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RADONIFYING OPERATORS AND INFINITELY DIVISIBLE WIENER INTEGRALS

In this article we illustrate the relation between the existence of Wiener integrals with respect to a Lévy process in a separable Banach space and radonifying operators. For this purpose, we introduce the class of ϑ-radonifying operators, i.e. operators which map a cylindrical measure ϑ to a genuine Radon measure. We study this class of operators for various examples of infinitely divisible cylindrical measures ϑ and highlight the differences from the Gaussian case.

1. Introduction

Starting with the work by Gel'fand [8], Gross [10] and Segal [23] the canonical Gaussian cylindrical measure has gained much attention in different areas of mathematics and applications. It is not only of interest from a theoretical point of view but it is also of importance in various applications such as filtering problems in Bensoussan [2], small ball probabilities in Li and Linde [12], interest rate models in Carmona and Tehranchi [6] and stochastic integration in Banach spaces in van Neerven, Veraar and Weis [26].

In his seminal work [10] Gross studies norms on a Hilbert space H such that the canonical Gaussian cylindrical measure γ extends to a σ-additive probability measure on the completion of H with respect to the norm. This directly leads to the class $\mathcal{R}(\gamma)$ of γ-radonifying operators, which consists of linear and bounded operators T from H to a Banach space V such that the cylindrical image measure $\gamma \circ T^{-1}$ extends to a σ-additive probability measure. The space $\mathcal{R}(\gamma)$ is known to have many desirable properties, such as the completeness under an appropriate norm, the ideal property and close relations to absolutely summing operators.

Recently, the space of γ-radonifying operators plays a fundamental role in the theory of stochastic integration in Banach spaces. In [26], van Neerven, Veraar and Weis develop a theory of stochastic integration for random operator-valued integrands with respect to cylindrical Wiener processes in UMD Banach spaces. Their approach is strongly based on the corresponding Wiener integrals for deterministic integrands introduced in [5] and [27], and those existence is naturally closely related to the class of γ-radonifying operators.

In our work [20], we extend the approach in [27] to Wiener integrals for deterministic integrands with respect to martingale-valued measures, in particular to Lévy processes. However, this work [20] was accomplished under the constraint not being able to use any of the fundamental properties of the space of γ-radonifying operators since the analogue

theory was not developed in a non-Gaussian setting. It became apparent, that if one would like to develop a theory of stochastic integration for random integrands similarly to the one in [26] but for Lévy processes, one needs to study an analog class of operators as γ-radonifying operators but radonifying an infinitely divisible cylindrical measure. This is the main motivation of this work where we show that one can introduce such a space of operators although it lacks many of the fundamental properties of γ-radonifying operators.

The canonical Gaussian cylindrical measure γ is distinguished among all Gaussian cylindrical measures by its characteristic function, which most often serves also as its definition. Equivalently, starting from a Gaussian cylindrical random variable X in an arbitrary Banach space V with covariance operator Q, one can explicitly construct a cylindrical random variable Θ in the reproducing kernel Hilbert space of Q whose cylindrical distribution equals the canonical Gaussian cylindrical distribution γ. This construction is based on the Karhunen-Loève expansion of X. We show in the first part of this work, that this construction of a canonically Gaussian distributed cylindrical random variable Θ on the reproducing kernel Hilbert space can be mimicked for each cylindrical random variable with second moments. Denoting the cylindrical distribution of Θ by ϑ, this construction motivates us to define the class $\mathcal{R}(\vartheta)$ of ϑ-radonifying operators in analogy to γ-radonifying operators as the space of operators T such that the image cylindrical measure $\vartheta \circ T^{-1}$ extends to a σ-additive probability measure on the Borel σ-algebra.

The class $\mathcal{R}(\vartheta)$ of ϑ-radonifying operators is only well studied if ϑ equals the canonical Gaussian cylindrical measure γ or a canonical stable cylindrical measure. In this work we show that for an arbitrary cylindrical measure ϑ the linear space $\mathcal{R}(\vartheta)$ can be equipped with a certain norm, introduced in this work, such that it becomes complete. However, already in the case of a canonical stable cylindrical measure ϑ, which might be considered as a non-Gaussian cylindrical measure most similar to the canonical Gaussian cylindrical measure γ, it is known that the space $\mathcal{R}(\vartheta)$ lacks many of the desirable properties of the space of γ-radonifying operators. We study the linear space $\mathcal{R}(\vartheta)$ for different examples of infinitely divisible cylindrical measures ϑ and compare it to the Gaussian situation.

In the last part of this work, we illustrate the relation of ϑ-radonifying operators and the existence of Wiener integrals with respect to a Lévy process. Although this is the underlying idea in the work [27] and to some extent in the generalisation [20], we are able to illustrate this relation more explicitly by defining a *cylindrical* integral, which in the case of stochastic integrability is induced by a genuine Banach space valued random variable. In particular for Lévy driven integrals, this rigorous relation between stochastic integrability and Banach space valued operators is novel, and it signficantly improves the description of integrable operators in [20].

2. Preliminaries

Throughout this paper, V is a separable Banach space with dual V^* and dual pairing $\langle \cdot, \cdot \rangle$. The Borel σ-algebra is denoted by $\mathcal{B}(V)$. If U is another separable Banach space the space of bounded and linear operators is denoted by $\mathcal{L}(U, V)$ equipped with the uniform operator norm $\|\cdot\|_{U \to V}$. An operator $T \in \mathcal{L}(U, V)$ is called *p-absolutely summing* if there exists a constant $c > 0$ such that for each $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in U$ it obeys

$$(2.1) \qquad \sum_{k=1}^{n} \|Tu_k\|^p \leqslant c^p \sup_{\|u^*\| \leqslant 1} \sum_{k=1}^{n} |\langle u_k, u^* \rangle|^p.$$

The space of all p-absolutely summing operators is denoted by $\Pi^p(U, V)$ and it is a Banach space under the norm $\|T\|_{\Pi^p} := \pi_p(T)$ where $\pi_p(T)$ is the smallest constant c satisfying (2.1).

For a measurable space $(S, \mathcal{S}, m)$ and $p \geqslant 1$ we shall denote the Lebesgue-Bochner space by $L_m^p(S; V)$. A probability space is denoted by $(\Omega, \mathcal{A}, P)$ and $L_P^0(\Omega; \mathbb{R})$ denotes the space of equivalence classes of measurable functions equipped with the topology of convergence in probability.

