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INTEGRAL EQUATION FOR THE TRANSITION DENSITY OF THE MULTIDIMENSIONAL MARKOV RANDOM FLIGHT

We consider the Markov random flight $\mathbf{X}(t)$ in the Euclidean space $\mathbb{R}^m, \ m \geq 2$, starting from the origin $\mathbf{0} \in \mathbb{R}^m$ that, at Poisson-paced times, changes its direction at random according to arbitrary distribution on the unit $(m-1)$-dimensional sphere $S^m(\mathbf{0}, 1)$ having absolutely continuous density. For any time instant $t > 0$, the convolution-type recurrent relations for the joint and conditional densities of the process $\mathbf{X}(t)$ and of the number of changes of direction, are obtained. Using these relations, we derive an integral equation for the transition density of $\mathbf{X}(t)$ whose solution is given in the form of a uniformly convergent series composed of the multiple double convolutions of the singular component of the density with itself. Two important particular cases of the uniform distribution on $S^m(\mathbf{0}, 1)$ and of the circular Gaussian law on the unit circle $S^2(\mathbf{0}, 1)$ are considered separately.

1. Introduction

Continuous-time random walks are an important field of stochastic processes (see, for instance, [1], [15, 16] and bibliography therein). An important case of such processes, random motions at finite speed in the multidimensional Euclidean spaces $\mathbb{R}^m, \ m \geq 2$, also called random flights, became the subject of intense research in last decades. The majority of published works deal with the case of isotropic Markov random flights when the motions are controlled by a homogeneous Poisson process and their directions are taken uniformly on the unit $(m-1)$-dimensional sphere [3–8], [14], [19, 20]. The limiting behaviour of a Markov random flight with a finite number of fixed directions in $\mathbb{R}^m$ was examined in [2]. In recent years the non-Markovian multidimensional random walks with Erlang- and Dirichlet-distributed displacements were studied in a series of works [10–13], [17, 18]. Such random motions at finite velocities are of a great interest due to their major theoretical importance and numerous fruitful applications in physics, chemistry, biology and other fields.

When studying such a motion, its explicit distribution is, undoubtedly, the most attractive aim of the research. However, despite many efforts, the closed-form expressions for the distributions of Markov random flights were obtained only in a few cases. In the spaces of low even dimensions such distributions were obtained in explicit forms by different methods (see [20], [14], [6], [8] for the Euclidean plane $\mathbb{R}^2$, [7] for the space $\mathbb{R}^4$ and [3] for the space $\mathbb{R}^6$). Moreover, in the spaces $\mathbb{R}^2$ and $\mathbb{R}^4$ such distributions are surprisingly expressed in terms of elementary functions, while in the space $\mathbb{R}^6$ the distribution has the form of a series composed of some polynomials. As far as the random flights in the odd-dimensional Euclidean spaces are concerned, their analysis is much more complicated in comparison with the even-dimensional cases. A formula for the transition density of the symmetric Markov random flight with unit speed in the space $\mathbb{R}^3$ was given in [19], however it has a very complicated form of an integral with variable limits, involving the inverse hyperbolic tangent function of the integration

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variable (see [19, formulas (1.3) and (4.21)]). Moreover, the density presented in this work evokes some questions since its absolutely continuous (integral) part is discontinuous at the origin $\mathbf{0} \in \mathbb{R}^3$ and this fact seems fairly strange.

The characteristic functions of the multidimensional random flights are much more convenient objects to analyse than their densities. This is due to the following fact: if the densities are functions with compact support then their characteristic functions (Fourier transforms) are analytical real functions defined everywhere in $\mathbb{R}^m$. That is why these characteristic functions were the subject of vast research, whose results were published in [4] and [9]. In particular, in [4] the time-convolutional recurrent relations for the joint and conditional characteristic functions of a Markov random flight in the Euclidean space $\mathbb{R}^m$ of arbitrary dimension $m \geq 2$ were obtained. By using these recurrent relations, the Volterra integral equation of second kind with continuous kernel for the unconditional characteristic function was derived and a closed-form expression for its Laplace transform was given. Such convolutional structure of the characteristic functions suggests a similar one for the respective densities. The discovery of convolutional relations for the densities of Markov random flights in $\mathbb{R}^m$, $m \geq 2$, is the main subject of this article.

