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A NOTE ON CONVERGENCE TO STATIONARITY OF RANDOM PROCESSES WITH IMMIGRATION

Let $X_1, X_2, \dots$ be random elements of the Skorokhod space $D(\mathbb{R})$ and $\xi_1, \xi_2, \dots$ positive random variables such that the pairs $(X_1, \xi_1), (X_2, \xi_2), \dots$ are independent and identically distributed. The random process $Y(t) := \sum_{k\geq 0} X_{k+1}(t - \xi_1 - \dots - \xi_k)\, \mathbb{1}_{\{\xi_1+\dots+\xi_k \leq t\}}$, $t \in \mathbb{R}$, is called random process with immigration at the epochs of a renewal process. Assuming that the distribution of ξ_1 is nonlattice and has finite mean while the process X_1 decays sufficiently fast, we prove weak convergence of $(Y(u+t))_{u\in\mathbb{R}}$ as $t \to \infty$ on $D(\mathbb{R})$ endowed with the J_1-topology. The present paper continues the line of research initiated in [2, 3]. Unlike the corresponding result in [3] arbitrary dependence between X_1 and ξ_1 is allowed.

1. Introduction

Denote by $D(\mathbb{R})$ the Skorokhod space of right-continuous real-valued functions which are defined on $\mathbb{R}$ and have finite limits from the left at each point of $\mathbb{R}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (X, ξ) be a random element in $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $D(\mathbb{R}) \times [0, \infty)$. More precisely, $X := (X(t))_{t\in\mathbb{R}}$ is a random process with paths in $D(\mathbb{R})$ which satisfies $X(t) = 0$ for all $t < 0$, and ξ is a positive random variable. X and ξ are allowed to be arbitrarily dependent. Further, on $(\Omega, \mathcal{F}, \mathbb{P})$ define the sequence $(X_1, \xi_1), (X_2, \xi_2), \dots$ of i.i.d. copies of the pair (X, ξ). Let $(S_n)_{n\in\mathbb{N}_0}$ (we use the notation $\mathbb{N}_0$ for the set of nonnegative integers $\{0, 1, 2, \dots\}$) be the zero-delayed random walk with increments ξ_k, i.e.,

$$S_0 := 0, \qquad S_n := \xi_1 + \dots + \xi_n, \quad n \in \mathbb{N}.$$

Denote by $(\nu(t))_{t\in\mathbb{R}}$ the associated first-passage time process given by $\nu(t) := \inf\{k \in \mathbb{N}_0 : S_k > t\}$ for $t \in \mathbb{R}$. The process $Y := (Y(t))_{t\in\mathbb{R}}$ defined by

$$Y(t) := \sum_{k\geq 0} X_{k+1}(t - S_k) = \sum_{k=0}^{\nu(t)-1} X_{k+1}(t - S_k), \quad t \in \mathbb{R}$$

is called *random process with immigration at the epochs of a renewal process* or just *random process with immigration.* These processes were introduced in [2, 3]. Since the paper at hand is intended to be a part of this series, we refrain from giving a detailed discussion here and refer the reader to the introduction in [2] for the motivation, bibliographic comments as well as the explanation of the term "processes with immigration".

In this note we are interested in weak convergence of random processes with immigration in the situation when $\mathbb{E}[|X(t)|]$ is finite and, in some sense to be specified later, integrable on $[0, \infty)$ and $\mathbb{E}\xi < \infty$. Under these assumptions and the additional assumption that X and ξ are independent, a functional limit theorem has been obtained recently in [3]. In the present paper we prove a counterpart of that result discarding the independence assumption.

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Before we formulate our main result, some preliminary work has to be done. In Sections 1.1 and 1.2 we recall some necessary information regarding the Skorokhod space $D(\mathbb{R})$ and the space of marked point processes. In Section 1.3 we present the construction of the stationary marked renewal process.

1.1. **Convergence in** $D(\mathbb{R})$**.** [1] Consider the subset D_0 of the Skorokhod space $D(\mathbb{R})$ composed of those functions $f \in D(\mathbb{R})$ which have finite limits $f(-\infty) := \lim_{t\to-\infty} f(t)$ and $f(\infty) := \lim_{t\to+\infty} f(t)$. For $a, b \in \mathbb{R}$, $a < b$ let $d_0^{a,b}$ be the Skorokhod metric on $D[a, b]$, i.e.,

$$d_0^{a,b}(x, y) = \inf_{\lambda\in\Lambda_{a,b}} \Big(\sup_{t\in[a,b]} |x(\lambda(t)) - y(t)| \vee \sup_{s\neq t} \Big| \log \Big(\frac{\lambda(t) - \lambda(s)}{t - s}\Big)\Big|\Big)$$

where $\Lambda_{a,b} = \{\lambda : \lambda$ is a strictly increasing and continuous function on $[a, b]$ with $\lambda(a) = a, \lambda(b) = b\}$. Following [5, Section 3], for $f, g \in D_0$, put

$$d_0(f, g) := d_0^{0,1}(\overline{\phi}(f), \overline{\phi}(g)),$$

where

$$\phi(t) := \log(t/(1 - t)),\ \ t \in (0, 1),\ \ \phi(0) := -\infty,\ \ \phi(1) := +\infty$$

and

$$\overline{\phi} : D_0 \to D[0, 1],\ \ \overline{\phi}(x)(\cdot) := x(\phi(\cdot)),\ \ x \in D_0.$$

Then (D_0, d_0) is a complete separable metric space. Mimicking the argument given in [5, Section 4] and using d_0 as a basis one can construct a metric d (its explicit form is of no importance here) on $D(\mathbb{R})$ such that $(D(\mathbb{R}), d)$ is a complete separable metric space. We shall need the following characterization of the convergence on $(D(\mathbb{R}), d)$, see Theorem 1(b) in [5] and Theorem 12.9.3(ii) in [12] for the convergence on $D[0, \infty)$.

Proposition 1.1. *Suppose $f_n, f \in D(\mathbb{R})$, $n \in \mathbb{N}$. The following conditions are equivalent:*

(i) *$f_n \to f$ in $(D(\mathbb{R}), d)$ as $n \to \infty$;*

(ii) *there exist $\lambda_n \in \Lambda$, where*

$$\Lambda := \{\lambda : \lambda \textit{ is a strictly increasing and continuous function on } \mathbb{R} \textit{ with } \lambda(\pm\infty) = \pm\infty\},$$

such that, for any finite a and b, $a < b$,

$$\lim_{n\to\infty} \max\Big\{ \sup_{u\in[a,\,b]} |f_n(\lambda_n(u)) - f(u)|,\ \sup_{u\in[a,\,b]} |\lambda_n(u) - u|\Big\} = 0;$$

(iii) *for any finite a and b, $a < b$ which are continuity points of f it holds that $f_n|_{[a,\,b]} \to f|_{[a,\,b]}$ in $(D[a, b], d_0^{a,b})$ as $n \to \infty$, where $g|_{[a,\,b]}$ denotes the restriction of $g \in D(\mathbb{R})$ to $[a, b]$.*

1.2. **Marked point processes and their convergence.** In this section we recall the notion of a marked point process on $\mathbb{R}$ along with the corresponding canonical spaces. We refer to the books [6] and [9] for the comprehensive exposition of the theory of marked point processes.

Let (K, ρ_K) be an arbitrary complete separable metric space and let $(\mathbb{R} \times K, \rho)$ be the product of $\mathbb{R}$ and K endowed with the product topology:

$$\rho((x_1, k_1), (x_2, k_2)) = |x_1 - x_2| + \rho_K(k_1, k_2),\quad x_1, x_2 \in \mathbb{R},\quad k_1, k_2 \in K.$$

Let M_K be the set of integer-valued measures m on $(\mathbb{R} \times K, \mathcal{B}(\mathbb{R} \times K))$ such that $m(\mathbb{R} \times K) = \infty$ and, for every bounded set $A \in \mathcal{B}(\mathbb{R})$,

$$m(A \times K) < \infty,$$

[1]This material was borrowed from [3] and is given here for the ease of reference.

where $\mathcal{B}(\mathbb{X})$ denotes the Borel sigma-algebra of the metric space $\mathbb{X}$. An arbitrary $m \in M_K$ can be represented as a countable sum of Dirac point measures on $\mathbb{R} \times K$:

$$m = \sum_{n\in\mathbb{Z}} \delta_{(t_n, k_n)}, \tag{1}$$

where the first coordinates can be arranged in the non-decreasing order:

$$\dots \le t_{-2} \le t_{-1} < 0 \le t_0 \le t_1 \le t_2 \cdots . \tag{2}$$

The elements of M_K are called *marked point processes* ("mpp" in what follows). A mpp m is called simple if the corresponding sequence $(t_n)_{n\in\mathbb{Z}}$ is strictly increasing in which case representation (1) subject to constraints (2) is unique. The sequence $(t_n)_{n\in\mathbb{Z}}$ represents the arrival epochs and the sequence (k_n) represents the marks, so that k_n is the mark brought by the arrival at time t_n. The space K is called *the mark space.*

It is known (see, for instance, [6, Chapter 1.15]) that M_K can be endowed with a structure of complete separable metric space, more precisely there exists a metric ρ_{M_K} such that:

- (M_K, ρ_{M_K}) is a complete separable metric space;
- $\rho_{M_K}(m_n, m) \to 0$ as $n \to \infty$ iff
$$\int f(x) m_n(\mathrm{dx}) \to \int f(x) m(\mathrm{dx}), \quad n \to \infty$$
for every continuous function $f : \mathbb{R} \times K \to \mathbb{R}^+$ with bounded support.

