Theory of Stochastic Processes
Vol. 19 (35), no. 2, 2014, pp. 52–63

Dedicated to the 50-th anniversary of the
Department of Theory of Stochastic Processes
Institute of Mathematics, National Academy of Sciences of Ukraine

ANDREY PILIPENKO AND MAKSYM TANTSIURA

ON THE STRONG EXISTENCE AND UNIQUENESS TO A SOLUTION OF A COUNTABLE SYSTEM OF SDES WITH MEASURABLE DRIFT

We consider a countable system of stochastic differential equations that describes a motion of an interacting particles in a random environment. A theorem on existence and uniqueness of a strong solution is proved if the drift term is a bounded measurable function that satisfies finite radius interaction condition.

Introduction

The aim of this paper is to prove existence and uniqueness for strong solution of a countable system of stochastic differential equations (SDEs) describing a motion of infinite system of interacting particles. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ be a filtered probability space, $\{w_k(t), t \geq 0\}_{k\in\mathbb{Z}}$ be independent $\mathcal{F}_t$-adapted Wiener processes, $\{u_k | k \in \mathbb{Z}\}$ be a nondecreasing sequence such that $\lim_{k\to+\infty} u_k = +\infty,\ \lim_{k\to-\infty} u_k = -\infty$.

Denote by $\mathbf{M}$ the space of all locally finite measures on $\mathbb{R}$ with a vague topology τ defined by

$$\nu_n \xrightarrow{\tau} \nu \Leftrightarrow \forall f \in C_c(\mathbb{R}) : \int_{\mathbb{R}} f d\nu_n \to \int_{\mathbb{R}} f d\nu, n \to \infty,$$

where $C_c(\mathbb{R})$ is a set of all continuous functions with compact support.

Consider the following infinite system of SDEs

$$(1) \qquad \begin{cases} dX_k(t) = a(X_k(t), \mu_t)dt + dw_k(t),\ k \in \mathbb{Z}, t \in [0, T], \\ \mu_t = \sum_{k\in\mathbb{Z}} \delta_{X_k(t)}, \\ X_k(0) = u_k,\ k \in \mathbb{Z}, \end{cases}$$

where $a : [0, T] \times \mathbf{M} \to \mathbb{R}$ is a measurable function.

Here $X_k(t)$ may be interpreted as a coordinate at the instant t of the k-th particle that started from u_k. If we assume that each particle has a unit mass, then the measure μ_t may be considered as the distribution of mass at the instant t.

Note that assumptions of theorems on existence and uniqueness of the strong solutions of SDEs may be much weaker than the ones for ODEs. For example, if $b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded and measurable function, $\{W(t), t \geq 0\}$ is an $\mathbb{R}^d$-valued Wiener process, then SDE

$$dX(t) = b(t, X(t))dt + dW(t),\ t \geq 0,$$

2000 *Mathematics Subject Classification.* 60H10; 60J60.

Key words and phrases. stochastic differential equation; strong solution; pathwise uniqueness; interacting particle system.

This work was partially supported by the Presidium of National Academy of Sciences of Ukraine as part of the joint scientific project with the Russian foundation of fundamental research, project number 09-01-14.

has a unique strong solution for any $\mathcal{F}_0$-adapted $X(0)$. This result in one-dimensional case was obtained by Zvonkin [1] and then it was generalized by Veretennikov [2] for the multidimensional situation (even for non-additive noise).

System (1) can be considered as an SDE in multidimensional space. The analogue of Veretennikov's result in a Hilbert space case was proved in [3], however their assumptions are not satisfied for (1).

We will assume that a function a from (1) satisfies a finite interaction radius condition. This will allow us to apply Veretennikov's result if the initial distribution $\{u_k\}$ has a lot of "big gaps". For example (Theorem 1.3), this condition is satisfied if $\{u_k\}$ are atoms of a Poisson point measure with constant intensity. Reasoning similar to ours were applied to a construction of a system of coalescing diffusions with changing mass in [4]. See also paper [5], where strong solutions to a countable system of interacted SDEs was investigated.

Equation (1) corresponds to a system, when each particle has a unit mass. The case when the total mass of a particle system is finite and may be non-atomic was considered, for example, in [6, 7, 8].

The question on existence and uniqueness of weak solutions for countable system of SDEs with interaction was considered by different approaches such as Dirichlet forms, Gibbs measures, etc., see, for example, [9, 10, 11, 12].

1. Main results

Theorem 1.1. *Suppose that*

1. a is a bounded measurable function:

$$\|a\|_\infty := \sup_{x\in\mathbb{R}} \sup_{\nu\in\boldsymbol{M}} |a(x,\nu)| < \infty;$$

2. the finite interaction radius condition is satisfied:

$$\exists d>0\ \forall x\in\mathbb{R}\ \forall\nu\in\boldsymbol{M}\colon a(x,\nu) = a(x, 1\!\!1_{(x-d,x+d)}\nu),$$

where $(1\!\!1_B\nu)(A) = \nu(A\cap B),\ A,B\in\mathcal{B}(\mathbb{R})$;

3. there exists a (random) sequence $\{y_n | n\in\mathbb{Z}\}$ such that

$$\forall n\in\mathbb{Z}\quad \inf_{i:u_i\ge y_n}\inf_{t\in[0,T]}\big(u_i+(w_i(t)\wedge 0)\big) - \sup_{i:u_i<y_n}\sup_{t\in[0,T]}\big(u_i+(w_i(t)\vee 0)\big) \ge 2\|a\|_\infty T + d$$

almost surely.