The paper is organized as follows. In Section 2 we introduce general Markov random flights in the Euclidean spaces $\mathbb{R}^m$, $m \geq 2$, with arbitrary dissipation function and describe the structure of their distribution. Some basic properties of the joint, conditional and unconditional characteristic functions of the process are also given. In Section 3 we derive the recurrent relations for the joint and conditional densities of the process and of the number of changes of direction in a form of double convolutions with respect to the space and time variables. Based on these recurrent relations, an integral equation for the transition density of the process is obtained in Section 4, whose solution is given in a form of uniformly converging series composed of the multiple double convolutions of the singular component of the density with itself. This solution is unique in the class of functions with compact support in $\mathbb{R}^m$. Two important particular cases of the uniform distribution on $S^m(\mathbf{0},1)$ and of the circular Gaussian law on the unit circle $S^2(\mathbf{0},1)$ are considered in Section 5.

2. Description of the process and its basic properties

Consider the following stochastic motion. A particle starts from the origin $\mathbf{0} = (0,\dots,0)$ of the Euclidean space $\mathbb{R}^m$, $m \geq 2$, at the initial time instant $t=0$ and moves with some constant speed c (note that c is treated as the constant norm of the velocity). The initial direction is a random m-dimensional vector with arbitrary distribution (also called the dissipation function) on the unit sphere

$$S^m(\mathbf{0},1) = \left\{ \mathbf{x} = (x_1,\dots,x_m) \in \mathbb{R}^m : \ \|\mathbf{x}\|^2 = x_1^2 + \dots + x_m^2 = 1 \right\}$$

having an absolutely continuous bounded density $\chi(\mathbf{x})$, $\mathbf{x} \in S^m(\mathbf{0},1)$. It should be emphasized that here and thereafter the upper index m means the dimension of the space, in which the sphere $S^m(\mathbf{0},1)$ is considered, but not the sphere's own dimension, which, clearly, is $m-1$. The motion is controlled by a homogeneous Poisson process $N(t)$ of rate $\lambda > 0$ as follows. At each Poissonian instant the particle instantaneously takes on a new random direction in $S^m(\mathbf{0},1)$ distributed with the same density $\chi(\mathbf{x})$, $\mathbf{x} \in S^m(\mathbf{0},1)$, independently of its previous motion, and keeps moving with the same speed c until the next Poisson event occurs. Then it takes on a new random direction again and so on.

Let $\mathbf{X}(t) = (X_1(t),\dots,X_m(t))$ be the particle's position at time $t > 0$ which is referred to as the m-dimensional Markov random flight. At arbitrary time instant $t > 0$ the particle, with probability 1, is located in the closed m-dimensional ball of radius ct centred at the origin $\mathbf{0}$:

$$\mathcal{B}^m(\mathbf{0},ct) = \left\{ \mathbf{x} = (x_1,\dots,x_m) \in \mathbb{R}^m : \ \|\mathbf{x}\|^2 = x_1^2 + \dots + x_m^2 \leq c^2t^2 \right\}.$$

Consider the probability distribution function

$$\Phi(\mathbf{x},t) = \Pr\left\{\mathbf{X}(t) \in d\mathbf{x}\right\}, \qquad \mathbf{x} \in \mathcal{B}^m(\mathbf{0},ct), \quad t \ge 0,$$

of the process $\mathbf{X}(t)$, where $d\mathbf{x}$ is an infinitesimal element in the space $\mathbb{R}^m$ with Lebesgue measure $\mu(d\mathbf{x}) = dx_1 \dots dx_m$. For arbitrary fixed $t > 0$, the distribution $\Phi(\mathbf{x},t)$ consists of two components.

The singular component corresponds to the case when no Poisson events occur in the time interval $(0,t)$ and it is concentrated on the sphere

$$S^m(\mathbf{0},ct) = \partial\mathcal{B}^m(\mathbf{0},ct) = \left\{\mathbf{x} = (x_1,\dots,x_m) \in \mathbb{R}^m : \ \|\mathbf{x}\|^2 = x_1^2 + \dots + x_m^2 = c^2t^2\right\}.$$