The following proposition gives another characterization of the convergence on space (M_k, ρ_{M_K}) (see Theorem D.1 and Corollary D.2 in [9, Appendix D]).

Proposition 1.2. *A sequence $m_n := \sum_{j\in\mathbb{Z}} \delta_{(t_j^{(n)}, k_j^{(n)})}$, $n \in \mathbb{N}$, of simple mpp's with the increasing enumeration of arrival epochs*

$$\dots < t_{-2}^{(n)} < t_{-1}^{(n)} < 0 \le t_0^{(n)} < t_1^{(n)} < t_2^{(n)} < \cdots, \quad n \in \mathbb{N}$$

converges in (M_K, ρ_{M_K}) to a simple mpp $m := \sum_{j\in\mathbb{Z}} \delta_{(t_j, k_j)}$ satisfying (2) *iff*

$$((t_{-q}^{(n)}, k_{-q}^{(n)}), \dots, (t_p^{(n)}, k_p^{(n)})) \to ((t_{-q}, k_{-q}), \dots, (t_p, k_p)), \quad n \to \infty$$

for every $p, q \in \mathbb{N}$.

1.3. **Stationary marked renewal point processes and stationary random processes with immigration.** Suppose that $\mu := \mathbb{E}\xi < \infty$ and that the distribution of ξ is nonlattice, i.e., it is not concentrated on any lattice $d\mathbb{Z}$, $d > 0$. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the sequence $(\xi_k, X_k)_{k\in\mathbb{N}}$ lives, define the following objects:

- an independent copy $(X_{-k}, \xi_{-k})_{k\in\mathbb{N}}$ of $(X_k, \xi_k)_{k\in\mathbb{N}}$;
- a pair (X_0, ξ_0) which is independent of $(X_k, \xi_k)_{k\in\mathbb{Z}\setminus\{0\}}$ and has joint distribution

$$\mathbb{P}\{\xi_0 \le x, X_0 \in \cdot\} = \frac{1}{\mu} \int_{[0,\, x]} y \mathbb{P}\{\xi \in \mathrm{d}y, X \in \cdot\}, \quad x \ge 0; \tag{3}$$

- a random variable U, independent of $(X_k, \xi_k)_{k\in\mathbb{Z}}$, with the uniform distribution on $[0, 1]$.

Set

$$S_{-k} := -(\xi_{-1} + \ldots + \xi_{-k}), \quad k \in \mathbb{N}$$

and

$$S_0^* := U\xi_0, \quad S_{-1}^* := -(1-U)\xi_0, \quad S_k^* := S_0^* + S_k, \quad S_{-k-1}^* := S_{-1}^* + S_{-k}, \quad k \in \mathbb{N}.$$

The (unmarked) point process

$$\mathcal{S} := \sum_{n\in\mathbb{Z}} \delta_{S_n^*}$$

is called *stationary renewal point process.* This process is shift invariant, i.e., $\sum_{n\in\mathbb{Z}} \delta_{S_n^*+t}$ has the same distribution as $\sum_{n\in\mathbb{Z}} \delta_{S_n^*}$ for all $t \in \mathbb{R}$ and its intensity measure is

$$\mathbb{E}\mathcal{S}(A) = \mu^{-1}|A|, \quad A \in \mathcal{B}(\mathbb{R}), \tag{4}$$

where $|A|$ is the Lebesgue measure of A. Now consider the process X_k as a mark brought by the point that arrived at time S_k^*, so that the mark space K is $D(\mathbb{R})$. It is natural to call the mpp

$$\mathcal{S}^{\mathcal{M}} := \sum_{n\in\mathbb{Z}} \delta_{(S_n^*,\, X_n)}$$

two-sided stationary marked renewal process. Note that it is simple with probability one in view of the assumption $\mathbb{P}\{\xi > 0\} = 1$. The process $\mathcal{S}^{\mathcal{M}}$ is stationary (see Example 1 on pp. 27-29 in [9] for the one-sided case) in the following sense:

$$\mathcal{S}^{\mathcal{M}}(\{A + t\} \times B) \overset{\mathrm{d}}{=} \mathcal{S}^{\mathcal{M}}(A \times B),$$

for arbitrary $A \in \mathcal{B}(\mathbb{R})$, $B \in \mathcal{B}(D(\mathbb{R}))$ and $t \in \mathbb{R}$, where $\{A + t\} := \{a + t : a \in A\}$ and $\overset{\mathrm{d}}{=}$ denotes equality of distributions. Equivalently,

$$\sum_{k\in\mathbb{Z}} \delta_{(S_k^*-t, X_k)} \overset{\mathrm{d}}{=} \sum_{k\in\mathbb{Z}} \delta_{(S_k^*, X_k)}, \quad t \in \mathbb{R}. \tag{5}$$

This fact is a simple consequence of Lemma 3.1 below (see Remark 3.1). It can also be deduced from the Palm theory of stationary point processes, see e.g. Chapter 4.8 in [10].

From the construction above it is clear that the distribution of the stationary renewal point process is symmetric around the origin:

$$\sum_{n\in\mathbb{Z}} \delta_{S_n^*} \overset{\mathrm{d}}{=} \sum_{n\in\mathbb{Z}} \delta_{-S_{-n-1}^*}. \tag{6}$$

For every $k \in \mathbb{Z}$ the mark X_k depends only on the interarrival time $\xi_k = S_k^* - S_{k-1}^*$. This observation together with (6) allows us to conclude that

$$\sum_{n\in\mathbb{Z}} \delta_{(S_n^*, X_n)} \overset{\mathrm{d}}{=} \sum_{n\in\mathbb{Z}} \delta_{(S_n^*, X_{n+1})}. \tag{7}$$

Fix any $u \in \mathbb{R}$. Since $\lim_{k\to-\infty} S_k^* = -\infty$ a.s. (almost surely), the sum

$$\sum_{k\le -1} X_k(u + S_k^*)\, \mathbb{1}_{\{S_k^* \ge -u\}}$$

is a.s. finite because the number of non-zero summands is a.s. finite. Define

$$Y^*(u) := \sum_{k\in\mathbb{Z}} X_k(u + S_k^*) = \sum_{k\in\mathbb{Z}} X_k(u + S_k^*)\, \mathbb{1}_{\{S_k^* \ge -u\}}$$

with the random variable $Y^*(u)$ being a.s. finite provided that the series $\sum_{k\ge 0} X_k(u + S_k^*)\, \mathbb{1}_{\{S_k \ge -u\}}$ converges in probability. The process $Y^* := (Y^*(u))_{u\in\mathbb{R}}$ is called *stationary random process with immigration.* In view of (5) the process Y^* is strictly stationary in the usual sense.

Note that from (7) we obtain

$$Y^*(\cdot) \overset{\mathrm{d}}{=} \sum_{k\in\mathbb{Z}} X_{k+1}(\cdot + S_k^*) = \sum_{k\in\mathbb{Z}} X_{k+1}(\cdot + S_k^*)\, \mathbb{1}_{\{S_k^* \ge -u\}}.$$

In [3] the stationary process with immigration has been defined by the series on the right-hand side of the last equality. In some situations this representation is more convenient. For example, it shows that $Y^*(u) \overset{\mathrm{d}}{=} Y^*(0) \overset{\mathrm{d}}{=} \sum_{k\ge 0} X_{k+1}(S_k^*)$. In particular, (4) readily implies

$$\mathbb{E}Y^*(u) = \mathbb{E}Y^*(0) = \int_0^\infty \mathbb{E}X(t)\mathrm{d}t,$$

since for $k \geq 0$ the mark X_{k+1} is independent[2] of S_k^*.

1.4. **Main result.** In the following we write '$Z_t \Rightarrow Z$ as $t \to \infty$ on (S, d^*)' to denote weak convergence of processes on a complete separable metric space (S, d^*) and '$\overset{d}{\to}$' to denote convergence in distribution of random variables or random vectors.

Let $\mathfrak{A}$ be the support of the discrete component of ξ, i.e., $\mathfrak{A} := \{t > 0 : \mathbb{P}\{\xi = t\} > 0\}$, and set

$$< \mathfrak{A} >:= \Big\{ \sum_i n_i a_i : a_i \in \mathfrak{A}, n_i \in \mathbb{N}_0 \Big\}.$$

Observe that $a \in < \mathfrak{A} >$ iff $(S_j)_{j\in\mathbb{N}_0}$ hits a with positive probability. Further, let

$$(8)\quad \mathcal{D} := \{t \in \mathbb{R} : \mathbb{P}\{X(t) \neq X(t-)\} > 0\},\ \mathcal{D}_\xi := \{t \in \mathbb{R} : \mathbb{P}\{X(\xi+t) \neq X(\xi+t-)\} > 0\}$$

and

$$\Delta_X := \mathcal{D}_\xi \ominus \mathcal{D} := \{a - b : a \in \mathcal{D}_\xi, b \in \mathcal{D}\}.$$

Theorem 1.1. *Suppose that $\mu := \mathbb{E}\xi < \infty$ and the distribution of ξ is nonlattice.*

(a) *If the function $G(t) := \mathbb{E}[|X(t)| \wedge 1]$ is directly Riemann integrable[3] (dRi) on $[0, \infty)$, then, for each $u \in \mathbb{R}$, the series $\sum_{k\geq 0} X_k(u + S_k^*)\,\mathbb{1}_{\{S_k^* \geq -u\}}$ is absolutely convergent with probability one, and, for any $n \in \mathbb{N}$ and any finite $u_1 < u_2 < \ldots < u_n$ such that Y^* is almost surely continuous at u_i,*

$$(9)\qquad \big(Y(t + u_1), \ldots, Y(t + u_n)\big) \overset{\mathrm{d}}{\to} \big(Y^*(u_1), \ldots, Y^*(u_n)\big), \quad t \to \infty.$$

(b) *If, for some $\varepsilon > 0$, the function $H_\varepsilon(t) := \mathbb{E}[\sup_{u\in[t,\,t+\varepsilon]} |X(u)| \wedge 1]$ is dRi on $[0, \infty)$, and*

$$(10)\qquad < \mathfrak{A} > \cap\, \Delta_X = \varnothing,$$

then

$$(11)\qquad Y(t + u) \Rightarrow Y^*(u), \quad t \to \infty \quad \textit{on } (D(\mathbb{R}), d).$$

Remark 1.1. Condition (10) needs to be checked only if the distribution of ξ has a discrete component. Otherwise, it holds automatically. If ξ and X are independent, condition (10) can replaced by the following simpler condition

$$(12)\qquad < \mathfrak{A} > \cap\, \Delta'_X = \{0\},$$

where $\Delta'_X := \mathcal{D} \ominus \mathcal{D}$. It can be verified that (12) is equivalent to formula (4) in [3].

