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ONE TYPE OF SINGULAR PERTURBATIONS OF A MULTIDIMENSIONAL STABLE PROCESS

A semigroup of linear operators on the space of all continuous bounded functions given on a d-dimensional Euclidean space $\mathbb{R}^d$ is constructed such that its generator can be written in the following form

$$\mathbf{A} + q(x)\delta_S(x)\mathbf{B}_\nu,$$

where $\mathbf{A}$ is the generator of a symmetric stable process in $\mathbb{R}^d$ (that is, a pseudo-differential operator whose symbol is given by $(-c|\xi|^\alpha)_{\xi\in\mathbb{R}^d}$, parameters $c>0$ and $\alpha\in(1,2]$ are fixed); $\mathbf{B}_\nu$ is the operator with the symbol $(2ic|\xi|^{\alpha-2}(\xi,\nu))_{\xi\in\mathbb{R}^d}$ ($i=\sqrt{-1}$ and $\nu\in\mathbb{R}^d$ is a fixed unit vector); S is a hyperplane in $\mathbb{R}^d$ that is orthogonal to ν; $(\delta_S(x))_{x\in\mathbb{R}^d}$ is a generalized function whose action on a test function consists in integrating the latter one over S (with respect to Lebesgue measure on S); and $(q(x))_{x\in S}$ is a given bounded continuous function with real values. This semigroup is generated by some kernel that can be given by an explicit formula. However, there is no Markov process in $\mathbb{R}^d$ corresponding to this semigroup because it does not preserve the property of a function to take on only non-negative values.

1. Introduction

Denote by $\hat{g}(t,x,y)$, $t>0$, $x\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$, transition probability density of a Wiener process in $\mathbb{R}^d$:

$$\hat{g}(t,x,y) = (2\pi t)^{-d/2}\exp\left\{-|y-x|^2/2t\right\}.$$

Let S be a hyperplane in $\mathbb{R}^d$ that is orthogonal to a fixed unit vector $\nu\in\mathbb{R}^d$: $S=\{x\in\mathbb{R}^d : (x,\nu)=0\}$ and let a continuous bounded function $(q(x))_{x\in S}$ be given. We define a function $\hat{G}$ of the arguments $t>0$, $x\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$ by the formula

$$\hat{G}(t,x,y) = \hat{g}(t,x,y) + \int_0^t d\tau \int_S \hat{g}(t-\tau,x,z)\frac{\partial \hat{g}(\tau,z,y)}{\partial \nu_z} q(z)\, d\sigma_z, \tag{1}$$

where $\frac{\partial \hat{g}(\tau,z,y)}{\partial \nu_z}$ is the derivative of $\hat{g}$ (as a function of the argument z) in the direction ν:

$$\frac{\partial \hat{g}(\tau,z,y)}{\partial \nu_z} = \frac{(y-z,\nu)}{\tau}\hat{g}(\tau,z,y)$$

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and the inner integral on the right hand side of (1) is a surface integral over S. It is well-known that the function $\hat{G}$ satisfies the Kolmogorov-Chapman equation

$$\hat{G}(s+t,x,y)=\int_{\mathbb{R}^d}\hat{G}(s,x,z)\hat{G}(t,z,y)\,dz \tag{2}$$

for all $s>0$, $t>0$, $x\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$; and the following condition

$$\int_{\mathbb{R}^d}\hat{G}(t,x,y)\,dy=1$$

holds true for all $t>0$, $x\in\mathbb{R}^d$ (see, for example, [13]). If additionally the function q is such that $|q(x)|\le 1$ for all $x\in S$, then the function $\hat{G}$ takes on only non-negative values. In this case there exists a continuous Markov process in $\mathbb{R}^d$ with the function $\hat{G}$ being its transition probability density and this process can be characterized as a solution to the following stochastic differential equation (see [13])

$$d\hat{x}(t)=\nu q(\hat{x}(t))\delta_S(\hat{x}(t))dt+dw(t),$$

where $(w(t))_{t\ge 0}$ is a d-dimensional Wiener process and the generalized function $(\delta_S(x))_{x\in\mathbb{R}^d}$ is determined by the relation

$$\langle\delta_S,\varphi\rangle=\int_S\varphi(x)\,d\sigma$$

valid for any continuous compactly supported function φ on $\mathbb{R}^d$.