In this case, at time instant t, the particle is located on the sphere $S^m(\mathbf{0},ct)$ and the probability of this event is

$$\Pr\left\{\mathbf{X}(t) \in S^m(\mathbf{0},ct)\right\} = e^{-\lambda t}.$$

If at least one Poisson event occurs before a time instant t, then the particle is located strictly inside the ball $\mathcal{B}^m(\mathbf{0},ct)$ and the probability of this event is

$$\Pr\left\{\mathbf{X}(t) \in int\,\mathcal{B}^m(\mathbf{0},ct)\right\} = 1 - e^{-\lambda t}.$$

The part of the distribution $\Phi(\mathbf{x},t)$ corresponding to this case is concentrated in the interior

$$int\,\mathcal{B}^m(\mathbf{0},ct) = \left\{\mathbf{x} = (x_1,\dots,x_m) \in \mathbb{R}^m : \ \|\mathbf{x}\|^2 = x_1^2 + \dots + x_m^2 < c^2t^2\right\}$$

of the ball $\mathcal{B}^m(\mathbf{0},ct)$ and forms its absolutely continuous component.

Let $p(\mathbf{x},t) = p(x_1,\dots,x_m,t), \ \ \mathbf{x} \in \mathcal{B}^m(\mathbf{0},ct), \ t > 0$, be the density of distribution $\Phi(\mathbf{x},t)$. It has the form

$$p(\mathbf{x},t) = p^{(s)}(\mathbf{x},t) + p^{(ac)}(\mathbf{x},t), \qquad \mathbf{x} \in \mathcal{B}^m(\mathbf{0},ct), \quad t > 0, \tag{2.1}$$

where $p^{(s)}(\mathbf{x},t)$ is the density (in the sense of generalized functions) of the singular component of $\Phi(\mathbf{x},t)$ concentrated on the sphere $S^m(\mathbf{0},ct)$ and $p^{(ac)}(\mathbf{x},t)$ is the density of the absolutely continuous component of $\Phi(\mathbf{x},t)$ concentrated in $int\,\mathcal{B}^m(\mathbf{0},ct)$.

The density $\chi(\mathbf{x}), \ \mathbf{x} \in S^m(\mathbf{0},1)$, on the unit sphere $S^m(\mathbf{0},1)$ generates the absolutely continuous and bounded (in $\mathbf{x}$ for any fixed t) density $\varrho(\mathbf{x},t), \mathbf{x} \in S^m(\mathbf{0},ct)$, on the sphere $S^m(\mathbf{0},ct)$ of radius ct according to the formula $\varrho(\mathbf{x},t) = \chi(\frac{1}{ct}\mathbf{x}), \ \mathbf{x} \in S^m(\mathbf{0},ct), \ t > 0$. Therefore, the singular part of density (2.1) has the form:

$$p^{(s)}(\mathbf{x},t) = e^{-\lambda t}\varrho(\mathbf{x},t)\delta(c^2t^2 - \|\mathbf{x}\|^2), \qquad t > 0, \tag{2.2}$$

where $\delta(x)$ is the Dirac delta-function.

The absolutely continuous part of density (2.1) has the form:

$$p^{(ac)}(\mathbf{x},t) = f^{(ac)}(\mathbf{x},t)\Theta(ct - \|\mathbf{x}\|), \qquad t > 0, \tag{2.3}$$

where $f^{(ac)}(\mathbf{x},t)$ is some positive function absolutely continuous in $int\,\mathcal{B}^m(\mathbf{0},ct)$ and $\Theta(x)$ is the Heaviside step function given by

$$\Theta(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases} \tag{2.4}$$

Let $\tilde{p}_n(\mathbf{x},t), \ n \ge 0$, be the conditional densities of the process $\mathbf{X}(t)$ conditioned by the random events $\{N(t) = n\}, \ n \ge 0$, where, as before, $N(t)$ is the number of the Poisson events that have occurred in the time interval $(0,t)$. Obviously $\tilde{p}_0(\mathbf{x},t) = \varrho(\mathbf{x},t)\delta(c^2t^2 - \|\mathbf{x}\|^2)$.

The conditional characteristic functions (Fourier transforms) of the process $\mathbf{X}(t)$ are:

$$\tilde{G}_n(\boldsymbol{\alpha},t) = \mathbb{E}\left\{e^{i\langle\boldsymbol{\alpha},\mathbf{X}(t)\rangle} \mid N(t) = n\right\}, \qquad n \ge 1, \tag{2.5}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ is the real m-dimensional vector and $\langle \boldsymbol{\alpha}, \mathbf{X}(t) \rangle$ is the inner product of the vectors $\boldsymbol{\alpha}$ and $\mathbf{X}(t)$.