For every $v_1^*, \ldots, v_n^* \in V^*$ and $n \in \mathbb{N}$ we define a linear map

$$\pi_{v_1^*, \ldots, v_n^*} \colon V \to \mathbb{R}^n, \qquad \pi_{v_1^*, \ldots, v_n^*}(v) = \big(\langle v, v_1^* \rangle, \ldots, \langle v, v_n^* \rangle\big).$$

For $n \in \mathbb{N}$ and $B \in \mathcal{B}(\mathbb{R}^n)$, sets of the form

$$C(v_1^*, \ldots, v_n^*; B) := \{v \in V : (\langle v, v_1^* \rangle, \ldots, \langle v, v_n^* \rangle) \in B\} = \pi_{v_1^*, \ldots, v_n^*}^{-1}(B)$$

are called *cylindrical sets*. If D is a subset of V^* then

$$\mathcal{Z}(V, D) := \left\{ \pi_{v_1^*, \ldots, v_n^*}^{-1}(B) : v_1^*, \ldots, v_n^* \in D, \ B \in \mathcal{B}(\mathbb{R}^n), \ n \in \mathbb{N} \right\},$$

defines the *cylindrical algebra generated by D*. The generated σ-algebra is denoted by $\mathcal{C}(V, D)$ and it is called the *cylindrical σ-algebra with respect to (V, D)*. If $D = V^*$ we write $\mathcal{Z}(V) := \mathcal{Z}(V, D)$ and $\mathcal{C}(V) := \mathcal{C}(V, D)$.

A function $\eta \colon \mathcal{Z}(V) \to [0, \infty]$ is called a *cylindrical measure on $\mathcal{Z}(V)$* if for each finite subset $D \subseteq V^*$ the restriction of η to the σ-algebra $\mathcal{C}(V, D)$ is a measure. A cylindrical measure η is called finite if $\eta(V) < \infty$ and a cylindrical probability measure if $\eta(V) = 1$. The characteristic function φ_η of a finite cylindrical measure η is defined by

$$\varphi_\eta \colon V^* \to \mathbb{C}, \qquad \varphi_\eta(v^*) := \int_V e^{i\langle v, v^* \rangle} \, \eta(dv).$$

We will always assume that the characteristic function is continuous, in which case the cylindrical measure η is called *continuous*. A cylindrical measure η has p-th weak moments if

$$\int_V |\langle v, v^* \rangle|^p \, \eta(dv) < \infty \qquad \text{for all } v^* \in V^*.$$

A cylindrical measure η is of cotype p if it has p-th weak moments and for each sequence $(v_n^*)_{n \in \mathbb{N}} \subseteq V^*$ the condition

$$\int_V |\langle v, v_n^* \rangle|^p \, \eta(dv) \to 0 \quad \text{for } n \to \infty,$$

implies that $\|v_n^*\| \to 0$.

A *cylindrical random variable Z in V* is a linear and continuous map

$$Z \colon V^* \to L_P^0(\Omega; \mathbb{R}).$$

The cylindrical random variable Z has weak p-th moments if $E[|Zv^*|^p] < \infty$ for all $v^* \in V^*$. In this case, the closed graph theorem implies that $Z \colon V^* \to L_P^p(\Omega; \mathbb{R})$ is continuous. The characteristic function of a cylindrical random variable Z is defined by

$$\varphi_Z \colon V^* \to \mathbb{C}, \qquad \varphi_Z(v^*) = E\big[\exp(iZv^*)\big].$$

By defining for each cylindrical set $C = C(v_1^*, \ldots, v_n^*; B) \in \mathcal{Z}(V)$ the mapping

$$\eta_Z(C) := P\big((Zv_1^*, \ldots, Zv_n^*) \in B\big),$$

we obtain a cylindrical probability measure η_Z, which is called the *cylindrical distribution of Z*. The characteristic functions φ_{η_Z} and φ_Z of η_Z and Z coincide. Conversely, for every cylindrical probability measure η on $\mathcal{Z}(V)$ there exist a probability space $(\Omega, \mathcal{A}, P)$ and a cylindrical random variable $Z \colon V^* \to L_P^0(\Omega; \mathbb{R})$ such that η is the cylindrical distribution of Z; see [24, VI.3.2].

A cylindrical random variable $Z\colon V^* \to L_P^0(\Omega;\mathbb{R})$ is called *induced by a random variable in* $L_P^p(\Omega;V)$ if there exists $Y \in L_P^p(\Omega;V)$ such that

$$\langle Y, v^* \rangle = Zv^* \qquad \text{for all } v^* \in V^*.$$

This is equivalent to the fact that the cylindrical distribution of Z extends to a probability measure on $\mathcal{B}(V)$; see Theorem IV.2.5 in [24].

3. Infinitely divisible cylindrical measures

The class of infinitely divisible cylindrical probability measures is introduced in [17]. A cylindrical probability measure η on $\mathcal{Z}(V)$ is called *infinitely divisible* if for each $k \in \mathbb{N}$ there exists a cylindrical probability measure η_k such that $\eta = \eta_k^{*k}$. Theorem 3.13 in [17] shows that a cylindrical probability measure η is infinitely divisible if and only if

$$\eta \circ \pi_{v_1^*,\dots,v_m^*}^{-1} \text{ is infinitely divisible on } \mathcal{B}(\mathbb{R}^m) \text{ for all } v_1^*,\dots,v_m^* \in V^* \text{ and } m \in \mathbb{N}.$$

In this equivalent description it is not sufficient only to take $n = 1$ as it is shown even in the case $V = \mathbb{R}^2$ in [14] and [9].

Let X be an infinitely divisible cylindrical random variable, that is its cylindrical distribution is infinitely divisible, and assume that X has weak second moments and $E[Xv^*] = 0$ for all $v^* \in V^*$. Define the covariance operator by

$$Q\colon V^* \to V^{**}, \qquad \langle Qv^*, w^* \rangle = E\big[(Xv^*)(Xw^*)\big].$$

The range of Q, i.e. the continuity of $Qv^* \colon V^* \to \mathbb{R}$, follows from the Cauchy-Schwarz inequality and the continuity of $X\colon V^* \to L_P^2(\Omega;\mathbb{R})$.

For the following we assume that Q is V-valued. This is guaranteed for example if X is a genuine random variable (see Theorem III.2.1 in [24]), in which case we set $Xv^* = \langle X, v^* \rangle$. Other examples of a V-valued covariance operator will be seen later in Section 5. As the covariance operator $Q\colon V^* \to V$ is positive and symmetric, it follows that there exists a Hilbert space H and $j \in L(H,V)$ such that $Q = jj^*$ and $j(H)$ is dense in V. Moreover, the Hilbert space H is unique up to isomorphism and separable as V is separable; see Section III.1.2 in [24].