Remark 1.2. The underlying idea which is used in [3] to derive a counterpart of Theorem 1.1 for independent X and ξ is based on the continuous mapping theorem applied to the joint convergence

$$(13)\qquad \Big(\sum_{k\geq 0} \delta_{t-S_k}, (X_{k+1})_{k\in\mathbb{Z}}\Big) \Rightarrow \Big(\sum_{k\in\mathbb{Z}} \delta_{S_k^*}, (X_{k+1})_{k\in\mathbb{Z}}\Big), \quad t \to \infty,$$

see (19) in [3]. For independent X and ξ the latter relation is a direct consequence of the convergence of point processes

$$\sum_{k\geq 0} \delta_{t-S_k} \Rightarrow \sum_{k\in\mathbb{Z}} \delta_{S_k^*}, \quad t \to \infty.$$

In the general situation one possible way to capture the dependency structure in the left-hand side of (13) is to consider the processes (X_k) as *marks* of the points $(t - S_k)$ of the point process $\sum \delta_{t-S_k}$. Using this approach we have been able to replace (13) with a convergence of marked point processes $\sum \delta_{(t-S_k, X_{k+1})}$, see Lemma 3.1 below, and to identify the limit as the two-sided stationary marked renewal process $\mathcal{S}^{\mathcal{M}}$ defined in

[2]However, X_{k+1} depends on S_k^* for $k < 0$.

[3]See Chapter 3.10.1 in [8] for the definition of direct Riemann integrability.

Section 1.3. Once this basic convergence is settled, the remaining proof goes along the same lines as the proof of Theorem 2.2 in [3], however with many technical modifications.

2. Examples and applications

In this section we give a few examples from several areas of applied probability in which the random processes with immigration appear. Further examples can be found in Section 4 of [3].

2.1. $GI/G/\infty$ **queues.** Let (ξ, η) be a random vector with positive components and let $((\xi_k, \eta_k))_{k\in\mathbb{N}}$ be a sequence of independent copies of (ξ, η). Assuming that ξ and η are independent and interpreting ξ_k as the interarrival time between k-th and $(k+1)$-st customer and η_k as the service time of the k-th customer in the queuing system with infinitely many servers, we obtain the classical $GI/G/\infty$ queue. However, in real world situation the assumption that ξ_k and η_k are independent is not always adequate and one may consider, for example, the model with positively correlated ξ_k and η_k, i.e., the larger service time of the last arrived customer, the more time it takes for the next customer to arrive. One of several quantities of interest is the number $Y(t)$ of busy servers at the system at time t

$$Y(t) = \sum_{k\geq 0} \mathbb{1}_{\{S_k \leq t < S_k + \eta_{k+1}\}}$$

which is a particular instance of the random process with immigration with $X(t) := \mathbb{1}_{\{\eta > t \geq 0\}}$. Assuming that the distribution of ξ is nonlattice, has finite mean and condition (10) is fulfilled we can apply our Theorem 1.1 to deduce that (11) holds iff $\mathbb{E}\eta < \infty$. If the latter holds the queuing system possesses a stationary regime. In the case of independent ξ and η a one-dimensional version of this result has initially been proved in [4].

2.2. **Divergent perpetuities.** Let (ξ, η) be a random vector with $\mathbb{P}\{\xi > 0\} = 1$ and put $X(t) := \eta e^{-at}$, $t \geq 0$, where $a > 0$ is a fixed constant. Then, for every fixed $t \geq 0$, $Y(t)$ is the normalized truncated perpetuity given by

$$Y(t) = e^{-at} \sum_{k=0}^{\nu(t)-1} \eta_{k+1} e^{aS_k} = e^{-at} \sum_{k=0}^{\nu(t)-1} \eta_{k+1} \prod_{i=1}^{k} e^{a\xi_i}, \quad t \geq 0.$$

Assuming that $\mathbb{E}\xi < \infty$ and that the distribution of ξ is non-lattice we infer that (11) holds if $\mathbb{E}(\log^+ |\eta|) < \infty$. Indeed, since $\mathcal{D} = \{0\}$ and $\mathcal{D}_\xi = -\mathfrak{A}$ we conclude that $\Delta_X = -\mathfrak{A}$ which implies that (10) holds. Further, for every $\varepsilon > 0$, the function

$$H_\varepsilon(t) = \mathbb{E}[\sup_{u\in[t,t+\varepsilon]} |X(u)| \wedge 1] = \mathbb{E}[|\eta| e^{-at} \wedge 1], \quad t \geq 0$$

is non-increasing, hence dRi iff it is Lebesgue integrable iff $\mathbb{E}(\log^+ |\eta|) < \infty$. Whenever the latter holds the one-dimensional distribution of the limiting process can be characterized via the distribution of a suitable convergent perpetuity. By stationarity of Y^* we have, for $u > 0$,

$$Y^*(u) \overset{\mathrm{d}}{=} Y^*(0) = \sum_{k\geq 0} \eta_k e^{-aS_k^*} = \eta_0 e^{-aS_0^*} + e^{-aS_0^*} \sum_{k\geq 1} \eta_k e^{-aS_k} =: \eta_0 e^{-aS_0^*} + e^{-aS_0^*} A_\infty.$$

Here (η_0, S_0^*) is independent of A_∞, a perpetuity which satisfies the distributional equality

$$A_\infty \overset{\mathrm{d}}{=} \eta e^{-a\xi} + e^{-a\xi} A'_\infty,$$

where $A'_\infty \overset{\mathrm{d}}{=} A_\infty$ and A'_∞ is independent of (ξ, η).

2.3. **Continuous time random walks.** A continuous time random walk (*ctrw*, in short) $(C(t))_{t\in\mathbb{R}}$ is defined by

$$C(t) = \sum_{k=0}^{\nu(t)-1} J_{k+1},$$

where $(\nu(t))_{t\in\mathbb{R}}$ is as before and $(J_k)_{k\in\mathbb{N}}$ is a sequence of random displacements which is usually assumed to be comprised of i.i.d. random variables. $C(t)$ is then interpreted as the position at time t of a particle performing jumps of sizes J_{k+1} at the epochs S_k, $k = 0, \dots, \nu(t) - 1$. If J_k and ξ_k are independent for every $k \in \mathbb{N}$ the *ctrw* is called uncoupled, otherwise it is coupled. For the up-to-date exposition of the limit theory for *ctrw* we refer to [7] and references therein.

If X in the definition of the random process with immigration takes the form $X(t) \equiv \eta$, $t \geq 0$, for some random variable η, then Y is the classical *ctrw*. In general, the random process with immigration can be thought of as a generalized ctrw with time-inhomogeneous jumps in which the displacement size depends on the time. The situation considered in our Theorem 1.1 covers a very special class of such "generalized coupled continuous time random walks" with finite mean of the waiting times and rapidly decreasing displacement process X.

3. Proof of Theorem 1.1

Our proof relies on the sequence of auxiliary lemmas given next along with the continuous mapping arguments. In what follows we write M as a shorthand for $M_{D(\mathbb{R})}$ and ρ_M for the corresponding metric so that $(M_{D(\mathbb{R})}, \rho_m)$ is a complete separable metric space (see Section 1.2 above).

Lemma 3.1. *Assume that $\mathbb{E}\xi < \infty$ and that the distribution of ξ is nonlattice. Then, as $t \to \infty$,*

$$\sum_{k\in\mathbb{Z}} \delta_{(t-S_k, X_{k+1})} \Rightarrow \mathcal{S}^{\mathcal{M}} = \sum_{k\in\mathbb{Z}} \delta_{(S_k^*, X_k)} \tag{14}$$

on (M, ρ_M).

In view of Lemma 5.1 in [3], the result stated in Lemma 3.1 is quite expected. However, the rigorous proof is technically involved. Hence it is relegated to the Appendix.