According to [4, formula (6.4)], the functions (2.5) are given by the formula:

$$\tilde{G}_n(\boldsymbol{\alpha}, t) = \frac{n!}{t^n} \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \dots \int_{\tau_{n-1}}^t d\tau_n \left\{ \prod_{j=1}^{n+1} \psi(\boldsymbol{\alpha}, \tau_j - \tau_{j-1}) \right\}, \qquad n \geq 1, \tag{2.6}$$

where

$$\psi(\boldsymbol{\alpha}, t) = \mathcal{F}_{\mathbf{x}} \left[\varrho(\mathbf{x}, t) \delta(c^2 t^2 - \|\mathbf{x}\|^2) \right] (\boldsymbol{\alpha}) = \int_{S^m(\mathbf{0}, ct)} e^{i\langle \boldsymbol{\alpha}, \mathbf{x} \rangle} \, \varrho(\mathbf{x}, t) \, \nu(d\mathbf{x}) \tag{2.7}$$

is the characteristic function (Fourier transform) of density $\varrho(\mathbf{x}, t)$ concentrated on the sphere $S^m(\mathbf{0}, ct)$ of radius ct and $\nu(d\mathbf{x})$ is the Lebesgue measure on $S^m(\mathbf{0}, ct)$. Note that, here and thereafter, the symbol $\mathcal{F}_{\mathbf{x}}$ means the standard Fourier transform with respect to the spatial variable $\mathbf{x}$.

Consider the integral factor in (2.6) separately:

$$\mathcal{J}_n(\boldsymbol{\alpha}, t) = \int_0^t d\tau_1 \int_{\tau_1}^t d\tau_2 \dots \int_{\tau_{n-1}}^t d\tau_n \left\{ \prod_{j=1}^{n+1} \psi(\boldsymbol{\alpha}, \tau_j - \tau_{j-1}) \right\}, \qquad n \geq 1. \tag{2.8}$$

This function has a quite definite probabilistic sense, namely

$$G_n(\boldsymbol{\alpha}, t) = \mathcal{F}_{\mathbf{x}} \big[p_n(\mathbf{x}, t) \big] (\boldsymbol{\alpha}) = \frac{(\lambda t)^n \, e^{-\lambda t}}{n!} \, \tilde{G}_n(\boldsymbol{\alpha}, t) = \lambda^n e^{-\lambda t} \mathcal{J}_n(\boldsymbol{\alpha}, t), \tag{2.9}$$

$$\boldsymbol{\alpha} \in \mathbb{R}^m, \qquad t > 0, \qquad n \geq 1,$$

is the characteristic function (Fourier transform) of the joint probability density $p_n(\mathbf{x}, t)$ of the particle's position at time instant t and of the number $N(t) = n$ of the Poisson events that have occurred by this time moment t.

It is known (see [4, Theorem 5]) that, for arbitrary $n \geq 1$, functions (2.8) are connected with each other by the following recurrent relation:

$$\mathcal{J}_n(\boldsymbol{\alpha}, t) = \int_0^t \psi(\boldsymbol{\alpha}, t - \tau) \, \mathcal{J}_{n-1}(\boldsymbol{\alpha}, \tau) \, d\tau = \int_0^t \psi(\boldsymbol{\alpha}, \tau) \, \mathcal{J}_{n-1}(\boldsymbol{\alpha}, t - \tau) \, d\tau, \qquad n \geq 1, \tag{2.10}$$

where, by definition, $\mathcal{J}_0(\boldsymbol{\alpha}, t) \stackrel{\text{def}}{=} \psi(\boldsymbol{\alpha}, t)$. Formula (2.10) can also be represented in the following time-convolutional form:

$$\mathcal{J}_n(\boldsymbol{\alpha}, t) = \psi(\boldsymbol{\alpha}, t) \stackrel{t}{*} \mathcal{J}_{n-1}(\boldsymbol{\alpha}, t), \qquad n \geq 1, \tag{2.11}$$

where the symbol $\stackrel{t}{*}$ means the convolution operation with respect to time variable t.