If $d=1$ (in this case $S=\{0\}$ and $q(0)$ is a constant q from the segment $[-1,1]$), then $\hat{G}$ is transition probability density of the so-called skew Brownian motion

$$\hat{G}(t,x,y)=(2\pi t)^{-1/2}\left[\exp\left\{-(y-x)^2/2t\right\}+q\,\mathrm{sign}\,y\exp\left\{-(|y|+|x|)^2/2t\right\}\right],$$

$t>0$, $x\in\mathbb{R}^1$, and $y\in\mathbb{R}^1$ (see [6], [13]).

Formula (1) gives thus some transformation of a d-dimensional Wiener process. The aim of this article is to transform likewise a d-dimensional symmetric stable process, that is a Markov process in $\mathbb{R}^d$ with its transition probability density given by

$$g(t,x,y)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\exp\left\{i(y-x,\xi)-ct|\xi|^\alpha\right\}d\xi,\quad t>0,\ x\in\mathbb{R}^d,\ y\in\mathbb{R}^d$$

(parameters $c>0$ and $\alpha\in(1,2)$ will be fixed throughout this article).

If we put into formula (1) the function g instead of $\hat{g}$, we arrive at the situation when the integrals on the right hand side of (1) do not exist. This suggests an idea to change at the same time the operator $\frac{\partial}{\partial\nu_z}$ in (1) by a weaker one in some sense. It is not difficult to comprehend that the operator $\mathbf{B}_\nu$ with its symbol given by $(2ic|\xi|^{\alpha-2}(\xi,\nu))_{\xi\in\mathbb{R}^d}$ is suitable.

So, we arrive at the formula

$$G(t,x,y)=g(t,x,y)+\int_0^t d\tau\int_S g(t-\tau,x,z)g_\nu(\tau,z,y)q(z)\,d\sigma_z, \tag{3}$$

where $t>0$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$, $g_\nu(\tau,z,y)=\mathbf{B}_\nu g(\tau,\cdot,y)(z)$.

We will show that the function G is well-defined and the family of operators $(T_t)_{t>0}$ defined for any bounded continuous function φ on $\mathbb{R}^d$ by the equality

$$T_t\varphi(x)=\int_{\mathbb{R}^d}G(t,x,y)\varphi(y)\,dy,\quad t>0,\ x\in\mathbb{R}^d, \tag{4}$$

indeed constitutes a semigroup generated (in some sense) by the operator

$$\mathbf{A}+q(x)\delta_S(x)\mathbf{B}_\nu,$$

where $\mathbf{A}$ is the generator of a symmetric stable process in $\mathbb{R}^d$, that is, a pseudo-differential operator with its symbol given by $(-c|\xi|^\alpha)_{\xi\in\mathbb{R}^d}$. The case of $d=1$ was considered in

[10]. The generalization of that result on a multidimensional situation is not trivial for the simple reason that a multidimensional stable process, unlike a Wiener one, is not a set of independent one-dimensional processes.

Symmetric stable processes (and some more general ones) were perturbed by terms of the type $(a(x), \nabla)$ with a more or less singular function $(a(x))_{x\in\mathbb{R}^d}$ in various papers that can be considered as close to this article (see, for example, [3, 5, 8, 9, 11, 12, 14]).

The article is organized as follows. In Section 2 we formulate some assertions about multidimensional symmetric stable distributions, in Section 3 the operator $\mathbf{B}$ is introduced, Section 4 is devoted to describing the properties of the function G, in Section 5 the corresponding semigroup of operators is described, and finally, in Section 6 the pseudo-differential equation for the semigroup is found.

2. Multidimensional stable distributions

Denote by $(h_d(x))_{x\in\mathbb{R}^d}$ for an integer d the function given by

$$h_d(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp\{-i(x,\xi) - c|\xi|^\alpha\}\, d\xi.$$

The function g defined above can be written in the form

$$g(t,x,y) = t^{-d/\alpha} h_d((y-x)t^{-1/\alpha})$$

for all $t > 0$, $x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$.

The following representation of h_d is well-known (see, for example, [1]):

$$h_d(x) = (2\pi)^{-d/2}|x|^{1-d/2} \int_0^\infty e^{-c\rho^\alpha} \rho^{d/2} J_{d/2-1}(\rho|x|)\, d\rho, \quad x \in \mathbb{R}^d, \tag{5}$$

where J_μ is the Bessel function of order μ:

$$J_\mu(z) = \frac{(z/2)^\mu}{\sqrt{\pi}\Gamma(\mu+1/2)} \int_{-1}^1 (1-u^2)^{\mu-1/2} \cos(zu)\, du$$

for $\operatorname{Re}\mu > -1/2$ and $J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z$.