Then there exists a unique strong solution of (1).

We postpone all proofs to the next sections.

Remark 1.1. Condition 2 means that if distance between two particles is greater than d, then they do not interact.

Remark 1.2. Condition 3 yields that system of particles can be divided into a countable number of finite subsystems so that distance between any two subsystems is greater than d for every $t\in[0,T]$. Hence, these subsystems do not interact.

Condition 3 in Theorem 1.1 is the hardest one to check. The following theorems give sufficient conditions ensuring condition 3.

Denote

$$p_w(t,x) = P(\sup_{s\in[0,t]} w(s)\ge x) = 2\int_{x\vee 0}^{\infty}\frac{1}{\sqrt{2\pi t}}\exp(-y^2/2t)dy,\ x\in\mathbb{R}, \tag{2}$$

where w is a Wiener process in $\mathbb{R}$.

Theorem 1.2. *Suppose that there exists a deterministic increasing sequence $\{z_n | n \in \mathbb{Z}\}$ such that:*

1. $\lim_{n\to\infty} z_n = +\infty,\ \lim_{n\to-\infty} z_n = -\infty.$

2. $\exists\tilde{\varepsilon} > 0\ \forall n \in \mathbb{Z}:\ \prod_{i\in\mathbb{Z}}\big(1 - p_w(T, |z_n - u_i| - \|a\|_\infty T - d/2)\big) > \tilde{\varepsilon}$

Then condition 3 of Theorem 1.1 is satisfied.

Remark 1.3. If the sequence $\{x_k(t)\}$ is indexed by $k \in \mathbb{N}$ instead of $k \in \mathbb{Z}$ and $\{u_k, k \geq 1\}$ is a nondecreasing sequence such that

$$\lim_{k\to+\infty} u_k = +\infty,$$

then Theorems 1.1 and 1.2 are also true.

Theorem 1.3. *Let $\mu_0 = \mu$, where $\mu = \sum_j \delta_{u_j}$ is a Poisson point measure with intensity measure m. Suppose that μ is independent of $\{w_k,\ k \in \mathbb{Z}\}$ and*

$$\exists C_m\ \forall[a, b] \subset \mathbb{R}:\ m([a, b]) \leq C_m(b - a + 1).$$

If conditions 1 and 2 of Theorem 1.1 are satisfied, then there exist a unique strong solution to equation (1) for every $T > 0$.

For a locally finite measure ν denote

$$\Lambda(\nu) := \limsup_{n\to\infty} \frac{\nu([-n, n])}{2n}.$$

The value $\Lambda(\nu)$ is an upper bound for the “average density” of the measure ν.

For any $\lambda > 0$ denote

$$M_\lambda = \{\nu |\ \Lambda(\nu) \leq \lambda\}.$$

Theorem 1.4. *Suppose that $\mu_0 = \sum_{k\in\mathbb{Z}} \delta_{u_k} \in M_\lambda$ with $\lambda d < 1$ and conditions 1 and 2 of Theorem 1.1 are satisfied. Then there exists a unique strong solution of the equation (1) for any $T > 0$.*

2. Proof of Theorem 1.1

Veretennikov's theorem [2] yields that if a function $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and measurable, then stochastic differential equation

$$\begin{cases} dY(t) = b(t, Y(t))dt + dW(t), t \in [0, T], \\ Y(0) = Y_0 \end{cases} \tag{3}$$

has a unique strong solution. Here $W(t), t \in [0, T]$, is a Wiener process in $\mathbb{R}^d$.

It follows from condition 3 that we can represent the set $\{u_k\}$ as a union of disjoint subsets

$$\{u_k\} = \cup_n \left\{u_{k_n}, u_{k_n+1}, ..., u_{k_{n+1}-1}\right\} \tag{4}$$

such that for any $n \neq m$, and for any $k_n \leq i < k_{n+1},\ k_m \leq j < k_{m+1}$ processes $X_i(t)$ and $X_j(t)$ do not interact (provided that a solution of (1) exists). Note that representation (4) is random and anticipating. Hence Theorem 1.1 does not follow directly from Veretennikov's theorem and we need to make some additional justification.

For every $n \in \mathbb{N}$ consider a system of equations

$$\begin{cases} dX_i^n(t) = a(X_i^n(t), \mu_t)dt + dw_i(t),\ -n \leq i \leq n,\ t \in [0, T], \\ dX_i^n(t) = 0,\ |i| > n, \\ \mu_t^n = \sum_{k\in\mathbb{Z}} \delta_{X_k^n(t)}, \\ X_i^n(0) = u_i,\ i \in \mathbb{Z}. \end{cases} \tag{5}$$

It is easy to see that system (5) is equivalent to a finite system of stochastic differential equations. So Veretennikov's theorem yields that there exists a unique solution of the equation (5).