From (2.11) it follows that

$$\mathcal{J}_n(\boldsymbol{\alpha}, t) = \left[\psi(\boldsymbol{\alpha}, t) \right]^{\stackrel{t}{*}(n+1)}, \qquad n \geq 1, \tag{2.12}$$

where $\stackrel{t}{*}(n+1)$ means the $(n+1)$-multiple convolution in t. Applying Laplace transformation $\mathcal{L}_t$ (in time variable t) to (2.12), we arrive at the formula

$$\mathcal{L}_t \left[\mathcal{J}_n(\boldsymbol{\alpha}, t) \right] (s) = \big(\mathcal{L}_t \left[\psi(\boldsymbol{\alpha}, t) \right] (s) \big)^{n+1}, \qquad n \geq 1. \tag{2.13}$$

It is also known (see [4, Corollary 5.3]) that conditional characteristic functions (2.6) satisfy the following recurrent relation

$$\tilde{G}_n(\boldsymbol{\alpha}, t) = \frac{n}{t^n} \int_0^t \tau^{n-1} \psi(\boldsymbol{\alpha}, t - \tau) \, \tilde{G}_{n-1}(\boldsymbol{\alpha}, \tau) \, d\tau, \qquad \tilde{G}_0(\boldsymbol{\alpha}, t) \stackrel{\text{def}}{=} \psi(\boldsymbol{\alpha}, t), \quad n \geq 1. \tag{2.14}$$

The unconditional characteristic function

$$G(\boldsymbol{\alpha}, t) = E\left\{e^{i\langle\boldsymbol{\alpha}, \mathbf{X}(t)\rangle}\right\} \tag{2.15}$$

of process $\mathbf{X}(t)$ satisfies the Volterra integral equation of second kind (see [4, Theorem 6]):

$$G(\boldsymbol{\alpha}, t) = e^{-\lambda t}\psi(\boldsymbol{\alpha}, t) + \lambda\int_0^t e^{-\lambda(t-\tau)}\psi(\boldsymbol{\alpha}, t-\tau)\; G(\boldsymbol{\alpha}, \tau)\; d\tau, \qquad t \geq 0, \tag{2.16}$$

or in the convolutional form

$$G(\boldsymbol{\alpha}, t) = e^{-\lambda t}\psi(\boldsymbol{\alpha}, t) + \lambda\big[\big(e^{-\lambda t}\psi(\boldsymbol{\alpha}, t)\big) \overset{t}{*} G(\boldsymbol{\alpha}, t)\big]. \tag{2.17}$$

This is the renewal equation for the Markov random flight $\mathbf{X}(t)$.

In the class of continuous functions integral equation (2.16) (or (2.17)) has the unique solution given by the uniformly converging series

$$G(\boldsymbol{\alpha}, t) = e^{-\lambda t}\sum_{n=0}^{\infty}\lambda^n\; \left[\psi(\boldsymbol{\alpha}, t)\right]^{\overset{t}{*}(n+1)}\; . \tag{2.18}$$

From (2.17) we obtain the general formula for the Laplace transform of characteristic function (2.15):

$$\mathcal{L}_t\left[G(\boldsymbol{\alpha}, t)\right](s) = \frac{\mathcal{L}_t\left[\psi(\boldsymbol{\alpha}, t)\right](s+\lambda)}{1 - \lambda\,\mathcal{L}_t\left[\psi(\boldsymbol{\alpha}, t)\right](s+\lambda)}\;, \qquad \operatorname{Re} s > 0. \tag{2.19}$$

These properties will be used in the next section for deriving the recurrent relations for the joint and conditional densities of Markov random flight $\mathbf{X}(t)$.

3. Recurrent relations

Consider the joint probability densities $p_n(\mathbf{x}, t),\; n \geq 0,\; \mathbf{x} \in \mathcal{B}^m(\mathbf{0}, ct),\; t > 0$, of the particle's position $\mathbf{X}(t)$ at time instant $t > 0$ and of the number of the Poisson events $\{N(t) = n\}$ that have occurred before this instant t. For $n = 0$, we have

$$p_0(\mathbf{x}, t) = p^{(s)}(\mathbf{x}, t) = e^{-\lambda t}\varrho(\mathbf{x}, t)\delta(c^2t^2 - \|\mathbf{x}\|^2), \qquad t > 0, \tag{3.1}$$

where, as before, $p^{(s)}(\mathbf{x}, t)$ is the singular part of density (2.1) concentrated on the surface of the sphere $S^m(\mathbf{0}, ct) = \partial\mathcal{B}^m(\mathbf{0}, ct)$ and it is given by (2.2).