The formula (5) implies the following statement characterizing the behavior of $h_d(x)$ for large x (see [1]):

$$\lim_{|x|\to\infty} |x|^{d+\alpha} h_d(x) = c\alpha 2^{\alpha-1}\pi^{-d/2-1} \sin\frac{\pi\alpha}{2} \Gamma\left(\frac{d+\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}\right) \tag{6}$$

It follows from the relation (6) that there exists a constant $N > 0$ such that

$$h_d(x) \le N \frac{1}{(1+|x|)^{d+\alpha}}$$

for all $x \in \mathbb{R}^d$. This inequality implies the following one

$$g(t,x,y) \le N \frac{t}{(t^{1/\alpha} + |x-y|)^{d+\alpha}} \tag{7}$$

valid for all $t > 0$, $x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$.

The inequality (7) as well as similar ones for (fractional) derivatives of g can be found in [4]. We will have below the opportunities to use them.

The next assertion seems to be almost evident from the probabilistic point of view, nevertheless, it will be provided by an analytical proof.

Proposition 2.1. *Let $d \ge 2$, ν be a fixed unit vector in $\mathbb{R}^d$ and $\tilde{x}$ be an arbitrary vector in $\mathbb{R}^d$ orthogonal to ν. Then for any $\xi \in \mathbb{R}^1$ the following formula*

$$\int_{\mathbb{R}^1} e^{i\lambda\xi} h_d(\lambda\nu + \tilde{x})\, d\lambda = (2\pi)^{-\frac{d-1}{2}} |\tilde{x}|^{-\frac{d-3}{2}} \int_0^\infty e^{-c(\xi^2+\rho^2)^{\alpha/2}} \rho^{\frac{d-1}{2}} J_{\frac{d-3}{2}}(\rho|\tilde{x}|)\, d\rho \tag{8}$$

holds true.

Proof. The integral on the left hand side of (8)(denote it by I) can be written as follows

$$I = \frac{2}{(2\pi)^{d/2}} \int_0^\infty e^{-c\rho^\alpha} \rho^{d/2}\, d\rho \int_0^\infty J_{\frac{d-2}{2}}\left(\rho\sqrt{\lambda^2+b^2}\right)\left(\sqrt{\lambda^2+b^2}\right)^{-\frac{d-2}{2}} \cos(\lambda\xi)\, d\lambda,$$

where $b = |\tilde{x}|$. The inner integral here can be calculated (see [2, Ch. III, §16]). Namely

$$\int_0^\infty J_{\frac{d-2}{2}}\left(\rho\sqrt{\lambda^2+b^2}\right)\left(\sqrt{\lambda^2+b^2}\right)^{-\frac{d-2}{2}} \cos(\lambda\xi)\, d\lambda =$$

$$= \begin{cases} 0 \text{ if } |\xi| > \rho; \\ \sqrt{\frac{\pi}{2}}\rho^{-\frac{d-2}{2}} J_{\frac{d-3}{2}}\left(b\sqrt{\rho^2-\xi^2}\right)\left(\frac{\sqrt{\rho^2-\xi^2}}{b}\right)^{\frac{d-3}{2}} \quad \text{if } |\xi| < \rho. \end{cases}$$

Hence,

$$I = (2\pi)^{-\frac{d-1}{2}} b^{-\frac{d-3}{2}} \int_{|\xi|}^\infty e^{-c\rho^\alpha} \rho J_{\frac{d-3}{2}}\left(b\sqrt{\rho^2-\xi^2}\right)\left(\sqrt{\rho^2-\xi^2}\right)^{\frac{d-3}{2}} d\rho.$$

Integrating here by the substitution $\rho' = \sqrt{\rho^2-\xi^2}$, we arrive at formula (8) □

Corollary 2.1. *Let $\mathbb{L}$ be a subspace of $\mathbb{R}^d$, $\dim \mathbb{L} = k$, $1 \le k < d$. For any $\xi \in \mathbb{L}$ and $\tilde{x} \in \mathbb{L}^\perp$ the formula*

$$\int_{\mathbb{L}} e^{i(x,\xi)} h_d(x+\tilde{x})\, dx = (2\pi)^{-\frac{d-k}{2}} |\tilde{x}|^{-\frac{d-k-2}{2}} \int_0^\infty e^{-c(|\xi|^2+\rho^2)^{\alpha/2}} \rho^{\frac{d-k}{2}} J_{\frac{d-k-2}{2}}(\rho|\tilde{x}|)\, d\rho$$

is valid.