Since the range of j^* is dense in H we can choose an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H with $e_k \in j^*(V^*)$. Thus, there exist some elements $v_k^* \in V^*$ obeying $j^* v_k^* = e_k$ for all $k \in \mathbb{N}$. It follows that

$$(3.1) \qquad Xv^* = \sum_{k=1}^{\infty} \langle je_k, v^* \rangle Xv_k^* \qquad \text{for all } v^* \in V^*,$$

where the sum converges in $L_P^2(\Omega;\mathbb{R})$. This representation is the Karhunen-Loève representation and is in this form established in [1]. Define a cylindrical random variable by

$$(3.2) \qquad \Theta_X \colon H \to L_P^2(\Omega;\mathbb{R}), \qquad \Theta_X h = \sum_{k=1}^{\infty} \langle e_k, h \rangle Xv_k^*.$$

The fact, that Θ_X is well defined and is a cylindrical random variable follows from the following lemma where we collect some simple properties of Θ_X and its cylindrical probability distribution ϑ_X.

Lemma 3.1. *For a cylindrical random variable* $X\colon V^* \to L_P^2(\Omega;\mathbb{R})$ *let* ϑ_X *denote the cylindrical distribution of* Θ_X *defined in* (3.2). *Then we have:*

(a) $E\left[|\Theta_X h|^2\right] = \|h\|^2$ *for all* $h \in H$.

(b) *the cylindrical distribution of* X *equals* $\vartheta_X \circ j^{-1}$.

(c) ϑ_X *is of cotype 2.*

(d) ϑ_X *is infinitely divisible.*

Proof. (a) The identity

$$E[(Xv_k^*)(Xv_\ell^*)] = \langle Qv_k^*, v_\ell^* \rangle = \langle e_k, e_\ell \rangle \tag{3.3}$$

implies that Xv_k^* and Xv_ℓ^* are uncorrelated for $k \neq \ell$ and $E[|Xv_k^*|^2] = 1$. Thus, part (a) follows from (3.2). This also shows that Θ_X is a well defined cylindrical random variable. (b) Due to (3.1) and (3.2) we have $\Theta_X(j^*v^*) = Xv^*$ for all $v^* \in V^*$ which establishes the claim. Part (c) follows from part (a).

(d) For $h_1, \ldots, h_m \in H$ and $n \in \mathbb{N}$ define

$$u_j^{(n)} := \sum_{k=1}^n \langle e_k, h_j \rangle v_k^* \qquad \text{for } j = 1, \ldots, m.$$

It follows from (3.2) for each $j = 1, \ldots, m$ that

$$\Theta_X h_j = \lim_{n \to \infty} X u_j^{(n)} \quad \text{in } L_P^2(\Omega; \mathbb{R}),$$

which yields

$$(\Theta_X h_1, \ldots, \Theta_X h_m) = \lim_{n \to \infty} \left(X u_1^{(n)}, \ldots, X u_m^{(n)} \right) \qquad \text{in probability in } \mathbb{R}^m.$$

The probability distribution of the random vector on the right hand side is infinitely divisible as it is given by $\eta \circ \pi_{u_1^{(n)}, \ldots, u_m^{(n)}}^{-1}$, where η denotes the cylindrical distribution of the infinitely divisible cylindrical random variable X. Consequently, the random vector on the left hand side is infinitely divisible, which shows that Θ_X is an infinitely divisible cylindrical random variable. $\square$

Example 3.2. Assume that X is a Gaussian cylindrical random variable that is Xv^* is Gaussian for all $v^* \in V^*$. In this case, it follows from (3.3) that $(Xv_k^*)_{k \in \mathbb{N}}$ is a sequence of independent, Gaussian random variables with $E[|Xv_k^*|^2] = 1$, which yields for the characteristic function φ_{Θ_X} of Θ_X:

$$\varphi_{\Theta_X}(h) = \prod_{k=1}^\infty \exp\left(-\tfrac{1}{2}\langle e_k, h \rangle^2\right) = \exp\left(-\tfrac{1}{2}\|h\|^2\right) \qquad \text{for all } h \in H. \tag{3.4}$$

Consequently, the cylindrical random variable Θ_X is distributed according to the canonical Gaussian cylindrical measure γ in this case.

4. Radonifying operators

Let H be a separable Hilbert space, V be a separable Banach space and ϑ be a cylindrical probability measure on $\mathcal{Z}(H)$. An operator $T \in \mathcal{L}(H, V)$ is called ϑ-*radonifying* if the image cylindrical measure $\vartheta \circ T^{-1}$ extends to a probability measure on $\mathcal{B}(V)$. If the extended measure has finite p-th moments, T is called ϑ-radonifying of order p. We define the space

$$\mathcal{R}^p(\vartheta) := \mathcal{R}_{H,V}^p(\vartheta) := \{T \in \mathcal{L}(H, V) : T \text{ is } \vartheta\text{-radonifying of order } p\}.$$

Let Θ denote a cylindrical random variable with cylindrical distribution ϑ. Theorem VI.3.1 in [24] guarantees that an operator $T \in \mathcal{L}(H, V)$ is in $\mathcal{R}^p(\vartheta)$ if and only if the cylindrical random variable $T(\Theta)$ defined by

$$T(\Theta) \colon V^* \to L_P^0(\Omega; \mathbb{R}), \qquad T(\Theta)v^* = \Theta(T^*v^*)$$

is induced by a genuine random variable in $L_P^p(\Omega; V)$. If S and T are in $\mathcal{R}^p(\vartheta)$ and $\alpha \in \mathbb{R}$ then we obtain for all $v^* \in V^*$ that

$$(\alpha S + T)(\Theta)v^* = \Theta((\alpha S + T)^*v^*) = \alpha S(\Theta)v^* + T(\Theta)v^*.$$

Since $S(\Theta)$ and $T(\Theta)$ are induced by genuine random variables in $L_P^p(\Omega; V)$, respectively, it follows that $(\alpha S + T)(\Theta)$ is also induced by a genuine random variable in $L_P^p(\Omega; V)$, and thus $\alpha S + T \in \mathcal{R}^p(\vartheta)$. For $T \in \mathcal{R}^p(\vartheta)$ define

$$\|T\|_p := \left(\int_V \|v\|^p \ (\vartheta \circ T^{-1})(dv) \right)^{1/p}.$$