Denote

$$\tau_{n_1,n_2} = \inf\Big\{t \in [0,T] \mid \exists k_1 \in (n_1,n_2]\ \exists k_2 \notin (n_1,n_2]$$
$$|u_{k_1} + w_{k_1}(t) - u_{k_2} - w_{k_2}(t)| \vee |u_{k_1} - u_{k_2} - w_{k_2}(t)| \vee$$
$$\vee |u_{k_1} + w_{k_1}(t) - u_{k_2}| \vee |u_{k_1} - u_{k_2}| \le 2\|a\|_\infty T + d\Big\} \wedge T.$$

Let us prove that

$$\forall t \in [0,\tau_{n_1,n_2}]\ \forall n \ge |n_1| \vee |n_2| :\ X_i^{|n_1|\vee|n_2|}(t) = X_i^n(t) \quad \text{a.s.} \tag{6}$$

It follows from the definition of τ_{n_1,n_2} that for any $n \ge |n_1| \vee |n_2|,\ k \in (n_1,n_2]$

$$\forall j \notin (n_1,n_2]\ \forall s \in [0,\tau_{n_1,n_2}] : |X_j^n(s) - X_k^n(s)| \ge d.$$

Hence,

$$\mu_s^n = \mathbb{1}_{[X_k^n(s)-d,X_k^n(s)+d]}\mu_s^n = \sum_{n_1<j\le n_2} \mathbb{1}_{[X_k^n(s)-d,X_k^n(s)+d]}\delta_{X_j^n(s)},\ \ s \in [0,\tau_{n_1,n_2}].$$

So, condition 2 of the Theorem implies that for $n \ge |n_1| \vee |n_2|,\ t \le \tau_{n_1,n_2},\ k \in (n_1,n_2]$

$$X_k^n(t) = u_k + w_k(t) + \int_0^t a(X_k^n(s),\mu_s^n)ds =$$

$$u_k + w_k(t) + \int_0^t a(X_k^n(s), \mathbb{1}_{[X_k^n(s)-d,X_k^n(s)+d]}\mu_s^n)ds. \tag{7}$$

Hence,

$$X_k^n(t) = u_k + w_k(t) + \int_0^t a(X_k^n(s), \sum_{n_1<j\le n_2} \delta_{X_j^n(s)})ds,\ t \in [0,\tau_{n_1,n_2}].$$

Therefore $X_k^n(t),\ t \in [0,\tau_{n_1,n_2}],\ k \in (n_1,n_2]$ and $X_k^{|n_1|\vee|n_2|}(t),\ t \in [0,\tau_{n_1,n_2}],\ k \in (n_1,n_2]$, are solutions of the same system of stochastic differential equations. Thus by Veretennikov's theorem we have

$$X_k^{|n_1|\vee|n_2|} = X_k^n(t), t \in [0,\tau_{n_1,n_2}] \ \text{ a.s.} \tag{8}$$

Remark 2.1. It can be checked that Veretennikov's theorem is also true for solutions defined up to a stopping time.

Condition 3 yields that with probability 1

$$\forall k \in \mathbb{Z}\ \ \exists n_1 < k\ \ \exists n_2 \ge k : \tau_{n_1,n_2} = T. \tag{9}$$

It follows from (8) and (9) that with probability 1 there exists n_0 such that for any fixed $k \in \mathbb{Z}$ solutions $\{X_k^n(t),\ t \in [0,T]\}$ coincide for $n \ge n_0$.

Denote

$$X_k(t) = \lim_{n\to\infty} X_k^n(t). \tag{10}$$

So,

$$X_k(t) = u_k + w_k(t) + \int_0^t a(X_k(s), \sum_{n_1<j\le n_2} \delta_{X_j(s)})ds, k \in (n_1,n_2], t \in [0,\tau_{n_1,n_2}]. \tag{11}$$

Now, using (9), we obtain that $X_k(t), k \in \mathbb{Z}, t \in [0,T]$ is a strong solution of (1).

Suppose $\{Y_k(\cdot),\ k \in \mathbb{Z}\}$ is another solution of (1). Then by the same arguments as above, for any $n_1 < n_2$ we have a.s. the equality

$$X_k(s) = Y_k(s), k \in (n_1,n_2],\ s \in [0,\tau_{n_1,n_2}].$$

As above, using (9), we get that solutions $\{X_k(\cdot),\ k \in \mathbb{Z}\}$ and $\{Y_k(\cdot),\ k \in \mathbb{Z}\}$ coincide. The Theorem is proved.

3. Proof of Theorem 1.2

At first let us prove that

$$\forall k \geq 1 \ \ P(\sup_{i:u_i<z_k} \sup_{t\in[0,T]} (u_i + w_i(t)) < \infty) = 1. \tag{12}$$

Let $k \in \mathbb{N}$ be arbitrary. It follows from condition 2 of the Theorem that

$$\sum_{i:u_i<z_k} p_w(T, |z_k - u_i| - \|a\|_\infty T - d/2) < +\infty, \tag{13}$$

so

$$\sum_{i:u_i<z_k} P\big(\sup_{t\in[0,T]} (u_i + w_i(t)) \geq z_k - \|a\|_\infty T - d/2\big) < +\infty. \tag{14}$$

It follows from the Borel-Cantelli lemma that with probability 1 there is only a finite number of integers i such that $u_i < z_k$ and $\sup_{t\in[0,T]}(u_i + w_i(t)) \geq z_k - \|a\|_\infty T - d/2$. Therefore, (12) is satisfied.

Analogously,

$$\forall k \geq 1 \ \ P\big(\inf_{i:u_i>z_k} \inf_{t\in[0,T]} (u_i + w_i(t)) > -\infty\big) = 1.$$

Denote

$$\xi_k = \max_{t\in[0,T]} (u_k + w_k(t)), \tag{15}$$

$$\eta_k = \min_{t\in[0,T]} (u_k + w_k(t)), \tag{16}$$

and

$$A_k = \{\sup_{i:u_i<z_k} \xi_i \leq z_k - d/2 - \|a\|_\infty T, \ \inf_{i:u_i>z_k} \eta_i \geq z_k + d/2 + \|a\|_\infty T\}. \tag{17}$$

To prove the Theorem it is sufficient to verify that

$$P(\limsup_{k\to\infty} A_k) = 1 \tag{18}$$

and

$$P(\limsup_{k\to-\infty} A_k) = 1. \tag{19}$$

We prove only (18). Formula (19) can be proved similarly.