Remark 3.1. Fix $u \in \mathbb{R}$. The shift-mapping $\theta_u : M \to M$ defined by $\theta_u(m(A \times B)) = m(\{A + u\} \times B)$, $A \in \mathcal{B}(\mathbb{R})$, $B \in \mathcal{M}(D(\mathbb{R}))$ is obviously continuous with respect to ρ_M for every $u \in \mathbb{R}$. Therefore, as $t \to \infty$,

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \delta_{(S_k^*, X_k)} \Leftarrow \sum_{k\in\mathbb{Z}} \delta_{(t-u-S_k, X_{k+1})} \quad &= \quad \theta_u\Big(\sum_{k\in\mathbb{Z}} \delta_{(t-S_k, X_{k+1})}\Big) \\ &\Rightarrow \quad \theta_u\Big(\sum_{k\in\mathbb{Z}} \delta_{(S_k^*, X_k)}\Big) = \sum_{k\in\mathbb{Z}} \delta_{(S_k^*-u, X_k)} \end{aligned}$$

on (M, ρ_M) which shows that $\mathcal{S}^{\mathcal{M}}$ is stationary.

Lemma 3.2. *The mapping $T : \mathbb{R} \times D(\mathbb{R}) \to D(\mathbb{R})$ defined by*

$$T(t_0, f(\cdot)) = f(t_0 + \cdot)$$

is measurable with respect to $\mathcal{B}(\mathbb{R}\times D(\mathbb{R}))$ and $\mathcal{B}(\mathbb{R})$ and continuous on $\mathbb{R}\times D(\mathbb{R})$ endowed with the product topology.

Proof. To show the measurability note that by Lemma 2.7 in [11] it is enough to show that $(t_0, f(\cdot)) \mapsto f(t_0 + t)$ is measurable for every $t \in \mathbb{R}$, but this is obvious since this mapping is a composition of two measurable mappings $(t_0, t) \mapsto t_0 + t$ and $f \mapsto f(t)$. The continuity has been proved in [3], see Lemma 5.2 therein. $\square$

For fixed $c > 0$, $l \in \mathbb{N}$ and $(u_1, \ldots, u_l) \in \mathbb{R}^l$, define the mapping $\phi_c^{(l)} : M \to \mathbb{R}^l$ by

$$\phi_c^{(l)}\Big(\sum_{n\in\mathbb{Z}} \delta_{(t_n, f_n(\cdot))}\Big) := \Big(\sum_{n\in\mathbb{Z}} f_n(t_n + u_j)\, \mathbb{1}_{\{|t_n|\le c\}}\Big)_{j=1,\ldots,l}$$

and the mapping $\phi_c : M \to D(\mathbb{R})$ by

$$\phi_c\Big(\sum_{n\in\mathbb{Z}} \delta_{(t_n, f_n(\cdot))}\Big) := \sum_{n\in\mathbb{Z}} f_n(t_n + \cdot)\, \mathbb{1}_{\{|t_n|\le c\}}\,.$$

For $f \in D(\mathbb{R})$, denote by $\mathrm{Disc}(f)$ the set of discontinuity points of f on $\mathbb{R}$. Clearly, both $\phi_c^{(l)}$ and ϕ_c are measurable as finite sums of measurable mappings. The continuity is provided by the next lemma.

Lemma 3.3. *The mapping $\phi_c^{(l)}$ is continuous at all points $m = \sum_{n\in\mathbb{Z}} \delta_{(t_n, f_n(\cdot))}$ such that m is simple, $m(\{-c, c\} \times D(\mathbb{R})) = 0$ and for which $u_1, \ldots, u_l$ are continuity points of $f_k(t_k + \cdot)$ for all $k \in \mathbb{Z}$. ϕ_c is continuous at all points $m = \sum_{n\in\mathbb{Z}} \delta_{(t_n, f_n(\cdot))}$ such that m is simple, $m(\{-c, c\} \times D(\mathbb{R})) = 0$ and $\mathrm{Disc}(f_k(t_k+\cdot)) \cap \mathrm{Disc}(f_j(t_j+\cdot)) = \varnothing$ for $k \ne j$.*

Proof. Let $c > 0$ and suppose that

$$\sum_{j\in\mathbb{Z}} \delta_{(t_j^{(n)}, f_j^{(n)})} =: m_n \to m =: \sum_{j\in\mathbb{Z}} \delta_{(t_j, f_j)}, \quad n \to \infty \tag{15}$$

on (M, ρ_M) where $m(\{-c, c\} \times D(\mathbb{R})) = 0$. Let $p, q \in \mathbb{N}_0$ be such that

$$t_{-p-1} < -c < t_{-p}, \qquad t_q < c < t_{q+1}. \tag{16}$$

From (15), using Proposition 1.2, we conclude that for $-p \le k \le q$

$$(t_k^{(n)}, f_k^{(n)}) \to (t_k, f_k), \quad n \to \infty \tag{17}$$

on $\mathbb{R} \times D(\mathbb{R})$ and also for large enough n

$$t_{-p-1}^{(n)} < -c < t_{-p}^{(n)}, \qquad t_q^{(n)} < c < t_{q+1}^{(n)}. \tag{18}$$

By Lemma 3.2 convergence (17) yields

$$f_k^{(n)}(t_k^{(n)} + \cdot) \to f_k(t_k + \cdot), \quad n \to \infty \tag{19}$$

on $D(\mathbb{R})$.

Now assume that $u_1, \ldots, u_l$ are continuity points of $f_k(t_k + \cdot)$ for all $k \in \mathbb{Z}$, then (19) implies that

$$\big(f_k^{(n)}(t_k^{(n)} + u_1), \ldots, f_k^{(n)}(t_k^{(n)} + u_l)\big) \;\to\; \big(f_k(t_k + u_1), \ldots, f_k(t_k + u_l)\big), \;\; n \to \infty$$

for $-p \le k \le q$. Summation of these relations over $k = -p, \ldots, q$ proves the continuity of $\phi_c^{(l)}$ in view of (16) and (18).

Theorem 4.1 in [11] tells us that addition on $D(\mathbb{R}) \times D(\mathbb{R})$ is continuous at those (x, y) for which $\mathrm{Disc}(x) \cap \mathrm{Disc}(y) = \varnothing$. Since this immediately extends to any finite number of summands we conclude that relations (19) entail

$$\phi_c(m_n) = \sum_{k=-p}^{q} f_k^{(n)}(t_k^{(n)} + \cdot) \;\to\; \sum_{k=-p}^{q} f_k(t_k + \cdot) = \phi_c(m), \;\; n \to \infty$$

on $D(\mathbb{R})$ provided that $\mathrm{Disc}(f_k(t_k + \cdot)) \cap \mathrm{Disc}(f_j(t_j + \cdot)) = \varnothing$ for $k \ne j$. □

Remark 3.2. The assumption that m is simple could have been omitted at the expense of a more involved proof. The present version of Lemma 3.3 serves our needs.

The next lemma relates the integrability of X to pathwise properties of Y^*, the stationary process with immigration.

Lemma 3.4. *Assume that $\mathbb{E}\xi < \infty$ and that the law of ξ is nonlattice.*

(i) *If* $G(t)=\mathbb{E}[|X(t)|\wedge 1]$ *is Lebesgue integrable on* $[0,\infty)$*, then* $|Y^*(u)|<\infty$ *for every* $u\in\mathbb{R}$ *almost surely.*

(ii) *If, for some (hence all)* $\varepsilon>0$*, the function* $H_\varepsilon(t)=\mathbb{E}[\sup_{u\in[t,t+\varepsilon]}|X(u)|\wedge 1]$ *is dRi on* $[0,\infty)$*, then* $Y^*(u)$ *takes values in* $D(\mathbb{R})$ *almost surely.*

Proof of (i). Put $\widehat{G}(t):=\mathbb{E}[|X(\xi+t)|\wedge 1]$ and note that

$$\int_0^\infty \mathbb{E}[|X(\xi+t)|\wedge 1]\mathrm{d}t=\mathbb{E}\int_0^\infty[|X(t)|\wedge 1]\mathrm{d}t-\mathbb{E}\int_0^\xi[|X(t)|\wedge 1]\mathrm{d}t.$$

Since $\mathbb{E}\int_0^\xi[|X(t)|\wedge 1]\mathrm{d}t<\mathbb{E}\xi<\infty$ we see that the Lebesgue integrability of G is equivalent to the Lebesgue integrability of $\widehat{G}$. Fix $u\in\mathbb{R}$ and set $\mathcal{Z}_k:=X_k(u+S_k^*)\,\mathbb{1}_{\{S_k^*\geq -u\}}$, $k\in\mathbb{N}$. We have

$$\begin{aligned}
\sum_{k\in\mathbb{N}}\mathbb{E}[|\mathcal{Z}_k|\wedge 1] &= \sum_{k\in\mathbb{N}}\mathbb{E}[(|X_k(\xi_k+u+S_{k-1}^*)|\wedge 1)\,\mathbb{1}_{\{S_k^*\geq -u\}}]\\
&= \sum_{k\in\mathbb{N}}\mathbb{E}[(|X_k(\xi_k+u+S_{k-1}^*)|\wedge 1)\,\mathbb{1}_{\{S_{k-1}^*\geq -u\}}]\\
&+ \sum_{k\in\mathbb{N}}\mathbb{E}[(|X_k(\xi_k+u+S_{k-1}^*)|\wedge 1)\,\mathbb{1}_{\{S_k^*\geq -u>S_{k-1}^*\}}]\\
&\leq \sum_{k\in\mathbb{Z}}\mathbb{E}[\widehat{G}(u+S_{k-1}^*)]\,\mathbb{1}_{\{S_{k-1}^*\geq -u\}}]+\sum_{k\in\mathbb{N}}\mathbb{P}\{S_k^*\geq -u>S_{k-1}^*\}\\
&= \frac{1}{\mu}\int_0^\infty\widehat{G}(s)\mathrm{d}s+\mathbb{P}\{-u>S_0^*\}
\end{aligned}$$

having utilized (4) for the last equality and independence of X_k and S_{k-1}^* for $k\in\mathbb{N}$. Therefore the following two series

$$\sum_{k\in\mathbb{N}}\mathbb{P}\{|\mathcal{Z}_k|\geq 1\}\quad\text{and}\quad\sum_{k\in\mathbb{N}}\mathbb{E}(|\mathcal{Z}_k|\,\mathbb{1}_{\{|\mathcal{Z}_k|\leq 1\}})$$

converge. The convergence of the first series together with the Borel-Cantelli lemma implies that $|\mathcal{Z}_k|\geq 1$ for only finitely many k a.s., while the convergence of the second implies that $\sum_{k\in\mathbb{N}}|\mathcal{Z}_k|\,\mathbb{1}_{\{|\mathcal{Z}_k|\leq 1\}}<\infty$ a.s. Hence $\sum_{k\in\mathbb{N}}|\mathcal{Z}_k|<\infty$ a.s. and $|Y^*(u)|<\infty$ a.s. since $\sum_{k\leq 0}\mathcal{Z}_k$ contains only finitely many summands for every fixed $u\in\mathbb{R}$.