Since $\|T\|_p^p = E[\|T(\Theta)\|^p]$ it follows that $\|\cdot\|_p$ defines a semi-norm on $\mathcal{R}^p(\vartheta)$. We obtain a norm on $\mathcal{R}^p(\vartheta)$ by defining

$$\|T\|_{\mathcal{R}^p} := \|T\|_p + \|T\|_{H \to V}.$$

Theorem 4.1. *For every cylindrical probability measure ϑ on $\mathcal{Z}(H)$ the space $\mathcal{R}^p(\vartheta)$ equipped with $\|\cdot\|_{\mathcal{R}^p}$ is a Banach space for each $p \geqslant 1$.*

Proof. Let $(T_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{R}^p(\vartheta)$. Denote for each $n \in \mathbb{N}$ by Y_n the induced random variable in $L_P^p(\Omega; V)$ with probability distribution $\vartheta \circ T_n^{-1}$. Since $(T_n)_{n \in \mathbb{N}}$ is also a Cauchy sequence in $\mathcal{L}(H, V)$ it follows that there exists $T \in \mathcal{L}(H, V)$ such that $\|T_n - T\|_{H \to V} \to 0$ for $n \to \infty$. Moreover, the equality $\|Y_m - Y_n\|_{L_P^p} = \|T_m - T_n\|_p$ implies that there exists a random variable $Y \in L_P^p(\Omega; V)$ such that $\|Y_n - Y\|_{L_P^p} \to 0$ for $n \to \infty$. The continuity of $\Theta \colon H \to L_P^0(\Omega; \mathbb{R})$ implies for every v^* that we have in $L_P^0(\Omega; \mathbb{R})$:

$$\lim_{n \to \infty} |\langle Y_n, v^* \rangle - \Theta(T^* v^*)| = \lim_{n \to \infty} \left| \Theta\big((T_n^* - T^*)v^*\big) \right| = 0.$$

Since $\langle Y_n, v^* \rangle \to \langle Y, v^* \rangle$ in $L_P^2(\Omega; \mathbb{R})$ we obtain $\langle Y, v^* \rangle = \Theta(T^* v^*)$ for all $v^* \in V^*$, which completes the proof. $\square$

Example 4.2. Let ϑ be given by the canonical Gaussian cylindrical measure γ on $\mathcal{Z}(H)$. Due to Fernique's theorem, each γ-radonifying operator is of any order $p \geqslant 1$. Thus, the space $\mathcal{R}^p(\gamma)$ coincides with the space of γ-radonifying operators. This class of operators is well studied, and is recently surveyed in [25].

In the special setting of Section 3 we obtain the following simplification of the norm in $\mathcal{R}^p(\vartheta)$ for $p \geqslant 2$:

Proposition 4.3. *For a cylindrical random variable X in V with weak second moments let the cylindrical random variable Θ_X and its cylindrical distribution ϑ_X be defined by (3.2). If $p \geqslant 2$ then $T \in \mathcal{R}^p(\vartheta_X)$ satisfies*

$$\|T\|_{H \to V} \leqslant \|T\|_p,$$

that is the norm $\|\cdot\|_{\mathcal{R}^p}$ is equivalent to $\|\cdot\|_p$.

Proof. For $T \in \mathcal{R}^p(\vartheta_X)$ let Y denote the induced random variable in $L_P^p(\Omega; V)$ with probability distribution $\vartheta_X \circ T^{-1}$ on $\mathcal{Z}(V)$. Lemma 3.1 implies:

$$\|T\|_{H \to V}^2 = \sup_{\|v^*\| \leqslant 1} \|T^* v^*\|^2 = \sup_{\|v^*\| \leqslant 1} E\left[|\Theta_X T^* v^*|^2 \right]$$

$$\leqslant E\left[\sup_{\|v^*\| \leqslant 1} |\langle Y, v^* \rangle|^2 \right] = E\left[\|Y\|^2 \right] \leqslant \left(E\left[\|Y\|^p \right] \right)^{2/p},$$

which completes the proof. $\square$

The following result is a straightforward conclusion of a result by Schwartz [22] and Kwapien [11], but it shows an important class of operators which are ϑ-radonifying.

Proposition 4.4. *If ϑ is a cylindrical probability measure of weak order $p \geqslant 1$ then we have*

$$\Pi^p(H, V) \subseteq \mathcal{R}^p_{H,V}(\vartheta).$$

Proof. The space of p-absolutely summing operators $\Pi^p(H, V)$ coincides with the space of p-radonifying operators; see Theorem VI.5.4 in [24] for $p > 1$ and Corollary in VI.5.4 in [24] for $p = 1$. The space of p-radonifying operators are operators $T \in \mathcal{L}(H, V)$ such for each cylindrical measure η on $\mathcal{Z}(H)$ with weak p-moments the image cylindrical measure $\eta \circ T^{-1}$ extends to a measure on $\mathcal{B}(V)$ with finite p-moment. $\square$

Proposition 4.5. *Let $\Theta\colon H \to L^0_P(\Omega; \mathbb{R})$ be a cylindrical random variable with cylindrical distribution ϑ. Then for $T \in \mathcal{L}(H, V)$ and $p \geqslant 1$ the following are equivalent:*

(a) $T \in \mathcal{R}^p(\vartheta)$.

(b) *there exists a random variable $Y \in L^p_P(\Omega; V)$ such that for all (some) orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of H:*

$$\langle Y, v^* \rangle = \sum_{k=1}^{\infty} \langle Te_k, v^* \rangle \Theta e_k \quad \text{in } L^0_P(\Omega; \mathbb{R}) \text{ for all } v^* \in V^*.$$

Proof. The operator T is in $\mathcal{R}^p(\vartheta)$ if and only if the cylindrical random variable $T(\Theta)$ is induced by a V-valued random variable Y in $L^p_P(\Omega; V)$, that is $\Theta(T^*v^*) = \langle Y, v^* \rangle$ for all $v^* \in V^*$. The continuity of $\Theta\colon H \to L^0_P(\Omega; \mathbb{R})$ implies in $L^0_P(\Omega; \mathbb{R})$:

$$\langle Y, v^* \rangle = \Theta(T^*v^*) = \Theta\left(\sum_{k=1}^{\infty} \langle T^*v^*, e_k \rangle e_k \right) = \sum_{k=1}^{\infty} \langle v^*, Te_k \rangle \Theta e_k$$

for all $v^* \in V^*$. $\square$

Remark 4.6. If ϑ is the canonical Gaussian cylindrical measure γ on H, then the random variables $(\Theta e_k)_{k \in \mathbb{N}}$ are independent and symmetric. Thus, Itô-Nisio's Theorem guarantees that part (b) in Proposition 4.5 is equivalent to

(c) *there exists a random variable $Y \in L^p_P(\Omega; V)$ such that for all (some) orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of H:*

$$Y = \sum_{k=1}^{\infty} Te_k \, \Theta e_k \quad \text{in } L^2_P(\Omega; V).$$

Often this property is taken as a definition of γ-radonifying operators in the literature.