If $n \geq 1$, then, according to (2.3), joint densities $p_n(\mathbf{x}, t)$ have the form:

$$p_n(\mathbf{x}, t) = f_n(\mathbf{x}, t)\Theta(ct - \|\mathbf{x}\|), \qquad n \geq 1, \quad t > 0, \tag{3.2}$$

where $f_n(\mathbf{x}, t),\; n \geq 1$, are some positive functions absolutely continuous in $int\,\mathcal{B}^m(\mathbf{0}, ct)$ and $\Theta(x)$ is the Heaviside step function.

The joint density $p_{n+1}(\mathbf{x}, t)$ can be expressed through the previous one $p_n(\mathbf{x}, t)$ by means of a recurrent relation. This result is given by the following theorem.

Theorem 1. *The joint densities $p_n(\mathbf{x}, t),\; n \geq 1$, are connected with each other by the following recurrent relation:*

$$p_{n+1}(\mathbf{x}, t) = \lambda\int_0^t\big[p_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*} p_n(\mathbf{x}, \tau)\big]\, d\tau, \;\; n \geq 1,\; \mathbf{x} \in int\,\mathcal{B}^m(\mathbf{0}, ct),\; t > 0. \tag{3.3}$$

Proof. Applying Fourier transformation to the right-hand side of (3.3), we have:

$$\begin{aligned}
&\lambda\,\mathcal{F}_{\mathbf{x}}\left[\int_0^t\big[p_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*} p_n(\mathbf{x}, \tau)\big]\, d\tau\right](\boldsymbol{\alpha}) \qquad (3.4)\\
&\quad = \lambda\int_0^t \mathcal{F}_{\mathbf{x}}\left[p_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*} p_n(\mathbf{x}, \tau)\right](\boldsymbol{\alpha})\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \lambda \int_0^t \mathcal{F}_{\mathbf{x}}\big[p_0(\mathbf{x}, t-\tau)\big](\boldsymbol{\alpha}) \, \mathcal{F}_{\mathbf{x}}\big[p_n(\mathbf{x},\tau)\big](\boldsymbol{\alpha}) \, d\tau \\
&= \lambda \int_0^t e^{-\lambda(t-\tau)} \mathcal{F}_{\mathbf{x}}\big[\varrho(\mathbf{x}, t-\tau)\delta(c^2(t-\tau)^2 - \|\mathbf{x}\|^2)\big](\boldsymbol{\alpha}) \, \mathcal{F}_{\mathbf{x}}\big[p_n(\mathbf{x},\tau)\big](\boldsymbol{\alpha}) \, d\tau \\
&= \lambda \int_0^t e^{-\lambda(t-\tau)} \psi(\boldsymbol{\alpha}, t-\tau) \, \lambda^n e^{-\lambda\tau} \mathcal{J}_n(\boldsymbol{\alpha},\tau) \, d\tau \\
&= \lambda^{n+1} e^{-\lambda t} \int_0^t \psi(\boldsymbol{\alpha}, t-\tau) \, \mathcal{J}_n(\boldsymbol{\alpha},\tau) \, d\tau \\
&= \lambda^{n+1} e^{-\lambda t} \mathcal{J}_{n+1}(\boldsymbol{\alpha}, t) \\
&= \mathcal{F}_{\mathbf{x}}\big[p_{n+1}(\mathbf{x}, t)\big](\boldsymbol{\alpha}),
\end{aligned}
$$

where we have used formulas (2.7), (2.9), (2.10). Thus, both the functions on the left- and right-hand sides of (3.3) have the same Fourier transform and, therefore, they coincide.