In particular, if $\nu \in \mathbb{R}^d$ is a fixed unit vector and $S = \{x \in \mathbb{R}^d : (x,\nu) = 0\}$, then for $\xi \in S$ and $\lambda \in \mathbb{R}^1$ the following relation

$$\int_S e^{i(x,\xi)} h_d(x+\lambda\nu)\, dx = \frac{1}{\pi}\int_0^\infty e^{-c(|\xi|^2+\rho^2)^{\alpha/2}} \cos(\rho\lambda)\, d\rho \tag{9}$$

is held.

3. The operator B

Denote by $\mathbf{B}$ the operator whose symbol is a vector-valued function given by $2ic|\xi|^{\alpha-2}\xi$ for $\xi \in \mathbb{R}^d$. This means that the result of its action on a function $(\varphi(x))_{x\in\mathbb{R}^d}$ given by the Fourier transform

$$\varphi(x) = \int_{\mathbb{R}^d} e^{i(x,\xi)} \Phi(\xi)\, d\xi, \quad x \in \mathbb{R}^d,$$

is a vector-valued function expressed by the integral

$$\mathbf{B}\varphi(x) = 2ic \int_{\mathbb{R}^d} e^{i(x,\xi)} |\xi|^{\alpha-2}\xi\Phi(\xi)\, d\xi$$

under the assumption that this integral is well-defined. This operator will play the role analogous to that of the gradient in classical theories. Notice, by the way, that $\mathbf{A} = \frac{1}{2}\mathbf{div}\,\mathbf{B}$, and the operator $\mathbf{A}$ can be thought of as an analogy to Laplacian.

We put $\varkappa = -\dfrac{2\pi^{\frac{d-1}{2}}\Gamma(2-\alpha)\Gamma\left(\frac{\alpha+1}{2}\right)\cos\frac{\pi\alpha}{2}}{\alpha(\alpha-1)\Gamma\left(\frac{d+\alpha}{2}\right)}$. Then the action of the operators $\mathbf{A}$ and $\mathbf{B}$ on a function $(\varphi(x))_{x\in\mathbb{R}^d}$ can be given by the following integrals

$$\mathbf{A}\varphi(x) = \frac{c}{\varkappa}\int_{\mathbb{R}^d} [\varphi(x+y) - \varphi(x) - (\nabla\varphi(x), y)]\, |y|^{-d-\alpha}\, dy,$$

$$\mathbf{B}\varphi(x) = \frac{2c}{\alpha\varkappa}\int_{\mathbb{R}^d} [\varphi(x+y) - \varphi(x)]\, |y|^{-d-\alpha} y\, dy$$

under the assumption, of course, that the function φ is sufficiently smooth and bounded.

Denote by $\mathbf{B}_\nu$ the operator whose symbol is $(2ic|\xi|^{\alpha-2}(\xi,\nu))_{\xi\in\mathbb{R}^d}$ for a given unit vector $\nu\in\mathbb{R}^d$. This operator is an analogy to differentiating in the direction ν.

We introduce the following notation $g_\nu(\tau,z,y)=\mathbf{B}_\nu g(\tau,\cdot,y)(z)$ for all $\tau>0$, $z\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$. Then

$$g_\nu(\tau,z,y)=\frac{2ic}{(2\pi)^d}\int_{\mathbb{R}^d}\exp\{i(z-y,\xi)-c\tau|\xi|^\alpha\}\,|\xi|^{\alpha-2}(\xi,\nu)\,d\xi$$

and integrating by parts, we get the formula

$$g_\nu(\tau,z,y)=\frac{2}{\alpha}\frac{(y-z,\nu)}{\tau}g(\tau,z,y) \tag{10}$$

valid for $\tau>0$, $z\in\mathbb{R}^d$, and $y\in\mathbb{R}^d$. This formula is quite analogous to that for $\frac{\partial\hat g(\tau,z,y)}{\partial\nu_z}$ (see above).

4. Properties of the function G

We first verify that the function G is defined correctly by formula (3). In order to do this, we have to show that the integrals on the right hand side of (3) do exist.

Notice that $g_\nu(t,z,y)=0$ if $z\in S$ and $y\in S$, and the integral on the right hand side of (3) vanishes. So, we have to consider the case of $(y,\nu)\neq 0$.