Theorem 4.7. *If V is a separable Hilbert space and the cylindrical measure ϑ has weak p-th moments for some $p \geqslant 1$ and is of finite cotype then it follows:*

$$\mathcal{R}^p(\vartheta) = \{T \in \mathcal{L}(H, V) : T \text{ is Hilbert-Schmidt}\}$$

Proof. It is well known that in Hilbert spaces the class of p-radonifying operators (see proof of Proposition 4.4) coincides with the space of Hilbert-Schmidt operators. Thus, the class of Hilbert-Schmidt operators is a subset of $\mathcal{R}^p(\vartheta)$.

Let T be in $\mathcal{R}^p(\vartheta)$. Then the cylindrical measure $\vartheta \circ T^{-1}$ extends to a Radon measure of order p and $\vartheta \circ (T^{**})^{-1}$ is a $\sigma(V^{**}, V^*)$-Radon measure. Denote the cotype of ϑ by $q \in [0, \infty)$. If $q \leqslant p$ then ϑ is also of cotype p, and thus, we can assume that $q \geqslant 1$. Theorem VI.5.9 in [24] implies that T^* is q-absolutely summing, which is equivalent to the fact that T^* is Hilbert-Schmidt, since H and V are Hilbert spaces. $\square$

Remark 4.8. If ϑ is a genuine probability measure with p-th moments then each operator in $\mathcal{L}(H, V)$ is in $\mathcal{R}^p(\vartheta)$ and Theorem 4.7 cannot be true. This case is excluded since in

this case ϑ cannot be of finite cotype: if $(h_n)_{n\in\mathbb{N}}$ is a sequence in H which converges sequentially weakly to 0 then Lebesgue's theorem implies

$$\lim_{n\to\infty} \int_H \langle h_n, h\rangle^p \, \vartheta(dh) = 0.$$

But if H is infinite dimensional then we can choose $\|h_n\| = 1$ for all $n \in \mathbb{N}$.

Stable cylindrical measures: A cylindrical measure ϑ on $\mathcal{Z}(H)$ is called *stable of order* $\alpha \in (0,2]$ if there exists a measure space $(S, \mathcal{S}, m)$ and a linear bounded operator $F \colon H \to L_m^\alpha(S; \mathbb{R})$ such that the characteristic function φ_ϑ of ϑ obeys

$$(4.1) \qquad \varphi_\vartheta \colon H \to \mathbb{C}, \qquad \varphi_\vartheta(h) = \exp\left(-\|Fh\|_{L_m^\alpha}^\alpha\right).$$

The characteristic function of $\vartheta \circ T^{-1}$ for an arbitrary operator $T \in \mathcal{L}(H, V)$ is given by

$$\varphi_{\vartheta\circ T^{-1}} \colon V^* \to \mathbb{C}, \qquad \varphi_{\vartheta\circ T^{-1}}(v^*) = \exp\left(-\|(FT^*)v^*\|_{L_m^\alpha}^\alpha\right).$$

It follows that T is ϑ-radonifying if and only if $FT^* \in \Lambda_\alpha(V^*, L_m^\alpha)$, where

$$\Lambda_\alpha(V^*, L_m^\alpha) := \Big\{ R \in \mathcal{L}(V^*, L_m^\alpha) \colon v^* \mapsto \exp(-\|Rv^*\|_{L_m^\alpha}^\alpha)$$

$$\text{is the characteristic function of a Radon measure on } V \Big\}.$$

By taking into account that an α-stable measure has finite r-th moments for all $r < \alpha$ according to Theorem 3.2 in [7], we obtain for each $p < \alpha$ that T is in $\mathcal{R}^p(\vartheta)$ if only if $FT^* \in \Lambda_\alpha(V^*, L_m^\alpha)$. The spaces $\Lambda_\alpha(V^*, L_m^\alpha)$ are surveyed in [13, Se.7.8], however an explicit description is only known in a few case. A case which can be easily described is the following:

Example 4.9. Assume that $V = \ell^q$ for some $q \in [2, \infty)$ and $L_m^\alpha(S; \mathbb{R}) = \ell^\alpha$ for some $\alpha < q'$ where $q' := q/(q-1)$. Let ϑ be an α-stable measure on the Hilbert space H with characteristic function of the form (4.1). Then an operator $T \in \mathcal{L}(H, V)$ is in $\mathcal{R}^p(\vartheta)$ for $p < \alpha$ if and only if

$$\sum_{k=1}^\infty \left(\sum_{j=1}^\infty |\langle FT^* e_j, e_k\rangle|^{q'}\right)^{\frac{\alpha}{q'}} < \infty,$$

where $(e_k)_{k\in\mathbb{N}}$ denotes the canonical Schauder basis for the spaces of sequences.

Example 4.10. Assume that V is given by some L^q space for $q \in [2, \infty)$ and ϑ is an α-stable cylindrical measure on the Hilbert space H with characteristic function of the form (4.1). Then an operator $T \in \mathcal{L}(H, V)$ is in $\mathcal{R}^p(\vartheta)$ for $p < \alpha$ if and only if FT^* is r-absolutely summing for any $r \in (0, q')$, i.e. $FT^* \in \Pi^r(L^{q'}, L_m^\alpha)$; see Proposition 7.8.7 in [13].

Stable cylindrical probability measures might be considered as a subset of infinitely divisible cylindrical measures with elements which are the most similar ones to the canonical Gaussian cylindrical measure γ. Nevertheless, many properties known for γ-radonifying operators do not hold for radonifying operators of stable cylindrical measures. One of these is the ideal property which is true for γ-radonifying operators: let H and H' be Hilbert spaces and V and V' be Banach spaces. Then if $T \in \mathcal{R}_{H,V}^2(\gamma)$, $S_1 \in \mathcal{L}(H', H)$ and $S_2 \in \mathcal{L}(V, V')$ then $S_2 T S_1 \in \mathcal{R}_{H',V'}^2(\gamma)$. This result can be found in [25]. However, already for stable cylindrical measures it is known that the ideal property is not satisfied any more: for $q > 2$ there exists an operator $T \in \mathcal{R}_{L^p, L^q}^p(\vartheta)$ and $S \in \mathcal{L}(L^p, L^p)$ such that TS is not in $\mathcal{R}_{L^p, L^q}^p(\vartheta)$; see [13] for this result.