The change of integration order in the first step of (3.4) is justified because the convolution $p_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*} p_n(\mathbf{x},\tau)$ of the singular part $p_0(\mathbf{x}, t-\tau)$ of the density with the absolutely continuous one $p_n(\mathbf{x},\tau)$, $n \geq 1$, is an absolutely continuous (and, therefore, uniformly bounded in $\mathbf{x}$) function. From this fact it follows that, for any $n \geq 1$, the integral in square brackets on the left-hand side of (3.4) converges uniformly in $\mathbf{x}$ for any fixed t. The theorem is proved. □

Remark 1. In view of (2.2) and (2.3), formula (3.3) can be represented in the following expanded form:

$$
\begin{aligned}
p_{n+1}(\mathbf{x}, t) = \lambda \int_0^t & e^{-\lambda(t-\tau)} \\
& \times \left\{ \int \varrho(\mathbf{x}-\boldsymbol{\xi}, t-\tau)\delta(c^2(t-\tau)^2 - \|\mathbf{x}-\boldsymbol{\xi}\|^2) \, f_n(\boldsymbol{\xi},\tau)\Theta(c\tau - \|\boldsymbol{\xi}\|) \, \nu(d\boldsymbol{\xi}) \right\} d\tau, \\
& \qquad n \geq 1, \; \mathbf{x} \in int\, \mathcal{B}^m(\mathbf{0}, ct), \; t > 0, \qquad (3.5)
\end{aligned}
$$

where the function $f_n(\boldsymbol{\xi},\tau)$ is absolutely continuous in the variable $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ and $\nu(d\boldsymbol{\xi})$ is the Lebesgue measure. The integration area in the interior integral on the right-hand side of (3.5) is given by the system

$$
\boldsymbol{\xi} \in \mathbb{R}^m \; : \; \begin{cases} \|\mathbf{x}-\boldsymbol{\xi}\|^2 = c^2(t-\tau)^2, \\ \|\boldsymbol{\xi}\| < c\tau. \end{cases}
$$

The first relation of this system determines a sphere $S^m(\mathbf{x}, c(t-\tau))$ of radius $c(t-\tau)$ centred at point $\mathbf{x}$, while the second one represents an open ball $int\, \mathcal{B}^m(\mathbf{0}, c\tau)$ of radius $c\tau$ centred at the origin $\mathbf{0}$. Their intersection

$$
M(\mathbf{x},\tau) = S^m(\mathbf{x}, c(t-\tau)) \cap int\, \mathcal{B}^m(\mathbf{0}, c\tau), \qquad (3.6)
$$

which is a part of (or the whole) surface of sphere $S^m(\mathbf{x}, c(t-\tau))$ located inside the ball $\mathcal{B}^m(\mathbf{0}, c\tau)$, represents the integration area of dimension $m-1$ in the interior integral of (3.5). Note that the sum of the radii of $S^m(\mathbf{x}, c(t-\tau))$ and $int\, \mathcal{B}^m(\mathbf{0}, c\tau)$ is $c(t-\tau)+c\tau = ct > \|\mathbf{x}\|$, that is greater than the distance $\|\mathbf{x}\|$ between their centres $\mathbf{0}$ and $\mathbf{x}$. This fact, as well as some simple geometric reasonings, show that intersection (3.6) depends on $\tau \in (0,t)$ as follows.

- If $\tau \in (0, \; \frac{t}{2} - \frac{\|x\|}{2c}]$, then intersection (3.6) is empty, that is, $M(\mathbf{x},\tau) = \varnothing$.
- If $\tau \in (\frac{t}{2} - \frac{\|x\|}{2c}, \; \frac{t}{2} + \frac{\|x\|}{2c}]$, then intersection $M(\mathbf{x},\tau)$ is not empty and represents some hypersurface of dimension $m-1$.
- If $\tau \in (\frac{t}{2} + \frac{\|x\|}{2c}, \; t)$, then $S^m(\mathbf{x}, c(t-\tau)) \subset int\, \mathcal{B}^m(\mathbf{0}, c\tau)$ and, therefore, in this case $M(\mathbf{x},\tau) = S^m(\mathbf{x}, c(t-\tau))$.

Thus, formula (3.5), as well as (3.3), can be rewritten in the expanded form:

$$
\begin{aligned}
p_{n+1}(\mathbf{x},t) = \lambda \int\limits_{\frac{t}{2}-\frac{\|x\|}{2c}}^{\frac{t}{2}+\frac{\|x\|}{2c}} e^{-\lambda(t-\tau)} \left\{ \int\limits_{M(\mathbf{x},\tau)} \varrho(\mathbf{x}-\boldsymbol{\xi}, t-\tau)\; f_n(\boldsymbol{\xi},\tau)\; \nu(d\boldsymbol{\xi}) \right\} d\tau \\
+ \lambda \int\limits_{\frac{t}{2}+\frac{\|x\|}{2c}}^{t} e^{-\lambda(t-\tau)} \left\{ \int\limits_{S^m(\mathbf{x},c(t-\tau))} \varrho(\mathbf{x}-\boldsymbol{\xi}, t-\tau)\; f_n(\boldsymbol{\xi},\tau)\; \nu(d\boldsymbol{\xi}) \right\} d\tau
\end{aligned}
\tag{3.7}
$$

