neq l_2$. Denote

$$\mathcal{G}_m^{l_1,l_2,i}=\{\sigma\in\mathcal{G}_m:\sigma_m(l_1)=\sigma_m(l_2)=i\}.$$

Then

$$\sum_{T\in T_{m-1}^m,i_{m-1}=i}|T(u,y,t,s)|=$$

$$=\sum_{l_1<l_2}\sum_{\sigma\in\mathcal{G}_m^{l_1,l_2,i}}p_{t-s}(u_{l_1},y_i)p_{t-s}(u_{l_2},y_i)\frac{u_{l_2}-u_{l_1}}{t-s}(-1)^{\varepsilon(T)}\prod_{j\neq l_1,l_2}p_{t-s}(u_j,y_{\sigma(j)})=$$

$$=\sum_{l_1<l_2}p_{t-s}(u_{l_1},y_i)p_{t-s}(u_{l_2},y_i)\frac{u_{l_2}-u_{l_1}}{t-s}\sum_{\sigma\in\mathcal{G}_m^{l_1,l_2,i}}\prod_{j\neq l_1,l_2}p_{t-s}(u_j,y_{\sigma(j)})(-1)^{\varepsilon(T)}.$$

Note that

$$\sum_{\sigma\in\mathcal{G}_m^{l_1,l_2,i}}\prod_{j\neq l_1,l_2}p_{t-s}(u_j,y_{\sigma(j)})(-1)^{\varepsilon(T)}=(-1)^{l_1+l_2+1}\det\big(p_{t-s}(u_j,y_k)\big)_{\substack{j\neq l_1,l_2\\ k\neq i}},$$

so the equality (12) holds.

Hence, we can write the solution to the boundary value problem (9)–(11) in the form:

$$Q_{m,s}f(u)=\int_{\Delta_m} f(y)G_m(u,y,t,0)\,dy+$$

$$+\sum_{T\in T^m_{m-1}}\int_0^t\int_{\Delta_{m-1}}(-1)^{\varepsilon(T)}Q_{m-1,t_{m-1}}f_T(y^{(m-1)})|T(u,y^{(m-1)},t,t_{m-1})|\,dy^{(m-1)}\,dt_{m-1}.$$

We replace $Q_{m-1,t_{m-1}}f_T(y^{(m-1)})$ with the solution of the corresponding boundary value problem in the obtained formula and note that for $T\in T^m_{m-1}, T_1\in T^{m-1}_{m-2}$

$$T\cup T_1\in T^m_{m-2},\ (f_T)_{T_1}=f_{T\cup T_1},\ |T||T_1|=|T\cup T_1|;$$

so we get

$$Q_{m,t}f(u)=\int_{\Delta_m}f(u)G_m(u,y,t,0)dy+$$

$$+\sum_{T\in T^m_{m-1}}\int_0^t\int_{\Delta_{m-1}}\int_{\Delta_{m-1}}(-1)^{\varepsilon(T)}f_T(y)G_{m-1}(y^{(m-1)},y,t_{m-1},0)\cdot$$

$$\cdot|T(u,y^{(m-1)},t,t_{m-1})|dydy^{(m-1)}dt_{m-1}+$$

$$+\sum_{T\in T^m_{m-2}}\int_0^t\int_0^{t_{m-1}}\int_{\Delta_{m-1}}\int_{\Delta_{m-2}}(-1)^{\varepsilon(T)}Q_{m-2,t_{m-2}}f_T(y^{(m-2)})$$

$$|T(u,y^{(m-1)},y^{(m-2)},t,t_{m-1},t_{m-2})|dy^{(m-2)}dy^{(m-1)}dt_{m-2}dt_{m-1}.$$

We continue this iterative scheme by replacing $Q_{k,t_k}f, k=1,\dots,m-1$ with the solutions of the corresponding boundary value problems. The theorem is proved.

□

References

1. I.I. Nishchenko, *Discrete time approximation of coalescing stochastic flows on the real line*, Theory of Stochastic Processes **17(33)** (2011), no. 1, 70–78.
2. T.E. Harris, *Coalescing and noncoalescing stochastic flows in* $\mathbb{R}_1$, Stoch. Process and Their Applic. **17** (1984), 187–210
3. E.V. Glinyanaya, *Disordering asymptotics in the discrete approximation of an Arratia flow*, Theory of Stochastic Processes **18(34)** (2012), no. 2, 39–49.
4. A.A. Dorogovtsev and I.I. Nischenko, *An analysis of stochastic flows*, Communications on Stochastic Analysis, **8** (2014), no. 3, 331–342
5. A. Shamov, *Short-time asymptotics of one-dimensional Harris flows*, Communications on Stochastic Analysis, **5** (2011), no. 3, 527–539
6. Y. Le Jan and O. Raimond, *Flows, coalescence and noise*, Ann. Probab. **32** (2004), 1247–1315.
7. O. Kallenberg, *Foundation of Modern Probability*, second ed., Springer, New York, 2001.
8. S. Karlin and J.G. McGregor, *Coincidence probabilities*, Pacific J.Math. **9** (1959), 1141–1164.
9. A.D. Polyanin, *Handbook of linear partial differential equations for engineers and scientists*, CRC press, 2001.

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