Events A_k are dependent, so the second Borel-Cantelli lemma can't be directly applied. The idea of the proof is to approximate events from some subsequence $\{A_{k_n}\}$ by a sequence $\{A'_{k_n}\}$, where A'_{k_n} are defined in a similar way as A_{k_n} but supremum and infimum are taken over a finite set of indices. If for different n these sets of indices have empty intersection, then events A'_{k_n} will be independent.

Let us make the formal reasoning. Analogously to proof of (12) it can be proved that for every integer k the set $\{i \in \mathbb{Z} | u_i > z_k, \ \eta_i \leq z_k + \|a\|_\infty T + d/2\}$ is finite a.s. Hence,

$$\lim_{n\to+\infty} \xi_n = \lim_{n\to+\infty} \eta_n = +\infty \ \text{ a.s.}$$

So, for any $n \in \mathbb{Z}$ and $\varepsilon > 0$ there exists $l(n,\varepsilon)$ such that

$$\forall l \geq l(n,\varepsilon): \ P(\sup_{n<k\leq n+l} \xi_k \neq \sup_{k\leq n+l} \xi_k) < \varepsilon, \ P(\inf_{k>n} \eta_k \neq \inf_{n<k\leq n+l} \eta_k) < \varepsilon. \tag{20}$$

For any increasing sequence $\{m_k | k \in \mathbb{N}\} \subset \mathbb{Z}$ denote

$$i(k) = i_m(k) = \max\{i | u_i \leq z_{m_k}\}.$$

Let us select a sequence $\{m_k\}$ such that

$$\forall k \geq 1 \ \ i(k+1) - i(k) \geq \max\{l(i(k), 1/2^k), \ 2\}. \tag{21}$$

Then

$$P\left(\forall M\ \exists k>M:\ \sup_{i\le i(k)}\xi_i\le z_{m_k}-\frac{d}{2}-\|a\|_\infty T,\ \inf_{i>i(k)}\eta_i\ge z_{m_k}+\frac{d}{2}+\|a\|_\infty T\right)\ge$$

$$P\left(\forall M\ \exists k>M:\ \sup_{i\le i(k)}\xi_i=\sup_{i(k-1)<i\le i(k)}\xi_i,\ \inf_{i>i(k)}\eta_i=\inf_{i(k+1)\ge i>i(k)}\eta_i,\right.$$

$$\left.\sup_{i(k-1)<i\le i(k)}\xi_i\le z_{m_k}-d/2-\|a\|_\infty T,\ \inf_{i(k+1)\ge i>i(k)}\eta_i\ge z_{m_k}+d/2+\|a\|_\infty T\right)\ge$$

$$\ge P(B_1\cap B_2), \tag{22}$$

where

$$B_1=\left\{\exists M\ \forall k>M:\ \sup_{i\le i(k)}\xi_i=\sup_{i(k-1)<i\le i(k)}\xi_i,\ \inf_{i>i(k)}\eta_i=\inf_{i(k+1)\ge i>i(k)}\eta_i\right\}, \tag{23}$$

$$B_2=\left\{\forall M\ \exists k>M:\ \sup_{i(k-1)<i\le i(k)}\xi_i\le z_{m_k}-d/2-\|a\|_\infty T,\right.$$

$$\left.\inf_{i(k+1)\ge i>i(k)}\eta_i\ge z_{m_k}+d/2+\|a\|_\infty T\right\}. \tag{24}$$

Event B_1 means that all but a finite number of the events

$$\{\sup_{i\le i(k)}\xi_i=\sup_{i(k-1)<i\le i(k)}\xi_i,\ \inf_{i>i(k)}\eta_i=\inf_{i(k+1)\ge i>i(k)}\eta_i\}$$

occur. Event B_2 means that events

$$\{\sup_{i(k-1)<i\le i(k)}\xi_i\le z_{m_k}-d/2-\|a\|_\infty T,\ \inf_{i(k+1)\ge i>i(k)}\eta_i\ge z_{m_k}+d/2+\|a\|_\infty T\}$$

occur infinitely often.

From (20) and (21) we obtain that

$$\sum_{k\ge1}P\left(\{\sup_{i\le i(k)}\xi_i\ne\sup_{i(k-1)<i\le i(k)}\xi_i\}\cup\{\inf_{i>i(k)}\eta_i\ne\inf_{i(k+1)\ge i>i(k)}\eta_i\}\right)\le$$

$$\sum_{k\ge1}(1/2^k+1/2^k)<+\infty.$$

It follows from the Borel-Cantelli lemma that $P(B_1)=1$.

Consider now event B_2. Denote

$$C_k=\{\sup_{i(2k-1)<i\le i(2k)}\xi_i\le z_{m_{2k}}-d/2-\|a\|_\infty T,$$

$$\inf_{i(2k+1)\ge i>i(2k)}\eta_i\ge z_{m_{2k}}+d/2+\|a\|_\infty T\}$$

Events $\{C_k,k\ge1\}$ are mutually independent. If we prove that

$$\sum_{k\ge1}P(C_k)=+\infty, \tag{25}$$

then the Borel-Cantelli lemma will imply $P(B_2)=1$.