Compound Poisson cylindrical measures: a *compound Poisson cylindrical distribution* (see Example 3.5 in [1]) is an infinitely divisible cylindrical measure ϑ on $\mathcal{Z}(H)$ with characteristic function

$$(4.2) \qquad \varphi_\vartheta : H \to \mathbb{C}, \qquad \varphi_\vartheta(h) = \exp\left(c \int_H \left(e^{i\langle h,g \rangle} - 1 \right) \nu(dg) \right),$$

where ν is a cylindrical probability measure on $\mathcal{Z}(H)$ and $c > 0$ is a constant. It follows from the finite dimensional theory of infinite divisible distributions that ϑ has weak second moments if and only if ν has weak second moments.

Equivalently, one can introduce a compound Poisson cylindrical distribution by cylindrical random variables. Let $X_1, X_2, \ldots$ be independent, cylindrical random variables in H with identical cylindrical distribution ν and let N be an independent, integer-valued Poisson distributed random variable with intensity $c > 0$, all defined on the probability space $(\Omega, \mathcal{A}, P)$. Then

$$Y : H \to L_P^0(\Omega; \mathbb{R}), \qquad Yh := \begin{cases} 0, & \text{if } N = 0, \\ X_1 h + \cdots + X_N h, & \text{else}, \end{cases}$$

defines a cylindrical random variable Y with a characteristic function which is of the form (4.2).

Theorem 4.11. *For a compound Poisson cylindrical distribution ϑ with characteristic function (4.2) it follows for each $p \geqslant 1$ that*

$$\mathcal{R}^p(\vartheta) = \mathcal{R}^p(\nu).$$

Proof. Let $T \in \mathcal{R}^p(\vartheta)$. Then $\mu := \vartheta \circ T^{-1}$ is an infinitely divisible measure on $\mathcal{B}(V)$ with p-th moment. If ξ denotes the Lévy measure of μ then it follows that $\xi = c(\nu \circ T^{-1})$ on $\mathcal{Z}(V)$ due to the uniqueness of cylindrical Lévy measures. Thus, the image cylindrical measure $\nu \circ T^{-1}$ extends to the probability measure $c^{-1}\xi$. Let Y be a V-valued random variable with distribution μ and $(X_k)_{k \in \mathbb{N}}$ a family of independent, V-valued random variables with distribution $c^{-1}\xi$. It follows that

$$E\left[\|X_1\|^p \right] \leqslant \frac{e^c}{c} \sum_{k=1}^\infty E\left[\|X_1 + \cdots + X_k\|^p \right] \frac{c^k}{k!} e^{-c} = \frac{e^c}{c} E\left[\|Y\|^p \right] < \infty,$$

i.e. the Radon measure $c^{-1}\xi$ has moments of order p which shows $T \in \mathcal{R}^p(\nu)$.

If we assume $T \in \mathcal{R}^p(\nu)$ then $\nu \circ T^{-1}$ is a probability measure on $\mathcal{B}(V)$ and

$$\mu : \mathcal{B}(V) \to [0,1], \qquad \mu(C) := e^{-c} \sum_{k=0}^\infty \frac{c^k (\nu \circ T^{-1})^{*k}(C)}{k!}$$

defines a probability measure on $\mathcal{B}(V)$ with characteristic function

$$\varphi_\mu : V^* \to \mathbb{C}, \qquad \varphi_\mu(v^*) = \exp\left(c \int_V \left(e^{i\langle v, v^* \rangle} - 1 \right) (\nu \circ T^{-1})(dv) \right),$$

see [13, Pro.5.3.1]. Since $\varphi_\mu = \varphi_{\vartheta \circ T^{-1}}$ it follows that $\vartheta \circ T^{-1}$ extends to the Radon measure μ on $\mathcal{B}(V)$. As before, let Y and $X_1, X_2, \ldots$ denote independent random variables with distributions μ and $\nu \circ T^{-1}$. The measure μ has p-th moments since Minkowski's inequality implies

$$E\left[\|Y\|^p \right] = \sum_{k=1}^\infty E\left[\|X_1 + \cdots + X_k\|^p \right] \frac{c^k}{k!} e^{-c} \leqslant \sum_{k=1}^\infty k^p E\left[\|X_1\|^p \right] \frac{c^k}{k!} e^{-c} < \infty,$$

which shows that $T \in \mathcal{R}^p(\vartheta)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Example 4.12. (Cylindrical normally distributed jumps)
Models of share prices perturbed by a discontinuous noise with normally distributed jumps are considered in Financial Mathematics from its very early times; see for example the work [15] by Merton. Accordingly, let ν be the canonical Gaussian cylindrical measure γ and let $c > 0$ be a constant. Then the compound Poisson cylindrical distribution ϑ with characteristic function (4.2) obeys

$$\mathcal{R}^p(\vartheta) = \mathcal{R}^p(\gamma).$$

5. Application: Wiener integrals

In this section we apply the theory of radonifying operators developed above in order to introduce Wiener integrals with respect to a Lévy process L with weak second moments on a separable Banach space U. In fact, the same approach can be applied if L is only a cylindrical Lévy process, but we want to avoid any more technical complications here; see [19] for details.

Recall that the Lévy process L can be decomposed into $L(t) = b + W(t) + M(t)$ for all $t \geqslant 0$, where $b \in U$ and W is a Wiener process with a covariance operator $C \in \mathcal{L}(U^*, U)$ and M is a Lévy process with weak second moments and a Lévy measure μ. The Wiener integrals with respect to the Wiener process W are developed in the publications [5] and [27] with many sophisticated refinements and applied to the stochastic Cauchy problem. Our rather simplified presentation below for integration with respect to W illustrates the core idea in the approach developed in [27]. In our work [20], we extend the approach in [27] to develop a Wiener integral with respect to a martingale-valued measure. By the theory developed here, we are able to relate this integral to ϑ-radonifying operators.