For a vector $x\in\mathbb{R}^d$, denote by $\tilde x$ its orthogonal projection on S: $\tilde x=x-\nu(x,\nu)$. Formula (9) implies the relation

$$\int_S g(t-\tau,x,z)e^{i(z,\xi)}\,d\sigma_z=\frac{1}{2\pi}e^{i(x,\tilde\xi)}\int_{\mathbb{R}^1}\exp\left\{-i\rho(x,\nu)-c(t-\tau)(\rho^2+|\tilde\xi|^2)^{\alpha/2}\right\}d\rho$$

valid for $x\in\mathbb{R}^d$, $\xi\in\mathbb{R}^d$, and $0<\tau<t$. As a consequence of this and formula (10), we get

$$\begin{aligned}\int_S g(t-\tau,x,z)g_\nu(\tau,z,y)\,d\sigma_z=\\ =\frac{2(y,\nu)}{\alpha\tau(2\pi)^{d+1}}\int_{\mathbb{R}^1}\int_{\mathbb{R}^1}\int_{\mathbb{R}^{d-1}}\exp\{-i(\tilde y-\tilde x,\eta)-i\rho(x,\nu)-ir(y,\nu)-\\ -ca_{\tau,t}(\rho,r,\eta)\}\,d\rho\,dr\,d\eta,\end{aligned} \tag{11}$$

where $a_{\tau,t}(\rho,r,\eta)=\tau(r^2+|\eta|^2)^{\frac{\alpha}{2}}+(t-\tau)(\rho^2+|\eta|^2)^{\frac{\alpha}{2}}$ for all $r\in\mathbb{R}^1$, $\rho\in\mathbb{R}^1$, $\eta\in\mathbb{R}^{d-1}$, and $0<\tau<t$.

For fixed $\tau>0$ and $t>\tau$, the function $a_{\tau,t}$ of the arguments $(\rho,r,\eta)\in\mathbb{R}^1\times\mathbb{R}^1\times\mathbb{R}^{d-1}$ is a homogeneous one of the degree α. Its minimal value on the sphere

$$\{(\rho,r,\eta):\rho^2+r^2+|\eta|^2=1\}$$

is easily seen be equal to $\tau\wedge(t-\tau)$. Assuming $\tau\in(0,t/2)$ and making use of the substitution $\eta=\tau^{-1/\alpha}\xi$, $\rho=\tau^{-1/\alpha}\theta$, and $r=\tau^{-1/\alpha}\lambda$, we can rewrite the right hand side of (11) as follows

$$\begin{aligned}\frac{2(y,\nu)\tau^{-\frac{d+1+\alpha}{\alpha}}}{\alpha(2\pi)^{d+1}}\int_{\mathbb{R}^1}\int_{\mathbb{R}^1}\int_{\mathbb{R}^{d-1}}\exp\{-i\tau^{-1/\alpha}[\lambda(y,\nu)+\theta(x,\nu)+(\tilde y-\tilde x,\xi)]-\\ -c\hat a_{\tau,t}(\theta,\lambda,\xi)\}\,d\theta\,d\lambda\,d\xi,\end{aligned}$$

where $\hat a_{\tau,t}(\theta,\lambda,\xi)=(\theta^2+|\xi|^2)^{\alpha/2}\cdot\frac{t-\tau}{\tau}+(\lambda^2+|\xi|^2)^{\alpha/2}$. Since $\frac{t-\tau}{\tau}\geq 1$ for $\tau\in(0,t/2)$, we have $\inf\hat a_{\tau,t}(\theta,\lambda,\xi)=1$, where infimum is taken over the sphere mentioned above. Therefore, we can apply Lemma 4.1 from [4] to the last integral to obtain the estimate

$$\int_S g(t-\tau,x,z)|g_\nu(\tau,z,y)|\,d\sigma_z\leq\frac{N|(y,\nu)|}{[\tau^{1/\alpha}+((x,\nu)^2+(y,\nu)^2+|\tilde y-\tilde x|^2)^{1/2}]^{d+\alpha+1}}$$

valid for $\tau \in (0, t/2)$, $x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$, where N is a positive constant depending only on c and α. Similar reasons for the case of $\tau \in (t/2, t)$ lead us to the inequality

$$\int_S g(t-\tau, x, z)|g_\nu(\tau, z, y)|\, d\sigma_z \leq \frac{N|(y,\nu)|}{[(t-\tau)^{1/\alpha} + ((x,\nu)^2 + (y,\nu)^2 + |\tilde{y} - \tilde{x}|^2)^{1/2}]^{d+\alpha+1}}$$

held true for $\tau \in (t/2, t)$, $x \in \mathbb{R}^d$, and $y \in \mathbb{R}^d$. We have thus arrived at the estimation

$$\left| \int_0^t d\tau \int_S g(t-\tau, x, z) g_\nu(\tau, z, y) q(z)\, d\sigma_z \right| \leq \tag{12}$$

$$\leq 2N\|q\||(y,\nu)| \int_0^{t/2} [\tau^{1/\alpha} + ((x,\nu)^2 + (y,\nu)^2 + |\tilde{y} - \tilde{x}|^2)^{1/2}]^{-d-\alpha-1}\, d\tau,$$

where $\|q\| = \sup_{z \in S} |q(z)|$. If $(y, \nu) \neq 0$, then the last integral is finite, and the function G is indeed defined correctly.