Proof of (ii). Again we start with the proof that dRi of H_ε is equivalent to dRi of $\widehat{H}_\varepsilon(t):=\mathbb{E}[\sup_{u\in[t,t+\varepsilon]}|X(\xi+u)|\wedge 1]$. Note that $X(\xi+\cdot)$ with probability one takes values in $D(\mathbb{R})$, hence, using exactly the same arguments as in the second paragraph of Section 3 in [3], we conclude that dRi of $\widehat{H}_\varepsilon$ is equivalent to

$$\sum_{k\geq 0}\mathbb{E}\Big[\sup_{t\in[k,k+1]}(|X(\xi+t)|\wedge 1)\Big]<\infty. \tag{20}$$

We have[4]

$$\begin{aligned}
\sum_{k\geq 0}\mathbb{E}\Big[\sup_{t\in[k,k+1]}(|X(\xi+t)|\wedge 1)\Big] &= \sum_{k\geq 0}\mathbb{E}\Big[\sup_{t\in[\xi+k,\xi+k+1]}(|X(t)|\wedge 1)\Big]\\
&\leq \sum_{k\geq 0}\mathbb{E}\Big[\sup_{t\in[\lfloor\xi\rfloor+k,\lfloor\xi\rfloor+k+2]}(|X(t)|\wedge 1)\Big]=\mathbb{E}\sum_{k\geq\lfloor\xi\rfloor}\Big[\sup_{t\in[k,k+2]}(|X(t)|\wedge 1)\Big]\\
&= \mathbb{E}\sum_{k\geq 0}\Big[\sup_{t\in[k,k+2]}(|X(t)|\wedge 1)\Big]-\mathbb{E}\sum_{k=0}^{\lfloor\xi\rfloor-1}\Big[\sup_{t\in[k,k+2]}(|X(t)|\wedge 1)\Big]
\end{aligned}$$

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[4]We use the notation $\lfloor x\rfloor:=\sup\{k\in\mathbb{Z}:k\leq x\}$, $x\in\mathbb{R}$.

Since $\mathbb{E}\sum_{k=0}^{\lfloor\xi\rfloor-1}\Big[\sup_{t\in[k,k+2]}(|X(t)|\wedge 1)\Big] \le \mathbb{E}\lfloor\xi\rfloor \le \mathbb{E}\xi < \infty$ we conclude that (20) is equivalent to

$$\mathbb{E}\sum_{k\ge 0}\Big[\sup_{t\in[k,k+2]}(|X(t)|\wedge 1)\Big] < \infty$$

which, in its turn, is equivalent to dRi of H_ε (see formulae (7) and (8) in [3]). Hence (20) holds.

In order to show that Y^* takes values in $D(\mathbb{R})$ with probability one, we use the fact that locally uniform limits of elements from $D(\mathbb{R})$ are again in $D(\mathbb{R})$. Therefore it is enough to check that the series $Y^*(u) = \sum_{k\in\mathbb{Z}} X_k(u+S_k^*)$ converges uniformly on every compact interval a.s. which is a consequence of the a.s. convergence of the series

$$\sum_{k\in\mathbb{Z}}\sup_{u\in[a,b]}|X_k(u+S_k^*)| \tag{21}$$

for any fixed $a<b$. Arguing in the same vein as in the proof of part (i) above, we see that it suffices to show

$$\mathbb{E}\sum_{k\in\mathbb{Z}}\sup_{u\in[a,b]}(|X_k(\xi_k+u+S_{k-1}^*)|\wedge 1) < \infty,$$

In view of the independence of X_k and S_{k-1}^* for $k\in\mathbb{N}$, we observe that

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{k\in\mathbb{N}}\sup_{u\in[a,b]}(|X_k(\xi_k+u+S_{k-1}^*)|\wedge 1)\Big]\\
&\qquad = \mathbb{E}\Big[\sum_{k\in\mathbb{N}}\mathbb{E}\Big[\sup_{u\in[a,b]}(|X(\xi+u+S_{k-1}^*)|\wedge 1)\,\Big|\,S_{k-1}^*\Big]\Big]\\
&\qquad \le \mathbb{E}\Big[\sum_{k\in\mathbb{Z}}\mathbb{E}\Big[\sup_{u\in[a,b]}(|X(\xi+u+S_{k-1}^*)|\wedge 1)\,\Big|\,S_{k-1}^*\Big]\Big]\\
&\qquad = \frac{1}{\mu}\int_{\mathbb{R}}\mathbb{E}\Big[\sup_{u\in[a,b]}(|X(\xi+u+t)|\wedge 1)\Big]\,\mathrm{d}t\\
&\qquad \le \frac{1}{\mu}\sum_{k\in\mathbb{Z}}\sup_{t\in[k,\,k+1)}\mathbb{E}\Big[\sup_{u\in[a+t,\,b+t]}|X(\xi+u)|\wedge 1\Big]\\
&\qquad \le \frac{b-a+1}{\mu}\sum_{k\in\mathbb{Z}}\mathbb{E}\Big[\sup_{u\in[k,\,k+1)}|X(\xi+u)|\wedge 1\Big],
\end{aligned}$$

where the last equality is a consequence of (4) and the last inequality follows from (18) in [3]. The last series converges in view of (20). It remains to note that

$$\mathbb{E}\sum_{k<0}\sup_{u\in[a,b]}(|X_k(\xi_k+u+S_{k-1}^*)|\wedge 1) \le \sum_{k<0}\mathbb{P}\{b+S_k^*\ge 0\} < \infty.$$

Thus $\sum_{k\in\mathbb{Z}} X_k(u+S_k^*)$ converges uniformly on $[a,b]$ for all $a<b$ a.s. and therefore is $D(\mathbb{R})$-valued a.s. □

Lemma 3.5. *If* (10) *holds, then*

$$\mathbb{P}\{\mathrm{Disc}(X_i(S_i^*+\cdot))\cap\mathrm{Disc}(X_j(S_j^*+\cdot))\neq\varnothing\} = 0,\quad i,j\in\mathbb{Z},\quad i>j.$$

Proof. Define the following random sets

$$D_\xi^{(i)} := \mathrm{Disc}(X_i(\xi_i+\cdot)),\quad D^{(j)} := \mathrm{Disc}(X_j(\cdot)),\quad D^{(i,j)} := D_\xi^{(i)}\ominus D^{(j)},\quad i,j\in\mathbb{Z}$$

and observe that for every $t \in \mathbb{R}$ the events $\{t \in D_\xi^{(i)}\}$, $\{t \in D^{(j)}\}$ and $\{t \in D^{(i,j)}\}$ are measurable. Put

$$\mathcal{D}_\xi^{(i)} := \{t \in \mathbb{R} : \mathbb{P}\{t \in D_\xi^{(i)}\} > 0\}, \quad \mathcal{D}^{(j)} := \{t \in \mathbb{R} : \mathbb{P}\{t \in D^{(j)}\} > 0\},$$
$$\mathcal{D}^{(i,j)} := \{t \in \mathbb{R} : \mathbb{P}\{t \in D^{(i,j)}\} > 0\}, \quad i, j \in \mathbb{Z}.$$

Note that by definition

$$\mathbb{P}\{t \in D_\xi^{(i)} \setminus \mathcal{D}_i\} = \mathbb{P}\{t \in D^{(j)} \setminus \mathcal{D}_j\} = \mathbb{P}\{t \in D^{(i,j)} \setminus \mathcal{D}^{(i,j)}\} = 0, \quad i, j \in \mathbb{Z}, \tag{22}$$

for every fixed $t \in \mathbb{R}$. We have for $i > j$

$$\begin{aligned}
&\mathbb{P}\{\mathrm{Disc}(X_i(S_i^* + \cdot)) \cap \mathrm{Disc}(X_j(S_j^* + \cdot)) \neq \varnothing\} \\
&\quad = \mathbb{P}\{\text{there exists } u(\omega) \text{ such that } S_i^* + u(\omega) \in \mathrm{Disc}(X_i(\cdot)), S_j^* + u(\omega) \in \mathrm{Disc}(X_j(\cdot))\} \\
&\quad \leq \mathbb{P}\{S_i^* - S_j^* \in \mathrm{Disc}(X_i(\cdot)) \ominus \mathrm{Disc}(X_j(\cdot))\} \\
&\quad = \mathbb{P}\{\xi_i + \ldots + \xi_{j+1} \in \mathrm{Disc}(X_i(\cdot)) \ominus \mathrm{Disc}(X_j(\cdot))\} \\
&\quad = \mathbb{P}\{\xi_{i-1} + \ldots + \xi_{j+1} \in \mathrm{Disc}(X_i(\xi_i + \cdot) \ominus \mathrm{Disc}(X_j(\cdot))\} \\
&\quad = \mathbb{P}\{\xi_{i-1} + \ldots + \xi_{j+1} \in D^{(i,j)}\} \\
&\quad \leq \mathbb{P}\{\xi_{i-1} + \ldots + \xi_{j+1} \in \mathcal{D}^{(i,j)}\} + \mathbb{P}\{\xi_{i-1} + \ldots + \xi_{j+1} \in D^{(i,j)} \setminus \mathcal{D}^{(i,j)}\},
\end{aligned}$$

Since $\xi_{j+1}, \ldots, \xi_{i-1}$ are independent of $D^{(i,j)}$, the second term vanishes in view of (22).