Let us estimate probability $P(C_k)$ from below:

$$P(C_k)=P\left(\bigcap_{i=i(2k-1)+1}^{i(2k)}\{\max_{t\in[0,T]}(u_i+w_i(t))\le z_{m_k}-d/2-T\|a\|_\infty\}\cap\right.$$

$$\left.\bigcap_{i=i(2k)+1}^{i(2k+1)}\{\min_{t\in[0,T]}(u_i+w_i(t))\geq z_{m_k}+d/2+T\|a\|_\infty\}\right)=$$

$$\prod_{i=i(2k-1)+1}^{i(2k+1)}\big(1-p_w(T,|z_{m_k}-u_i|-d/2-T\|a\|_\infty)\big)\geq$$

$$\prod_{i\in\mathbb{Z}}\big(1-p_w(T,|z_{m_k}-u_i|-d/2-T\|a\|_\infty)\big)>\tilde{\varepsilon},$$

where $\tilde{\varepsilon}>0$ is from condition 2 of the Theorem. Hence (25) is satisfied and consequently $P(B_2)=1$.

Therefore, $P(B_1\cap B_2)=1$, and (22) yields that events A_k occur infinitely often a.s. when $k\to+\infty$. The case $k\to-\infty$ can be considered similarly. This completes the proof of the Theorem.

4. Proof of Theorem 1.3

If intensity of a Poisson point measure is a finite measure, then μ has a finite number of atoms a.s. In this case existence and uniqueness of a strong solution of (1) follows from Veretennikov's theorem. Further we consider the case when $m([0,+\infty))=+\infty$ and $m((-\infty,0])=+\infty$. The proof for the case when $m([0,+\infty))<\infty$ or $m((-\infty,0])<\infty$ is similar.

Let $\mu=\sum_{k\in\mathbb{Z}}\delta_{u_k}$. Without loss of generality we will assume that $\{u_k,k\in\mathbb{Z}\}$ is a non-decreasing sequence. Since $\{u_k,k\in\mathbb{Z}\}$ and $\{w_i,i\in\mathbb{Z}\}$ are independent, it suffices to construct a $\sigma(u_k,k\in\mathbb{Z})$-measurable sequence $\{z_n|n\in\mathbb{Z}\}$ that satisfies conditions of Theorem 1.2 a.s.

Denote

$$A=d/2+\|a\|_\infty T,$$

$$D_1(x,r)=\prod_{|u_i-x|>r}(1-p_w(T,|x-u_i|-A)),$$

$$D_2(x,r)=\prod_{|u_i-x|\leq r}(1-p_w(T,|x-u_i|-A)),$$

$$D(x)=\prod_{i\in\mathbb{Z}}(1-p_w(T,|x-u_i|-A)).$$

Lemma 4.1. *Let μ be a Poisson point measure that satisfies conditions of Theorem 1.3. Then for all $B>0,\ x\in\mathbb{R}:$*

$$\sum_{i\in\mathbb{Z}}p_w(T,|x-u_i|-B)<+\infty\quad a.s.$$

Proof of Lemma 4.1. Denote $f_T(y)=m([0,y))1\!\!1_{y\geq0}-m([y,0))1\!\!1_{y<0}$. Without loss of generality we can assume that

$$\mu([y_1,y_2))=\Pi([f_T(y_1),f_T(y_2))),\ [y_1,y_2)\subset\mathbb{R},$$

where Π is a Poisson point measure with intensity 1. The application of the strong law of large numbers yields that

$$P(\limsup_{n\to\infty}\mu([x-n,x+n])/2n\leq C_m)=1.$$

Hence,

$$P(\exists Q>0\ \forall n\in\mathbb{N}\ \ \mu([x-n,x+n])\leq 2Qn)=1.$$

Then with probability 1

$$\sum_{i\in\mathbb{Z}} p_w(T,|x-u_i|-B) = \sum_{k\in\mathbb{N}} \sum_{i:|x-u_i|\in[k-1,k)} p_w(T,|x-u_i|-B) \leq$$

$$\sum_{k\in\mathbb{N}} |\{i:|x-u_i|\in[k-1,k)\}| p_w(T,k-1-B) \leq \sum_{k\in\mathbb{N}} 2Qk p_w(T,k-1-B) < +\infty,$$

because

$$p_w(t,y) \sim \frac{\sqrt{2t}}{\sqrt{\pi} y} \exp\left(-y^2/2t\right),\ y\to+\infty. \tag{26}$$

The Lemma is proved.

Let $x\in\mathbb{Z}$ be fixed.

Denote

$$I_x(k) = \begin{cases} [x+k, x+k+1),\ k\geq 1, \\ (x-1,x+1),\ k=0, \\ (x-k-1,x-k],\ k\leq -1. \end{cases} \tag{27}$$

Let $\{\xi_k | k\in\mathbb{Z}\}$ be independent Poisson random variables, $\xi_k \sim Pois(2C_m - m(I_x(k)))$, where m is intensity of μ. Assume also that $\{\xi_k | k\in\mathbb{Z}\}$ are independent of μ. Set $\mu_x(\mathbb{R}\backslash\mathbb{Z}) = 0$ and $\mu_x(\{k\}) = \mu(I_x(k)) + \xi_k, k\in\mathbb{Z}$.