Since $\int_{\mathbb{R}^d} g_\nu(t, x, y)\, dy \equiv 0$, $t > 0$, $x \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} G(t, x, y)\, dy \equiv 1, \quad t > 0,\ x \in \mathbb{R}^d.$$

The fact that the function G satisfies the Kolmogorov-Chapman equation (see (2)) can be established in the same way as it is done in one-dimensional case (see [10]).

Remark 4.1. Considering the integral

$$I(t, y) = \int_0^t d\tau \int_S g_\nu(\tau, z, y)\, d\sigma_z, \quad t > 0,\ y \notin S,$$

one can observe that (see formula (9))

$$I(t, y) = \frac{2(y,\nu)}{\pi\alpha} \int_0^t \frac{d\tau}{\tau} \int_0^\infty e^{-c\tau\rho^\alpha} \cos(\rho(y,\nu))\, d\rho.$$

Integrating now by parts, we get for $t > 0$, $y \notin S$

$$I(t, y) = \frac{2c}{\pi} \int_0^t d\tau \int_0^\infty \rho^\alpha e^{-c\tau\rho^\alpha} \frac{\sin(\rho(y,\nu))}{\rho}\, d\rho =$$

$$= \lim_{\delta \to 0+} \frac{2}{\pi} \int_0^\infty e^{-c\delta\rho^\alpha} \frac{\sin(\rho(y,\nu))}{\rho}\, d\rho - \frac{2}{\pi} \int_0^\infty e^{-ct\rho^\alpha} \frac{\sin(\rho(y,\nu))}{\rho}\, d\rho.$$

Hence, the following formula

$$I(t, y) = \operatorname{sign}(y,\nu) - \frac{2}{\pi} \int_0^\infty e^{-ct\rho^\alpha} \frac{\sin(\rho(y,\nu))}{\rho}\, d\rho$$

holds true for $t > 0$ and $(y, \nu) \neq 0$.

As a consequence of this we have the following analogy to the classical theorem (see, for example, [13, Ch. III]).

$$\lim_{y \to x\pm} \int_0^t d\tau \int_S v(t-\tau, z) g_\nu(\tau, z, y)\, d\sigma_z = \pm v(t, x),$$

where $t > 0$, $x \in S$, and $(v(\tau, z))_{\tau > 0, z \in S}$ is a continuous bounded function; the symbol $y \to x\pm$ means that $y = x \pm \delta\nu$ for $x \in S$ and $\delta \to 0+$. In particular,

$$\lim_{\hat{y} \to y\pm} \int_0^t d\tau \int_S g(t-\tau, x, z) g_\nu(\tau, z, \hat{y}) q(z)\, d\sigma_z = \pm q(y) g(t, x, y)$$

for $t > 0$, $x \in \mathbb{R}^d$, and $y \in S$. This implies the formula

$$G(t, x, y\pm) = (1 \pm q(y)) g(t, x, y)$$

valid for all $t > 0$, $x \in \mathbb{R}^d$, and $y \in S$.

5. The semigroup of operators $(T_t)_{t>0}$

Denote by $\mathbb{C}_b(\mathbb{R}^d)$ the set of all real-valued continuous bounded functions on $\mathbb{R}^d$ with the norm $\|\varphi\| = \sup_{x\in\mathbb{R}^d} |\varphi(x)|$.

Proposition 5.1. *For any $t > 0$, the operator T_t defined by formula (4) is a linear bounded operator on $\mathbb{C}_b(\mathbb{R}^d)$ and the family of them $(T_t)_{t>0}$ forms a semigroup.*

Proof. The assertion can be deduced from the inequality (12), but we propose another way. Put

$$u(t, x, \varphi) = \int_{\mathbb{R}^d} g(t, x, y)\varphi(y)\, dy$$

and

$$u_\nu(t, x, \varphi) = \int_{\mathbb{R}^d} g_\nu(t, x, y)\varphi(y)\, dy$$

for $t > 0$, $x \in \mathbb{R}^d$, and $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$. Since g is a transition probability density, we have

$$|u(t, x, \varphi)| \le \|\varphi\|$$

for all $t > 0$, $x \in \mathbb{R}^d$. The function $u_\nu(\tau, z, \varphi)$ for $\tau > 0$, $z \in S$, and $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$ can be estimated as follows (we use the formula (10))

$$|u_\nu(\tau, z, \varphi)| \le \frac{2}{\alpha\tau}\|\varphi\| \int_{\mathbb{R}^d} |(y, \nu)| g(\tau, z, y)\, dy = \frac{2\|\varphi\|}{\alpha\tau^{1-1/\alpha}} \int_{\mathbb{R}^d} |(y, \nu)| h_d(y)\, dy.$$