Pick an arbitrary $t_0 \in \mathcal{D}^{(i,j)}$. By definition we have

$$\begin{aligned}
0 &< \mathbb{P}\{t_0 \in \mathrm{Disc}(X_i(\xi_i + \cdot)) \ominus \mathrm{Disc}(X_j(\cdot))\} \\
&\leq \mathbb{1}_{\{t_0 \in \mathcal{D}_\xi^{(i)} \ominus \mathcal{D}^{(j)}\}} + \mathbb{P}\{t_0 \in (\mathrm{Disc}(X_i(\xi_i + \cdot)) \setminus \mathcal{D}_\xi^{(i)}) \ominus (\mathrm{Disc}(X_j(\cdot)) \setminus \mathcal{D}^{(j)})\} \\
&+ \mathbb{P}\{t_0 \in \mathrm{Disc}(X_i(\xi_i + \cdot)) \ominus (\mathrm{Disc}(X_j(\cdot)) \setminus \mathcal{D}^{(j)})\} \\
&+ \mathbb{P}\{t_0 \in (\mathrm{Disc}(X_i(\xi_i + \cdot)) \setminus \mathcal{D}_\xi^{(i)}) \ominus \mathrm{Disc}(X_j(\cdot))\}.
\end{aligned}$$

The last three probabilities equal zero in view of the first two equalities in (22), the independence of $X_i(\xi_i + \cdot)$ and $X_j(\cdot)$ and the fact that both $D^{(j)}$ and $D_\xi^{(i)}$ are at most countable a.s. Hence $0 < \mathbb{1}_{\{t_0 \in \mathcal{D}_\xi^{(i)} \ominus \mathcal{D}^{(j)}\}}$ which implies $t_0 \in \mathcal{D}_\xi^{(i)} \ominus \mathcal{D}^{(j)}$ and thereupon

$$\mathcal{D}^{(i,j)} \subset \mathcal{D}_\xi^{(i)} \ominus \mathcal{D}^{(j)}.$$

Thus, we have proved that

$$\mathbb{P}\{\xi_{i-1} + \ldots + \xi_{j+1} \in \mathcal{D}^{(i,j)}\} \leq \mathbb{P}\{\xi_{i-1} + \ldots + \xi_{j+1} \in \mathcal{D}_\xi^{(i)} \ominus \mathcal{D}^{(j)}\}.$$

If $i, j \neq 0$, then $\mathcal{D}_\xi^{(i)} = \mathcal{D}_\xi$ and $\mathcal{D}^{(j)} = \mathcal{D}$ (see (8) for the definition of $\mathcal{D}_\xi$ and $\mathcal{D}$), and the probability on the right-hand side of the last centered inequality equals $\mathbb{P}\{\xi_{i-1} + \ldots + \xi_{j+1} \in \Delta_X\}$. The cases $i = 0$ or $j = 0$ require a separate treatment since the joint distribution of (X_0, ξ_0) is other than that of (X, ξ). We only treat the situation when $i = 0$ and show that $\mathcal{D}_\xi^{(0)} \subset \mathcal{D}_\xi$, the other case being similar. Assume that $t_0 \notin \mathcal{D}_\xi$ and consider the set $A_{t_0} := \{f(\cdot) \in D(\mathbb{R}) : f(t_0) \neq f(t_0-)\}$. Using (3) we have

$$\begin{aligned}
\mathbb{P}\{X_0(\xi_0 + \cdot) \in A_{t_0}\} &= \int_0^\infty \mathbb{P}\{X_0(\cdot) \in A_{t_0} \ominus \{y\}, \xi_0 \in \mathrm{d}y\} \\
&\overset{(3)}{=} \frac{1}{\mu} \int_0^\infty y \mathbb{P}\{\xi \in \mathrm{d}y, X(\cdot) \in A_{t_0} \ominus \{y\}\} \\
&= \frac{1}{\mu} \int_0^\infty y \mathbb{P}\{\xi \in \mathrm{d}y, X(\xi + \cdot) \in A_{t_0}\}.
\end{aligned}$$

The probability under the integral sign equals zero identically in view of $t_0 \notin \mathcal{D}_\xi$. Hence $\mathbb{P}\{X_0(\xi_0+\cdot)\in A_{t_0}\}=0$ which is equivalent to $t_0\notin \mathcal{D}_\xi^{(0)}$. This shows that $\mathcal{D}_\xi^{(0)}\subset\mathcal{D}_\xi$. In the same vein, we infer $\mathcal{D}^{(0)}\subset\mathcal{D}$ and therefore

$$\mathbb{P}\{\xi_{i-1}+\ldots+\xi_{j+1}\in\mathcal{D}_\xi^{(i)}\ominus\mathcal{D}^{(j)}\}\le\mathbb{P}\{\xi_{i-1}+\ldots+\xi_{j+1}\in\mathcal{D}_\xi\ominus\mathcal{D}\}$$

provided that $i=0$ or $j=0$. Combining pieces together we obtain

$$\mathbb{P}\{\mathrm{Disc}(X_i(S_i^*+\cdot))\cap\mathrm{Disc}(X_j(S_j^*+\cdot))\neq\varnothing\}\le\mathbb{P}\{\xi_{i-1}+\ldots+\xi_{j+1}\in\Delta_X\}.$$

If $j\ge 0$ or $i\le 0$, then $\xi_{i-1}+\ldots+\xi_{j+1}\overset{\mathrm{d}}{=}S_{i-j-1}$, whence $\mathbb{P}\{\xi_{i-1}+\ldots+\xi_{j+1}\in\Delta_X\}=0$ by (10). If $j<0<i$ the latter equality also holds since the support of the discrete component of ξ_0 is the same as that of ξ by (3). The proof of Lemma 3.5 is complete. $\square$

With these preparatory results at hand we are ready to prove Theorem 1.1.

Proof of (11). We shall use Lemma 3.3. To this end, observe that each S_j^* has an absolutely continuous distribution. In particular, $\mathcal{S}^{\mathcal{M}}$ is simple mpp and $\mathcal{S}^{\mathcal{M}}(\{-c,c\}\times D(\mathbb{R}))=0$ a.s. for every $c>0$. From Lemma 3.3 and Lemma 3.5, we see that ϕ_c is a.s. continuous at $\sum_{k\in\mathbb{Z}}\delta_{(S_k^*,X_k)}$ and therefore (14) implies

$$\begin{aligned}Y_c(t,\cdot) &:= \sum_{k\in\mathbb{Z}}X_{k+1}(t-S_k+\cdot)\,\mathbb{1}_{\{|t-S_k|\le c\}}=\phi_c\Big(\sum_{k\in\mathbb{Z}}\delta_{(t-S_k,X_{k+1})}\Big)\\ &\Rightarrow \phi_c\Big(\sum_{k\in\mathbb{Z}}\delta_{(S_k^*,X_k)}\Big)=\sum_{k\in\mathbb{Z}}X_k(S_k^*+\cdot)\,\mathbb{1}_{\{|S_k^*|\le c\}}=:Y_c^*(\cdot).\end{aligned}\tag{23}$$

on $(D(\mathbb{R}),d)$.

Using Proposition 1.1 we conclude that in order to prove (11) it suffices to check that

$$Y(t+u)\ \Rightarrow\ Y^*(u),\quad t\to\infty\tag{24}$$

on $(D[a,b],d_0^{a,b})$ for any a and b, $a<b$ which are not fixed discontinuities of Y^*. To this end, first observe that (23) implies

$$Y_c(t,\cdot)\ \Rightarrow\ Y_c^*(\cdot),\quad t\to\infty\tag{25}$$

on $(D[a,b],d_0^{a,b})$ for any a and b, $a<b$ which are not fixed discontinuities of Y_c^*.