Observe that μ_x is a Poisson point measure with intensity $2C_m\left(\delta_x + \sum_{k\in\mathbb{Z}}\delta_k\right)$ and

$$\forall r>0\ \ \mu([x-r,x+r]) \leq \mu_x([x-r,x+r]).$$

Let $\mu_x = \sum_{k\in\mathbb{Z}} \delta_{v_k(x)}$, where $\{v_k(x), k\in\mathbb{Z}\}$ is a non-decreasing sequence. Denote

$$D_1^v(x,r) = \prod_{|v_i(x)-x|>r} (1-p_w(T,|x-v_i(x)|-A)),$$

$$D_2^v(x,r) = \prod_{|v_i(x)-x|\leq r} (1-p_w(T,|x-v_i(x)|-A)),$$

$$D^v(x) = \prod_{i\in\mathbb{Z}} (1-p_w(T,|x-v_i(x)|-A)).$$

The distributions of $D_1^v(x,r),\ D_2^v(x,r)$ and $D^v(x)$ are independent of $x\in\mathbb{Z}$. Moreover,

$$\forall x\in\mathbb{Z}\ \forall r>0:\ D_1(x,r)\geq D_1^v(x,r),\ D_2(x,r)\geq D_2^v(x,r),\ D(x)\geq D^v(x). \tag{28}$$

It follows from Lemma 4.1 that $D_1^v(x,r)\to 1,\ r\to+\infty$ with probability 1. Hence, there exists an increasing sequence $\{r_i, i\geq 1\}$ (independent of x) such that

$$\forall i\in\mathbb{N}\ \forall x\in\mathbb{Z}:\ P(D_1^v(x,r_i)<1/2)<1/2^i. \tag{29}$$

Combining (28) and (29) we obtain

$$\forall i\in\mathbb{N}\ \forall x\in\mathbb{Z}:\ P(D_1(x,r_i)<1/2)<1/2^i.$$

Let us construct a sequence $\{x_i | i\geq 1\}\subset\mathbb{Z}$ in a following way:

$$x_1=0,\ x_{k+1}=x_k+r_k+r_{k+1}+1,\ k\geq 1.$$

Then it follows from the Borel-Cantelli lemma that

$$P(\exists k_0\ \forall k>k_0:\ D_1(x_k,r_k)>1/2)=1. \tag{30}$$

Choose $\tilde{A} > A = \|a\|_\infty T + d/2$ such that $p_w(T,\tilde{A}-A)<1/2$. Without loss of generality we can assume that $\{r_k, k\geq 1\}$ were chosen so that $\tilde{A}<r_k$ for all $k\in\mathbb{N}$. Let us estimate

$$P\left(D_2(x_k,r_k)>1/2\right)\geq$$

$$P\left(\mu([x_k-\tilde{A},x_k+\tilde{A}])=0, D_2(x_k,r_k)>1/2\right)\geq$$

$$P\left(\mu([x_k-\tilde{A},x_k+\tilde{A}])=0\right)P\left(\prod_{\tilde{A}<|v_i(x)-x|<r_k}(1-p_w(T,|x-v_i(x)|-A))>1/2\right)$$

$$\geq \exp(-C_m(2\tilde{A}+1))P(D_1^v(x,\tilde{A})>1/2). \tag{31}$$

Events $\{D_2(x_k,r_k)>1/2\}$ are independent and the estimate of probability (31) is independent of k. Hence, by the second Borell-Cantelli lemma

$$P(D_2(x_k,r_k)>1/2 \ \text{ i.o.})=1.$$

Combining this with (30) we obtain

$$P(D(x_k)>1/4 \ \text{ i.o.})=P(D_1(x_k,r_k)D_2(x_k,r_k)>1/4 \ \text{ i.o.})=1.$$

It follows from the construction of $\{x_k\}$ that there exists $\sigma\{u_k,\ k\in\mathbb{Z}\}$-measurable subsequence $z_k=x_{n_k}$ that a.s. satisfies conditions of Theorem 1.2 for $k\geq 0$ and $\tilde{\varepsilon}=1/4$. Negative indices can be considered analogously. Theorem 1.3 is proved.

5. Proof of Theorem 1.4

Let us verify conditions of Theorem 1.2 for some $T=T_0$. Denote $A_T=\|a\|_\infty T+d/2$, $f_T(x)=p_w(T,|x|-A_T)\wedge 1/2$.

It follows from (26) that

$$I(T):=\int_{\mathbb{R}}f_T(x)dx<+\infty,\ I_n(T):=\int_{|x|>n}f_T(x)dx<+\infty.$$

Denote

$$S_T(x)=\sum_{i\in\mathbb{Z}}f_T(x-u_i).$$

Then

$$\int_{-n}^{n}S_T(x)dx=\sum_{i\in\mathbb{Z}}\int_{-n}^{n}f_T(x-u_i)dx\leq\sum_{i:u_i\in[-n,n]}\int_{\mathbb{R}}f_T(x-u_i)dx+$$

$$\sum_{k\in\mathbb{N}}\sum_{i:|u_i|\in(k+n-1,k+n]}\int_{|x-u_i|\geq k}f_T(x-u_i)dx\leq \tag{32}$$

$$|\{i:u_i\in[-n,n]\}|I(T)+\sum_{k\in\mathbb{N}}|\{i:|u_i|\leq k+n\}|I_k(T).$$

Choose $\varepsilon>0$ such that $\lambda(1+\varepsilon)<1/d$. Since $\mu(0)\in M_\lambda$, there exists $n_0=n_0(\varepsilon)$ such that

$$|\{i:u_i\in[-n,n]\}|\leq 2n\lambda(1+\varepsilon),\ |\{i:|u_i|\leq k+n\}|\leq 2(k+n)\lambda(1+\varepsilon)$$

for all $n\geq n_0(\varepsilon)$.