It is well-known that the first absolute moment of the distribution h_d is finite. So, there exists a constant $K > 0$ such that

$$|u_\nu(\tau, z, \varphi)| \le K\|\varphi\|\tau^{-1+1/\alpha} \tag{13}$$

for all $\tau > 0$, $z \in S$, and $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$. In order to estimate the integral $\int_S g(t-\tau, x, z)\, d\sigma_z$, $0 < \tau < t$, $x \in \mathbb{R}^d$, we use formula (9)

$$\int_S g(t-\tau, x, z)\, d\sigma_z = \frac{1}{\pi}(t-\tau)^{-1/\alpha} \int_0^\infty e^{-c\rho^\alpha} \cos(\rho(x, \nu)(t-\tau)^{-1/\alpha})\, d\rho.$$

This implies the existence of a constant (we again denote it by K) such that

$$\int_S g(t-\tau, x, z)\, d\sigma_z \le K(t-\tau)^{-1/\alpha}. \tag{14}$$

As a consequence of (13) and (14), we have the following inequality ($t > 0$, $x \in \mathbb{R}^d$, $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$)

$$\left| \int_0^t d\tau \int_S g(t-\tau, x, z) u_\nu(\tau, z, \varphi) q(z)\, d\sigma_z \right| \le K\|\varphi\|\|q\|$$

where K is a constant (we can choose it the same as in (13) and (14)). Since

$$T_t\varphi(x) = u(t, x, \varphi) + \int_0^t d\tau \int_S g(t-\tau, x, z) u_\nu(\tau, z, \varphi) q(z)\, d\sigma_z,$$

we have thus proved boundedness of the operator T_t for fixed $t > 0$. The rest of properties of T_t claimed in Proposition 5.1 are obvious. □

6. The equation for the function $(T_t\varphi(x))_{t>0,x\in\mathbb{R}^d}$

It is well-known that the function $u(t,x,\varphi)$ (see Proposition 5.1) satisfies the equation

$$\frac{\partial u(t,x,\varphi)}{\partial t} = \mathbf{A}u(t,\cdot,\varphi)(x), \quad t>0,\ x\in\mathbb{R}^d \tag{15}$$

and the initial condition

$$\lim_{t\to 0+} u(t,x,\varphi)=\varphi(x), \quad x\in\mathbb{R}^d.$$

We now put for $t>0$, $x\in\mathbb{R}^d$, and $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$

$$V(t,x,\varphi)=\int_0^t d\tau\int_S g(t-\tau,x,z)u_\nu(\tau,z,\varphi)q(z)\,d\sigma_z.$$

Proposition 6.1. *The function V satisfies the equation (15) for $t>0$ and $x\notin S$.*

Proof. We first estimate the function $\mathbf{A}g(t,\cdot,y)(x)$. Represent it as follows

$$\mathbf{A}g(t,\cdot,y)(x) = -\frac{c}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x-y,\xi)}|\xi|^\alpha e^{-ct|\xi|^\alpha}\,d\xi =$$
$$= -\frac{c}{(2\pi)^d}t^{-\frac{d+\alpha}{\alpha}}\int_{\mathbb{R}^d} e^{i(x-y,\xi)t^{-1/\alpha}}|\xi|^\alpha e^{-c|\xi|^\alpha}\,d\xi.$$

Lemma 4.2 from [4] mentioned above allows us to obtain the estimation

$$|\mathbf{A}g(t,\cdot,y)(x)|\le \frac{N}{(t^{1/\alpha}+|x-y|)^{d+\alpha}},$$

where $t>0$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$, and N is a positive constant. Using this inequality we can write down the following chain of estimations ($t>0$, $x\notin S$).