We begin with defining a Wiener integral with respect to the discontinuous martingale M with Lévy measure μ. For $\rho := \lambda \otimes \mu$, where λ denotes the Lebesgue measure on $[0, T]$, define $H_M := L_\rho^2([0, T] \times U; \mathbb{R})$. Let V denote another separable Banach space and let $F \colon [0, T] \to \mathcal{L}(U, V)$ be a function satisfying

$$(5.1) \qquad \langle F(\cdot)\cdot, v^* \rangle \in H_M \qquad \text{for all } v^* \in V^*.$$

Then one can define a cylindrical random variable by

$$I_M \colon V^* \to L_P^2(\Omega; \mathbb{R}), \qquad I_M v^* = \int_0^T F^*(s) v^* \, dM(s).$$

Since the integrand is U^*-valued and M has weak second moments, the integral can be easily defined by following an Itô approach, see e.g. [18], or by the approach of Métivier and Pellaumail in [16], or as introduced by Rosiński in [21]. In all cases, it follows that I_M is an infinitely divisible cylindrical random variable with weak second moments. The covariance operator of I_M is given by

$$Q_M \colon V^* \to V, \qquad \langle Q_M v^*, w^* \rangle = \int_{[0,T] \times U} \langle F(s)u, v^* \rangle \langle F(s)u, w^* \rangle \, \rho(ds, du).$$

The mapping Q_M is V-valued and not only V^{**}-valued, since Pettis' measurability theorem guarantees due to (5.1) that $(t, u) \mapsto F(t)u$ is strongly measurable. The covariance operator Q_M can be factorised by

$$j_M \colon H_M \to V, \qquad \langle j_M f, v^* \rangle := \int_{[0,T] \times U} \langle F(s)u, v^* \rangle f(s, u) \, \rho(ds, du).$$

Since the adjoint operator is given by $j_M^* w^* = \langle F(\cdot)\cdot, w^* \rangle = F^*(\cdot) w^*$ for all $w^* \in V^*$ it follows that $Q_M = j_M j_M^*$, and thus $H_M = L_\rho^2([0, T] \times U; \mathbb{R})$ is established as the

reproducing kernel Hilbert space of Q_M. Define the cylindrical random variable

$$\Theta_M \colon H_M \to L_P^2(\Omega; \mathbb{R}), \qquad \Theta_M f = \int_{[0,T] \times U} f(s,u) \, M(ds, du),$$

and let ϑ_M denote the cylindrical distribution of Θ_M. Let $(f_k)_{k \in \mathbb{N}}$ be an orthonormal basis of H_M and choose $v_k^* \in V^*$ such that $f_k = j_M^* v_k^*$. By continuity of Θ_M it follows for all $v^* \in V^*$ that we have in $L_P^2(\Omega; \mathbb{R})$:

$$I_M v^* = \Theta_M(j_M^* v^*) = \Theta_M \left(\sum_{k=1}^{\infty} \langle f_k, j_M^* v^* \rangle f_k \right)$$

$$= \sum_{k=1}^{\infty} \langle j_M f_k, v^* \rangle \Theta_M f_k = \sum_{k=1}^{\infty} \langle j_M f_k, v^* \rangle I_M v_k^*.$$

In summary, we have explicitly derived the setting of Section 3: for the cylindrical random variable I_M we derived the reproducing kernel Hilbert space H_M of its covariance operator Q_M with embedding $j_M \colon H_M \to V$. In addition, we constructed the cylindrical random variable Θ_M in H_M, which is based on the Karhunen-Loève representation of I_M according to (3.1) and which satisfies $j_M(\Theta_M) = I_M$.

It follows from Lemma 3.1 that Θ_M is an infinitely divisible cylindrical random variable, and by approximating $f \in H_M$ by step functions, the characteristic function of Θ_M is given by

$$\varphi_{\Theta_M} \colon H_M \to \mathbb{C}, \qquad \varphi_{\Theta_M}(f) = \exp\left(\int_{H_M} \left(e^{i \langle f, g \rangle} - 1 - i \langle f, g \rangle \right) \nu(dg) \right),$$

where ν is a cylindrical measure on $\mathcal{Z}(H_M)$ satisfying $\nu \circ (\langle \cdot, f \rangle)^{-1} = \rho \circ f^{-1}$ for all $f \in H_M$.

Since the cylindrical distribution of I_M equals $\vartheta_M \circ j_M^{-1}$ according to Lemma 3.1, there exists a random variable $Y_M \in L_P^0(\Omega; V)$ obeying

$$(5.2) \qquad \langle Y_M, v^* \rangle = \int_0^T F^*(s) v^* \, M(ds) \qquad \text{for all } v^* \in V^*,$$

if and only if $j_M \colon H_M \to V$ is ϑ_M-radonifying.

The same approach can be applied to introduce the stochastic integral with respect to the Wiener process W with covariance operator C. Let K denote the reproducing kernel Hilbert space of C with embedding $i_C \colon K \to U$, i.e. $C = i_C i_C^*$. For functions $F \colon [0,T] \to \mathcal{L}(U, V)$ satisfying

$$(5.3) \qquad i_C^* F^*(\cdot) v^* \in L^2([0,T]; K) \qquad \text{for all } v^* \in V^*,$$

one can define a cylindrical random variable by

$$I_W \colon V^* \to L_P^2(\Omega; \mathbb{R}), \qquad I_W v^* = \int_0^T F^*(s) v^* \, dW(s).$$

The covariance operator of I_W is given by

$$Q_W \colon V^* \to V, \qquad \langle Q_W v^*, w^* \rangle = \int_0^T \langle i_C^* F^*(s) v^*, i_C^* F^*(s) w^* \rangle \, ds.$$

The covariance operator Q_W can be factorised through the Hilbert space $H_W := L^2([0,T]; K)$, and the embedding is given by

$$j_W \colon H_W \to V, \qquad \langle j_W f, v^* \rangle = \int_0^T \langle i_C^* F^*(s) v^*, f(s) \rangle \, ds,$$

with adjoint operator $j_W^* v^* = i_C^* F^*(\cdot) v^*$. If γ denotes the canonical Gaussian cylindrical measure on H_W it follows that $\gamma \circ j_W^{-1}$ is a cylindrical Gaussian distribution with covariance operator $j_W j_W^*$, that is $\gamma \circ j_W^{-1}$ coincides with the cylindrical distribution of I_W. Consequently, we obtain that there exists a random variable $Y_W \in L_P^0(\Omega; V)$ obeying

$$\langle Y_W, v^* \rangle = \int_0^T F^*(s) v^* \, dW(s) \qquad \text{for all } v^* \in V^*,$$

if and only if $j_W \colon H_W \to V$ is γ-radonifying.