Hence

$$\int_{-n}^{n}S_T(x)dx\leq 2n\lambda(1+\varepsilon)I(T)+\sum_{k\in\mathbb{N}}2(k+n)\lambda(1+\varepsilon)I_k(T)<+\infty. \tag{33}$$

The finiteness of (33) follows from the definitions of $I(T)$, $I_k(T)$, and relation (26).

Denote

$$f(x)=\lim_{T\to 0+}f_T(x)=\frac{1}{2}\mathbb{1}_{|x|\leq d/2}. \tag{34}$$

Notice that $I(T)\to d$ and $I_k(T)\to(d-2k)\mathbb{1}_{k<d/2}$ as $T\to 0+$.

Denote $S(x)=\sum_{i\in\mathbb{Z}}f(x-u_i)$.

Combining (33), (34), and using Lebesgue dominated convergence theorem, we obtain

$$\int_{-n}^{n} S(x)dx = \lim_{T\to 0} \int_{-n}^{n} S_T(x)dx.$$

Let us prove that

$$\lim_{T\to 0} \limsup_{n\to\infty} \frac{\int_{-n}^{n} (S_T(x) - S(x))dx}{2n} = 0. \tag{35}$$

Denote

$$K(T) = \int_{\mathbb{R}} (f_T(x) - f(x))dx,\ \ K_n(T) = \int_{|x|\in[n,n+1)} (f_T(x) - f(x))dx.$$

Analogously to (32) we have

$$\begin{aligned}
&\frac{1}{2n}\int_{-n}^{n} (S_T(x) - S(x))dx \le \frac{1}{2n}|\{i : u_i \in [-n, n]\}|K(T)+\\
&\frac{1}{2n}\sum_{k\in\mathbb{N}}\ \sum_{i:u_i\in(k+n-1,k+n]}\ \sum_{j\ge k} K_j(T) \le \frac{1}{2n}\lambda(1+\varepsilon)nK(T)+\\
&\frac{1}{2n}\sum_{j\in\mathbb{N}} K_j(T) \sum_{0\le\ k\le\ j} |\{i : |u_i| \in (k+n-1, k+n]\}| =\\
&\frac{1}{2n}\lambda(1+\varepsilon)nK(T) + \frac{1}{2n}\sum_{j\in\mathbb{N}} K_j(T)|\{i : |u_i| \in (n-1, j+n]\}| \le\\
&\frac{1}{2n}\lambda(1+\varepsilon)nK(T) + \frac{1}{2n}\sum_{j\in\mathbb{N}} K_j(T)|\{i : |u_i| \le j+n\}| \le\\
&\le \frac{1}{2n}\lambda(1+\varepsilon)nK(T) + \frac{1}{2n}\sum_{j\in\mathbb{N}} 2\lambda(1+\varepsilon)(j+n)K_j(T),\ \ n \ge n_0(\varepsilon).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{2n}\int_{-n}^{n} (S_T(x) - S(x))dx \le \lambda(1+\varepsilon)/2 \left(K(T) + \sum_{j\in\mathbb{N}} K_j(T)\right) +\\
&\frac{\lambda(1+\varepsilon)}{n}\sum_{j\in\mathbb{N}} jK_j(T) \le \lambda(1+\varepsilon)K(T) + \frac{\lambda(1+\varepsilon)}{n}\sum_{j\in\mathbb{N}} jK_j(T),\ \ n \ge n_0(\varepsilon).
\end{aligned}$$

It follows from Lebesgue dominated convergence theorem that $K(T) \to 0$ as $T \to 0$. Using the definition of $K_j(T)$ it is easy to see that $\sup_{T\in(0,1]}\sum_{j\in\mathbb{N}} jK_j(T) < +\infty$. So, (35) is proved.

Observe that

$$\lim_{n\to\infty} \frac{\int_{-n}^{n} S(x)dx}{2n} = \lambda d/2 < 1/2. \tag{36}$$

Denote $H(x) = \sup\{T > 0 \mid K(T) < x\}$. It follows from (35) and (36) that

$$\forall T_0 < H\left(\frac{1-\lambda d}{2\lambda(1+\varepsilon)}\right) : \ \ \limsup_{n\to\infty} \frac{\int_{-n}^{n} S_{T_0}(x)dx}{2n} < 1/2. \tag{37}$$

Set

$$T_0 = \frac{1}{2}H\left(\frac{1-\lambda d}{2\lambda(1+\varepsilon)}\right).$$

Then

$$\forall n_0\ \exists x, |x| > n_0 :\ S_{T_0}(x) < 1/2. \tag{38}$$

It follows from the definition of $S_T(x)$ that if $S_T(x) < 1/2$, then

$$\forall i \geq 1 \; p_w(T, |x-u_i| - A_T) < 1/2.$$

Note that

$$\ln(1-y) \geq -2y, \;\; y \in [0, 1/2].$$

Hence if $S_{T_0}(x) < 1/2$, then

$$\sum_{i\in\mathbb{Z}} \ln(1 - p_w(T_0, |u_i - x| - A_{T_0})) \geq -\sum_{i\in\mathbb{Z}} 2p_w(T_0, |u_i - x| - A_{T_0}) = -2S_{T_0}(x) \geq -1.$$

It follows from (38) that we can construct a sequence $\{z_n, n \in \mathbb{Z}\}$ such that

$$z_0 \in \{z \in \mathbb{R} | S_{T_0}(z) < 1/2\},$$
$$z_{n+1} \in \{z \geq z_n + 1 | S_{T_0}(z) < 1/2\}, \;\; n \geq 1,$$
$$z_{n-1} \in \{z \leq z_n - 1 | S_{T_0}(z) < 1/2\}, \;\; n < 0.$$

Then

$$\prod_{i\in\mathbb{Z}} \big(1 - p_w(T, |z_n - u_i| - \|a\|_\infty T - d/2)\big) \geq e^{-1}.$$

Theorem 1.2 yields that there exists a unique strong solution to (1) for $t \in [0, T_0]$.