From the proof of Lemma 3.4 we know that the series which defines $Y^*(u)$ converges locally uniformly in $u\in\mathbb{R}$. Hence for every fixed $t_0\in\mathbb{R}$ we have

$$\begin{aligned}\mathbb{P}\{t_0\in\mathrm{Disc}(Y^*(\cdot))\} &= \mathbb{P}\{\text{there exists } k\in\mathbb{Z}\text{ such that } t_0\in\mathrm{Disc}(X_k(S_k^*+\cdot))\}\\ &= \sum_{k\in\mathbb{Z}}\mathbb{P}\{t_0\in\mathrm{Disc}(X_k(S_k^*+\cdot))\}\\ &\ge \sum_{k\in\mathbb{Z}}\mathbb{P}\{t_0\in\mathrm{Disc}(X_k(S_k^*+\cdot)),|S_k^*|\le c\}\\ &= \mathbb{P}\{t_0\in\mathrm{Disc}(Y_c^*(\cdot))\}\end{aligned}$$

where the second and the last equalities follow from Lemma 3.5. The process $Y^*(\cdot)$ is stationary and is a.s. $D(\mathbb{R})$-valued by Lemma 3.4(ii). Hence $\mathbb{P}\{t_0\in\mathrm{Disc}(Y_c^*(\cdot))\}=\mathbb{P}\{t_0\in\mathrm{Disc}(Y^*(\cdot))\}=0$ which implies that (25) holds for all $a<b$.

Now (24) follows from Theorem 4.2 in [1] if we can prove that

$$Y_c^*\ \Rightarrow\ Y^*,\quad c\to\infty\tag{26}$$

on $(D[a,b],d_0^{a,b})$ and that

$$\lim_{c\to\infty}\limsup_{t\to\infty}\mathbb{P}\Big\{d_0^{a,b}(Y_c(t,\cdot),Y(t+\cdot))>\varepsilon\Big\}=0\tag{27}$$

for all $\varepsilon > 0$ and any $a, b \in \mathbb{R}$, $a < b$. Since $d_0^{a,b}$ is dominated by the uniform metric on $[a, b]$, relation (27) follows from

$$\lim_{c\to\infty} \limsup_{t\to\infty} \mathbb{P}\Big\{ \sup_{u\in[a,\,b]} \Big| \sum_{k\geq 0} X_{k+1}(u+t-S_k)\, \mathbb{1}_{\{|t-S_k|>c\}} + \sum_{k<0} X_{k+1}(u+t-S_k)\, \mathbb{1}_{\{|t-S_k|\leq c\}} \Big| > \varepsilon \Big\} = 0$$

for all $\varepsilon > 0$ and any $a, b \in \mathbb{R}$, $a < b$. The latter is a consequence of

$$\lim_{c\to\infty} \limsup_{t\to\infty} \mathbb{P}\Big\{ \sup_{u\in[a,\,b]} \Big| \sum_{k\geq 0} X_{k+1}(u+t-S_k)\, \mathbb{1}_{\{|t-S_k|>c\}} \Big| > \varepsilon \Big\} = 0 \tag{28}$$

and

$$\lim_{c\to\infty} \limsup_{t\to\infty} \mathbb{P}\Big\{ \sup_{u\in[a,\,b]} \Big| \sum_{k<0} X_{k+1}(u+t-S_k)\, \mathbb{1}_{\{|t-S_k|\leq c\}} \Big| > \varepsilon \Big\} = 0. \tag{29}$$

The proof of (28) given in Section 5 of [3], see formula (25) therein, works without any changes since X_{k+1} and S_k are independent for $k \geq 0$. The relation (29) is trivial since

$$\sum_{k<0} X_{k+1}(u+t-S_k)\, \mathbb{1}_{\{|t-S_k|\leq c\}} = 0 \tag{30}$$

for $t > c > 0$.

As for relation (26) we claim an even stronger statement

$$\sup_{u\in[a,b]} |Y_c^*(u) - Y^*(u)| \to 0, \quad c \to \infty \tag{31}$$

a.s. for all fixed a, b. Indeed,

$$\sup_{u\in[a,b]} |Y_c^*(u) - Y^*(u)| \leq \sum_{k\in\mathbb{Z}} \sup_{u\in[a,b]} |X_k(u + S_k^*)|\, \mathbb{1}_{\{|S_k^*|>c\}}$$

Invoking the monotone convergence theorem we deduce that the right-hand side tends to zero as $c \to \infty$ in view of (21).

Proof of (9). Fix $l \in \mathbb{N}$ and real numbers $\alpha_1, \ldots, \alpha_l$ and $u_1, \ldots, u_l$. According to Lemma 3.3, for every $c > 0$, the mapping $\phi_c^{(l)}$ is continuous at $\big(\sum_{k\in\mathbb{Z}} \delta_{S_k^*}, (X_{k+1})_{k\in\mathbb{Z}}\big)$ a.s. Now apply the continuous mapping theorem to (3.1) twice (first using the map $\phi_c^{(l)}$ and then the map $(x_1, \ldots, x_l) \mapsto \alpha_1 x_1 + \ldots + \alpha_l x_l$) to obtain

$$\sum_{i=1}^{l} \alpha_i Y_c(t, u_i) \ \overset{\mathrm{d}}{\to}\ \sum_{i=1}^{l} \alpha_i Y_c^*(u_i), \quad t \to \infty.$$

The proof of (9) is complete if we verify

$$\sum_{i=1}^{l} \alpha_i Y_c^*(u_i) \ \overset{\mathrm{d}}{\to}\ \sum_{i=1}^{l} \alpha_i Y^*(u_i), \quad c \to \infty \tag{32}$$

and

$$\lim_{c\to\infty} \limsup_{t\to\infty} \mathbb{P}\Big\{ \Big| \sum_{i=1}^{l} \Big(\alpha_i \sum_{k\geq 0} X_{k+1}(u_i+t-S_k)\, \mathbb{1}_{\{|t-S_k|>c\}} + \sum_{k<0} X_{k+1}(u_i+t-S_k)\, \mathbb{1}_{\{|t-S_k|\leq c\}} \Big) \Big| > \varepsilon \Big\} = 0 \tag{33}$$

for all $\varepsilon > 0$. As for (32), an even stronger statement holds: $Y_c^*(u) \to Y^*(u)$ as $c \to \infty$ a.s. for all $u \in \mathbb{R}$. This can be checked in the same way as (31) using the fact (see Lemma 3.4(i)) that

$$\mathbb{E}\Big[\sum_{k\in\mathbb{Z}} |X_k(u+S_k^*)| \wedge 1\Big] \ < \ \infty.$$

Further, (33) is a consequence of

$$\lim_{c\to\infty} \limsup_{t\to\infty} \mathbb{P}\Big\{\Big|\sum_{k\geq 0} X_{k+1}(u+t-S_k)\,\mathbb{1}_{\{|t-S_k|>c\}}\Big| > \varepsilon\Big\} = 0$$

which has been checked in [3] (see formula (27) therein and note that the independence between X and ξ was not used in that proof), and equality (30) which holds for $t > c > 0$. The proof of Theorem 1.1 is complete.

4. Appendix

Proof of Lemma 3.1. Observe that, for all $t \geq 0$,

$$\sum_{k\in\mathbb{Z}} \delta_{(t-S_k, X_{k+1})} = \sum_{k\in\mathbb{Z}} \delta_{(t-S_{\nu(t)-k-1}, X_{\nu(t)-k})}.$$

Hence, according to Proposition 1.2, it is enough to prove that, for all $p, q \in \mathbb{N}$, as $t \to \infty$,

(34)
$$\Big((t-S_{\nu(t)+p-1}, X_{\nu(t)+p}), \ldots, (t-S_{\nu(t)-q-1}, X_{\nu(t)-q})\Big) \xrightarrow{\mathrm{d}} \Big((S_{-p}^*, X_{-p}), \ldots, (S_q^*, X_q)\Big).$$

Let $y, z, y_1, \ldots, y_{p-1}, z_1, \ldots, z_q$ be arbitrary nonnegative numbers and $A_{-p}, \ldots, A_q$ arbitrary sets in $\mathcal{B}(D(\mathbb{R}))$. Define the events

$$\begin{aligned}
\mathcal{E}_1(t) &:= \{t-S_{\nu(t)-1} < z, t-S_{\nu(t)} \geq -y, X_{\nu(t)} \in A_0\},\\
\mathcal{E}_2(t) &:= \{\xi_{\nu(t)+1} \leq y_1, X_{\nu(t)+1} \in A_{-1}, \ldots, \xi_{\nu(t)+p-1} \leq y_{p-1}, X_{\nu(t)+p-1} \in A_{-p+1}\},\\
\mathcal{E}_3(t) &:= \{\xi_{\nu(t)-1} \leq z_1, X_{\nu(t)-1} \in A_1, \ldots, \xi_{\nu(t)-q} \leq z_q, X_{\nu(t)-q} \in A_q\},\\
\mathcal{E}_4(t) &:= \{X_{\nu(t)+p} \in A_{-p}\}.
\end{aligned}$$

Then (34) is equivalent to

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{P}\{\mathcal{E}_1(t)\cap\mathcal{E}_2(t)\cap\mathcal{E}_3(t)\cap\mathcal{E}_4(t)\} \ &= \ \mathbb{P}\{S_{-1}^* \geq -y, S_0^* < z, X_0 \in A_0\}\\
&\times\mathbb{P}\{\xi_{-1} \leq y_1, X_{-1} \in A_{-1}, \ldots, \xi_{-p+1} \leq y_{p-1}, X_{-p+1} \in A_{-p+1}\}\\
&\times\mathbb{P}\{\xi_1 \leq z_1, X_1 \in A_1, \ldots, \xi_q \leq z_q, X_q \in A_q\}\mathbb{P}\{X_{-p} \in A_{-p}\}\\
&= \mathbb{P}\{S_{-1}^* \geq -y, S_0^* < z, X_0 \in A_0\}\mathbb{P}\{X \in A_{-p}\}\\
&\times \prod_{i=1}^{p-1} \mathbb{P}\{\xi \leq y_i, X \in A_{-i}\} \prod_{i=1}^{q} \mathbb{P}\{\xi \leq z_i, X \in A_i\}.
\end{aligned}$$