$$\left|\int_0^t d\tau\int_S \mathbf{A}g(t-\tau,\cdot,z)(x)u_\nu(\tau,z,\varphi)q(z)\,d\sigma_z\right|\le$$
$$\le const\ \|\varphi\|\|q\|\int_0^t\frac{d\tau}{\tau^{1-1/\alpha}}\int_{\mathbb{R}^{d-1}}\frac{dz}{[(t-\tau)^{1/\alpha}+(|z|^2+(x,\nu)^2)^{1/2}]^{d+\alpha}}\le$$
$$\le const\ \|\varphi\|\|q\|\frac{t^{1/\alpha}}{|(x,\nu)|^{\alpha+1}}\int_{\mathbb{R}^{d-1}}\frac{dz}{[|z|^2+1]^{(d+\alpha)/2}}=$$
$$= const\ \|\varphi\|\|q\|\frac{t^{1/\alpha}}{|(x,\nu)|^{\alpha+1}}.$$

Therefore, for $x\notin S$ and $t>0$ the integral

$$\int_0^t d\tau\int_S\frac{\partial g(t-\tau,x,z)}{\partial t}u_\nu(\tau,z,\varphi)q(z)\,d\sigma_z$$

exists as well. The proposition will be proved if we show that the relation

$$\lim_{\varepsilon\to 0+}\int_S g(\varepsilon,x,z)u_\nu(t,z,\varphi)q(z)\,d\sigma_z=0 \tag{16}$$

holds true for fixed $t>0$ and $x\notin S$. We can estimate the integral in (16) in the following way

$$\left|\int_S g(\varepsilon,x,z)u_\nu(t,z,\varphi)q(z)\,d\sigma_z\right|\le\frac{const}{t^{1-1/\alpha}}\|q\|\int_S g(\varepsilon,x,z)\,d\sigma_z=$$
$$= const\ \|q\|t^{1/\alpha-1}\frac{1}{\pi\varepsilon^{1/\alpha}}\int_0^\infty e^{-c\rho^\alpha}\cos\frac{\rho(x,\nu)}{\varepsilon^{1/\alpha}}\,d\rho$$

(we have just used formula (9)). According to (6), we have asymptotical relation

$$\frac{1}{\pi\varepsilon^{1/\alpha}}\int_0^\infty e^{-c\rho^\alpha}\cos\frac{\rho(x,\nu)}{\varepsilon^{1/\alpha}}\,d\rho \sim const\,\frac{\varepsilon}{|(x,\nu)|^{\alpha+1}}$$

as $\varepsilon\to 0+$, and (16) has been established. □

Remark 6.1. One can easily see that for any compactly supported function $(\psi(x))_{x\in\mathbb{R}^d}$ the following relation

$$\lim_{\varepsilon\to 0+}\int_{\mathbb{R}^d}\psi(x)\,dx\int_S g(\varepsilon,x,z)u_\nu(t,z,\varphi)q(z)\,d\sigma_z = \int_S \psi(z)u_\nu(t,z,\varphi)q(z)\,d\sigma_z$$

holds.

This means that

$$\frac{\partial V(t,x,\varphi)}{\partial t} = \mathbf{A}V(t,\cdot,\varphi)(x) + q(x)\delta_S(x)u_\nu(t,x,\varphi).$$

Proposition 6.2. *For $t>0$ and $x\in S$, the following equalities*

$$V_\nu(t,x\pm,\varphi) = \mp q(x)u_\nu(t,x,\varphi)$$

hold true, where $V_\nu(t,x\pm,\varphi)=\lim_{\varepsilon\to0+}V_\nu(t,x\pm\varepsilon\nu,\varphi)$ and $V_\nu(t,z,\varphi)=\mathbf{B}_\nu V(t,\cdot,\varphi)(z)$.

Proof. This assertion can be proved in a way quite similar to the calculations in Remark 4.1. □

Now, it is a simple observation that the function δ_S is a symmetrical one in the following sense

$$\delta_S(x)V_\nu(t,x,\varphi)=0,\quad x\in\mathbb{R}^d.$$

It means that the function

$$U(t,x,\varphi)=u(t,x,\varphi)+V(t,x,\varphi),\quad t>0,\ x\in\mathbb{R}^d,$$

satisfies the equation

$$\frac{\partial U(t,x,\varphi)}{\partial t} = \mathbf{A}U(t,\cdot,\varphi)(x) + q(x)\delta_S(x)\mathbf{B}_\nu U(t,\cdot,\varphi)(x),\quad t>0,\ x\in\mathbb{R}^d$$

and the initial condition

$$\lim_{t\to0+}U(t,x,\varphi)=\varphi(x),\quad x\in\mathbb{R}^d.$$

In other words, the operator $\mathbf{A}+q(x)\delta_S\mathbf{B}_\nu$ generates the semigroup $(T_t)_{t>0}$.

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