Finally, let $F \colon [0, T] \to \mathcal{L}(U, V)$ be a function obeying (5.1), (5.3) and

(5.4) $$\qquad\qquad \langle F^*(\cdot) v^*, b \rangle \in L^1([0, T]; \mathbb{R}) \qquad \text{for all } v^* \in V^*.$$

Then one can define for each $A \in \mathcal{B}([0, T])$ a cylindrical random variable $I_A \colon V^* \to L_P^2(\Omega; \mathbb{R})$ by

(5.5)
$$I_A v^* = \int_0^T \mathbb{1}_A(s) \langle F^*(s) v^*, b \rangle \, ds + \int_0^T \mathbb{1}_A(s) F^*(s) v^* \, dW(s) + \int_0^T \mathbb{1}_A(s) F^*(s) v^* \, dM(s).$$

The function F is called *stochastically integrable with respect to* L if and only if for each $A \in \mathcal{B}([0, T])$ there exists a random variable $Y_A \in L_P^0(\Omega; V)$ such that

(5.6) $$\qquad\qquad \langle Y_A, v^* \rangle = I_A v^* \qquad \text{for all } v^* \in V^*.$$

By the derivation above one obtains the following result:

Theorem 5.1. *A function* $F \colon [0, T] \to L(U, V)$ *satisfying* (5.1), (5.3) *and* (5.4) *is stochastically integrable with respect to* $L(\cdot) = b + W(\cdot) + M(\cdot)$ *if and only if the following are satisfied:*

(i) $F(\cdot) b \colon [0, T] \to V$ *is Pettis integrable;*
(ii) $j_W \colon H_W \to V$ *is* γ-*radonifying;*
(iii) $j_M \colon H_M \to V$ *is* ϑ_M-*radonifying.*

Proof. If part: for $A \in \mathcal{B}([0, T])$ define the cylindrical random variables

$$I_b^A \colon V^* \to \mathbb{R}, \qquad I_b^A v^* = \int_A \langle v^*, F(s) b \rangle \, ds,$$

$$I_W^A \colon V^* \to L_P^2(\Omega; \mathbb{R}), \qquad I_W^A v^* = \int_0^T \mathbb{1}_A(s) F^*(s) v^* \, dW(s).$$

$$I_M^A \colon V^* \to L_P^2(\Omega; \mathbb{R}), \qquad I_M^A v^* = \int_0^T \mathbb{1}_A(s) F^*(s) v^* \, dM(s).$$

We have to show that these cylindrical random variables are induced by genuine random variables in $L_P^0(\Omega; V)$, respectively. Condition (i) implies that there exists $v^A \in V$ such that

(5.7) $$\qquad\qquad \langle v^A, v^* \rangle = I_b^A v^* \qquad \text{for all } v^* \in V^*.$$

The covariance operator of the cylindrical random variable I_W^A is given by

$$Q_W^A \colon V^* \to V, \qquad \langle Q_W v^*, w^* \rangle = \int_A \langle i_C^* F^*(s) v^*, i_C^* F^*(s) w^* \rangle \, ds.$$

It follows that

$$\langle Q_W^A v^*, v^* \rangle \leqslant \langle Q_W^{[0,T]} v^*, v^* \rangle \qquad \text{for all } v^* \in V^*.$$

Since Condition (ii) guarantees that $Q_W^{[0,T]}$ is the covariance operator of a Gaussian measure on $\mathcal{B}(V)$, Theorem 3.3.1 in [3] implies that there exists a Gaussian measure on

$\mathcal{B}(V)$ with covariance operator Q_W^A. Thus, Theorem IV.2.5 in [24] guarantees that there exists a random variable $Y_W^A \in L_P^0(\Omega; V)$ such that

$$(5.8) \qquad \langle Y_W^A, v^* \rangle = I_W^A v^* \qquad \text{for all } v^* \in V^*.$$

From the independent increments of M it follows that the cylindrical distribution of I_M^A is infinitely divisible and that its cylindrical Lévy measure ν_A is given by

$$\nu_A \colon \mathcal{Z}(V) \to [0, \infty], \qquad \nu_A(B) = \int_{A \times U} \mathbb{1}_B(F(s)u) \, \rho(ds, du).$$

Condition (iii) implies that $\nu_{[0,T]}$ is the genuine Lévy measure of an infinitely divisible probability measure on $\mathcal{B}(V)$. Since

$$\nu_A(B) \leqslant \nu_{[0,T]}(B) \qquad \text{for all } B \in \mathcal{Z}(V),$$

Theorem 3.4 in [19] implies that ν_A extends to a genuine Lévy measure on $\mathcal{B}(V)$. Theorem IV.2.5 in [24] guarantees that there exists a random variable $Y_M^A \in L_P^0(\Omega; V)$ such that

$$(5.9) \qquad \langle Y_M^A, v^* \rangle = I_M^A v^* \qquad \text{for all } v^* \in V^*.$$

It follows from (5.7), (5.8) and (5.9) that the random variable $Y_A := v^A + Y_W^A + Y_M^A$ satisfies (5.6).

Only if part: Let I_b, I_W and I_M denote the cylindrical random variables defined in the beginning of the proof for $A = [0, T]$. Stochastic integrability of F implies that there exists a random variable $Y \in L_P^0(\Omega; V)$ such that

$$\langle Y, v^* \rangle = I_b v^* + I_W v^* + I_M v^* \qquad \text{for all } v^* \in V^*.$$

Since $I_W v^*$ is symmetric and independent of $(I_b + I_M) v^*$ for all $v^* \in V^*$, it follows from Proposition 7.14.51 in [4] that the cylindrical distributions of I_W and $I_b + I_M$ extend to probability measures on $\mathcal{B}(V)$. Consequently, j_W is γ-radonifying and $I_b + I_M$ is induced by a random variable in $L_P^0(\Omega; V)$. Let M' denote an independent copy of M. Then $I_b + I_M - I_b - I_{M'}$ is induced by a random variable $X \in L_P^0(\Omega; V)$ and X is infinitely divisible. Denoting the Lévy measure of X by ξ it follows that

$$\nu_{[0,T]}(B) \leqslant \nu_{[0,T]}(B) + \nu_{[0,T]}(-B) = \xi(B) \qquad \text{for all } B \in \mathcal{Z}(V).$$

Theorem 3.4 in [19] implies that $\nu_{[0,T]}$ extends to a genuine Lévy measure on $\mathcal{B}(V)$, which yields that j_M is ϑ_M-radonifying. $\qquad\qquad\square$

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