Lemma 5.1. *Let $\{X_k(t) | t \in [0,T], k \in \mathbb{Z}\}$ be a solution of (1), $\Lambda(\mu_0) < +\infty$. Then*

$$P(\forall t \in [0,T]: \;\; \Lambda(\mu_0) = \Lambda(\mu_t)) = 1.$$

Proof of Lemma 5.1. Since $\Lambda(\mu_0) < +\infty$ we have

$$\liminf_{|k|\to+\infty} \frac{u_k}{k} = \liminf_{|k|\to+\infty} \frac{X_k(0)}{k} \geq \frac{2}{\Lambda(\mu_0)}.$$

Hence

$$\forall \delta > 0 \quad \sum_{k\in\mathbb{Z}} P\left(\sup_{t\in[0,T]} |w_k(t)| > \delta |X_k(0)|\right) < +\infty.$$

It follows from the Borel-Cantelli lemma that a.s.

$$\sup_{t\in[0,T]} |w_k(t)| \leq \delta |X_k(0)|$$

for all k except of maybe a finite number.

Hence

$$\sup_{t\in[0,T]} |X_k(t) - X_k(0)| \leq \|a\|_\infty T + \sup_{t\in[0,T]} |w_k(t)| \leq \|a\|_\infty T + \delta |X_k(0)|.$$

Therefore there a.s. exists n_0 such that for any $n \geq n_0$

$$\left\{k : |X_k(0)| \leq \frac{n}{1+2\delta}\right\} \subset \{k : |X_k(t)| \leq n\} \subset \{k : |X_k(0)| \leq n(1+2\delta)\}.$$

So,

$$\limsup_{n\to\infty} \frac{\mu_t([-n,n])}{2n} \leq \limsup_{n\to\infty} \frac{\mu_0([-n(1+2\delta), n(1+2\delta)])}{2n} = \Lambda(\mu_0)(1+2\delta).$$

Analogously

$$\limsup_{n\to\infty} \frac{\mu_t([-n,n])}{2n} \geq \frac{\Lambda(\mu_0)}{1+2\delta}.$$

Since $\delta > 0$ is arbitrary, we obtain $\Lambda(\mu_t) = \Lambda(\mu_0)$.

The Lemma is proved.

It follows from Lemma 5.1 that $\mu_{T_0} \in M_\lambda$. Hence the solution of the equation (1) can be extended in a unique way to the interval $t \in [T_0, 2T_0]$, then it can be extended to the interval $[2T_0, 3T_0]$, and so on. Theorem 1.4 is proved.

Thanks. This paper was partially prepared during a visit of authors to Potsdam University. Authors are grateful to Sylvie Roelly for the invitation, hospitality and helpful discussions.

References

1. A. K. Zvonkin, *A transformation of the phase space of a diffusion process that removes the drift*, *Mat. Sb. (N.S.)*, **93(135)** (1974), 129–149.
2. A. Yu. Veretennikov, *On strong solutions and explicit formulas for solutions of stochastic integral equations*, Mat. Sb. **111(153)** (1980), no. 3, 434–452.
3. G. Da Prato, F. Flandoli, E. Priola, M. Rockner, *Strong uniqueness for stochastic evolution equations in Hilbert spaces with bounded and measurable drift*, arXiv: 1109.0363v1 (2011).
4. V. V. Konarovskii, *On infinite system of diffusing particles with coalescing*, Theory of Probability and Its Applications, **55(1)** (2011), 134–144.
5. T. Ichiba, I. Karatzas, M. Shkolnikov, *Strong solutions of stochastic equations with rank-based coefficients*, Probability Theory and Related Fields, **156(1-2)** (2013), 229–248.
6. A. S. Sznitman, *Topics in propagation of chaos*, In Ecole d'Ete de Probabilites de Saint-Flour XIX, 1989, Springer Berlin Heidelberg, (1991) pp. 165–251.
7. A. A. Dorogovtsev, *Measure-valued processes and stochastic flows*, Proceedings of Institute of Mathematics of NAS of Ukraine. Mathematics and its Applications, v. 66, Kiev 290 pp. ISBN: 978-966-02-4540-2, 2007.
8. A. Pilipenko, *Measure-valued diffusions and continual systems of interacting particles in random media*, Ukrainian mathematical journal, **9**, (2005), 1289–1301.
9. A. V. Skorokhod, *Stochastic equations for complex systems*, Springer, **13** (1988).
10. H. Georgii, *Gibbs measures and phase transitions*, de Gruyter Studies in Mathematics, **9** (1988).
11. H. Osada, *Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions*, Communications in mathematical physics, **176(1)** (1996), 117–131.
12. F. Redig, S. Roelly, W. Ruszel, *Short-time Gibbsianness for infinite-dimensional diffusions with space-time interaction*, Journal of Statistical Physics, **138(6)** (2010) 1124–1144.

Institute of Mathematics, National Academy of Sciences of Ukraine, Tereshchenkivska str. 3, 01601, Kiev, Ukraine

Ukraine, 01601, Kyiv, Tereschenkivska 3, Institute of Mathematics
E-mail address: mtan@meta.ua