The probability on the left-hand side can be replaced with $p(t) := \mathbb{P}\{\mathcal{E}_1(t)\cap\mathcal{E}_2(t)\cap\mathcal{E}_3(t)\cap \mathcal{E}_4(t), \nu(t) > q\}$, for $\mathbb{P}\{\nu(t) \le q\} \to 0$ as $t \to \infty$.Conditioning on $\nu(t)$ we obtain

$$\begin{aligned} p(t) &= \sum_{k\ge q+1} \mathbb{P}\{\mathcal{E}_1(t)\cap\mathcal{E}_2(t)\cap\mathcal{E}_3(t)\cap\mathcal{E}_4(t), \nu(t) = k\} \\ &= \sum_{k\ge q+1} \mathbb{P}\{\mathcal{E}_1(t)\cap\mathcal{E}_2(t)\cap\mathcal{E}_3(t)\cap\mathcal{E}_4(t), S_{k-1} \le t, S_k > t\} \\ &= \sum_{k\ge q+1} \mathbb{P}\{\mathcal{E}_1(t)\cap\mathcal{E}_3(t), S_{k-1} \le t, S_k > t\} \\ &\qquad\qquad \times\mathbb{P}\{X_{k+p} \in A_{-p}\} \prod_{i=1}^{p-1} \mathbb{P}\{\xi_{k+i} \le y_i, X_{k+i} \in A_{-i}\}. \end{aligned}$$

Thus, it remains to show that

$$\begin{aligned} (35)\quad \lim_{t\to\infty} \sum_{k\ge q+1} &\mathbb{P}\{\mathcal{E}_1(t)\cap\mathcal{E}_3(t), S_{k-1} \le t, S_k > t\} \\ &= \mathbb{P}\{-S^*_{-1} \le y, S^*_0 < z, X_0 \in A_0\} \prod_{i=1}^{q} \mathbb{P}\{\xi \le z_i, X \in A_i\}. \end{aligned}$$

We continue as follows

$$\begin{aligned} &\mathbb{P}\{\mathcal{E}_1(t)\cap\mathcal{E}_3(t), S_{k-1} \le t, S_k > t\} \\ &\quad = \mathbb{P}\{t - z < S_{k-1} \le t, t < S_k \le t + y, X_k \in A_0, \\ &\qquad\quad \xi_{k-1} \le z_1, X_{k-1} \in A_1, \dots, \xi_{k-q} \le z_q, X_{k-q} \in A_q\} \\ &\quad = \int_{[0,\,t]} \mathbb{P}\{t - z - v < \xi_{k-1} + \ldots + \xi_{k-q} \le t - v, t - v < \xi_k + \ldots + \xi_{k-q} \le t + y - v, \\ &\qquad\quad X_k \in A_0, \xi_{k-1} \le z_1, X_{k-1} \in A_1, \dots, \xi_{k-q} \le z_q, X_{k-q} \in A_q\}\mathbb{P}\{S_{k-q-1} \in \mathrm{d}v\} \\ &\quad = \int_{[0,\,t]} \mathbb{P}\{t - z - v < \xi_1 + \ldots + \xi_q \le t - v, t - v < \xi_1 + \ldots + \xi_q + \hat{\xi} \le t + y - v, \\ &\qquad\quad \hat{X} \in A_0, \xi_1 \le z_1, X_1 \in A_1, \dots, \xi_q \le z_q, X_q \in A_q\}\mathbb{P}\{S_{k-q-1} \in \mathrm{d}v\}, \\ &\quad =: \int_{[0,\,t]} \hat{p}(t-v)\mathbb{P}\{S_{k-q-1} \in \mathrm{d}v\} \end{aligned}$$

where

$$\begin{aligned} \hat{p}(u) := \mathbb{P}\{u - z < S_q \le u, u < S_q + \hat{\xi} \le u + y, \hat{X} \in A_0, \\ \xi_1 \le z_1, X_1 \in A_1, \dots, \xi_q \le z_q, X_q \in A_q\}, \end{aligned}$$

and $(\hat{X}, \hat{\xi})$ has the same distribution as (X, ξ) and is independent of everything else.

The nonnegative function $\hat{p}(u)$ is dRi because it is locally Riemann integrable and bounded from above by a nonincreasing integrable function $u \mapsto \mathbb{P}\{S_q + \hat{\xi} > u\}$. By the key renewal theorem, we infer

$$\begin{aligned} \sum_{k\ge q+1} \mathbb{P}\{\mathcal{E}_1(t)\cap\mathcal{E}_3(t), S_{k-1} \le t, S_k > t\} &= \sum_{k\ge q+1} \int_{[0,\,t]} \hat{p}(t-v)\mathbb{P}\{S_{k-q-1} \in \mathrm{d}v\} \\ &\to \frac{1}{\mu}\int_0^\infty \hat{p}(u)\mathrm{d}u \end{aligned}$$

as $t \to \infty$. Hence proving (35) amounts to checking the equality

$$(36)\qquad \mathbb{P}\{-S^*_{-1} \le y, S^*_0 < z, X_0 \in A_0\} \prod_{i=1}^{q} \mathbb{P}\{\xi \le z_i, X \in A_i\} = \frac{1}{\mu}\int_0^\infty \hat{p}(u)\mathrm{d}u.$$

Set $\mathcal{E} := \{\xi_1 \le z_1, X_1 \in A_1, \dots, \xi_q \le z_q, X_q \in A_q\}$ and rewrite $\hat{p}(u)$ as follows:

$$\begin{aligned}\hat{p}(u) &= \mathbb{P}\{u - z\wedge\hat{\xi} < S_q \le u - (\hat{\xi}-y)_+, z\wedge\hat{\xi} \ge (\hat{\xi}-y)_+, \hat{X}\in A_0, \mathcal{E}\}\\ &= \mathbb{P}\{S_q + z\wedge\hat{\xi} > u, z\wedge\hat{\xi} \ge (\hat{\xi}-y)_+, \hat{X}\in A_0, \mathcal{E}\}\\ &- \mathbb{P}\{S_q + (\hat{\xi}-y)_+ > u, z\wedge\hat{\xi} \ge (\hat{\xi}-y)_+, \hat{X}\in A_0, \mathcal{E}\}.\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{\mu}\int_0^\infty \hat{p}(u)\mathrm{d}u &= \frac{1}{\mu}\mathbb{E}\Big((S_q + z\wedge\hat{\xi})\,\mathbb{1}_{\{z\wedge\hat{\xi}\ge(\hat{\xi}-y)_+, \hat{X}\in A_0, \mathcal{E}\}}\\ &\qquad -(S_q + (\hat{\xi}-y)_+)\,\mathbb{1}_{\{z\wedge\hat{\xi}\ge(\hat{\xi}-y)_+, \hat{X}\in A_0, \mathcal{E}\}}\Big)\\ &= \frac{1}{\mu}\mathbb{E}\Big(z\wedge\hat{\xi} - (\hat{\xi}-y)_+\Big)_+\mathbb{1}_{\{\hat{X}\in A_0, \mathcal{E}\}}\\ &= \frac{1}{\mu}\mathbb{E}\Big(z\wedge\hat{\xi} - (\hat{\xi}-y)_+\Big)_+\mathbb{1}_{\{\hat{X}\in A_0\}}\,\mathbb{P}\{\mathcal{E}\}\\ &= \frac{1}{\mu}\mathbb{E}\Big(z\wedge\hat{\xi} - (\hat{\xi}-y)_+\Big)_+\mathbb{1}_{\{\hat{X}\in A_0\}}\prod_{i=1}^q \mathbb{P}\{\xi\le z_i, X\in A_i\},\end{aligned}$$

where we have used that $\mathcal{E}$ is independent of $(\hat{X}, \hat{\xi})$. Applying formula (3) we arrive at

$$\begin{aligned}\mathbb{P}\{-S^*_{-1}\le y, S^*_0 < z, X_0\in A_0\} &= \mathbb{P}\{(1-U)\xi_0 \le y, U\xi_0 < z, X_0\in A_0\}\\ &= \int_0^1 \mathbb{P}\{\xi_0\le y(1-s)^{-1}, \xi_0 < zs^{-1}, X_0\in A_0\}\mathrm{d}s\\ &\overset{(3)}{=} \frac{1}{\mu}\int_0^1\int_{[0,\,y(1-s)^{-1}\wedge zs^{-1}]} t\mathbb{P}\{\hat{\xi}\in\mathrm{d}t, \hat{X}\in A_0\}\mathrm{d}s\\ &= \frac{1}{\mu}\int_{[0,\,\infty)}\Big(\int_{(1-y/t)_+}^{z/t\wedge 1} t\mathrm{d}s\Big)\mathbb{P}\{\hat{\xi}\in\mathrm{d}t, \hat{X}\in A_0\}\\ &= \frac{1}{\mu}\int_{[0,\,\infty)}\Big(z\wedge t - (t-y)_+\Big)_+\mathbb{P}\{\hat{\xi}\in\mathrm{d}t, \hat{X}\in A_0\}\\ &= \frac{1}{\mu}\mathbb{E}\Big(z\wedge\hat{\xi} - (\hat{\xi}-y)_+\Big)_+\mathbb{1}_{\{\hat{X}\in A_0\}},\end{aligned}$$

and (36) follows. The proof of Lemma 3.1 is complete.

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