and the expressions in curly brackets of (3.7) represent surface integrals over $M(\mathbf{x},\tau)$ and $S^m(\mathbf{x}, c(t-\tau))$.

Remark 2. By means of the double convolution of two arbitrary generalized functions $g_1(\mathbf{x},t), g_2(\mathbf{x},t) \in \mathscr{S}',\ \mathbf{x} \in \mathbb{R}^m,\ t > 0$,

$$
g_1(\mathbf{x},t) \overset{\mathbf{x}}{*} \overset{t}{*} g_2(\mathbf{x},t) = \int_0^t \int_{\mathbb{R}^m} g_1(\boldsymbol{\xi},\tau)\; g_2(\mathbf{x}-\boldsymbol{\xi}, t-\tau)\; d\boldsymbol{\xi}\; d\tau \tag{3.8}
$$

formula (3.3) can be represented in the succinct convolutional form

$$
p_{n+1}(\mathbf{x},t) = \lambda \left[p_0(\mathbf{x},t) \overset{\mathbf{x}}{*} \overset{t}{*} p_n(\mathbf{x},t) \right]. \tag{3.9}
$$

Taking into account the well-known connections between the joint and conditional densities, we can extract from Theorem 1 a convolution-type recurrent relation for the conditional probability densities $\tilde{p}_n(\mathbf{x},t),\ n \geq 1$.

Corollary 1. *The conditional densities* $\tilde{p}_n(\mathbf{x},t),\ n \geq 1$, *are connected with each other by the following recurrent relation:*

$$
\tilde{p}_{n+1}(\mathbf{x},t) = \frac{n+1}{t^{n+1}} \int_0^t \tau^n \left[\tilde{p}_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*}\ \tilde{p}_n(\mathbf{x},\tau) \right] d\tau, \quad n \geq 1,\ \mathbf{x} \in int\, \mathcal{B}^m(\mathbf{0}, ct),\ t > 0, \tag{3.10}
$$

where $\tilde{p}_0(\mathbf{x},t) = \varrho(\mathbf{x},t)\delta(c^2t^2 - \|\mathbf{x}\|^2)$ *is the conditional density corresponding to the case when no Poisson events occur before time instant* t.

Proof. The proof immediately follows from Theorem 1 and recurrent formula (2.14). □

Remark 3. Formulas (3.3) and (3.10) are also valid for $n = 0$. In this case, for arbitrary $t > 0$, they take the form:

$$
p_1(\mathbf{x},t) = \lambda \int_0^t \left[p_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*}\ p_0(\mathbf{x},\tau) \right] d\tau, \tag{3.11}
$$

$$
\tilde{p}_1(\mathbf{x},t) = \frac{1}{t} \int_0^t \left[\tilde{p}_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*}\ \tilde{p}_0(\mathbf{x},\tau) \right] d\tau, \tag{3.12}
$$

where, as before, the function $p_0(\mathbf{x},t)$, defined by (3.1), is the singular part of the density concentrated on the surface of the sphere $S^m(\mathbf{0}, ct)$. The derivation of (3.11) is a simple recompilation of the proof of Theorem 1 where one should take into account the boundedness of the density $p_0(\mathbf{x},t)$ that justifies the change of integration order in (3.4). Formula (3.12) follows from (3.11).

4. Integral equation for transition density

The transition probability density $p(\mathbf{x}, t)$ of the multidimensional Markov flight $\mathbf{X}(t)$ is defined by the formula

$$p(\mathbf{x}, t) = \sum_{n=0}^{\infty} p_n(\mathbf{x}, t), \qquad \mathbf{x} \in \mathcal{B}^m(\mathbf{0}, ct), \quad t > 0, \tag{4.1}$$

where the joint densities $p_n(\mathbf{x}, t)$, $n \geq 0$, are given by (3.1) and (3.2). The density (4.1) is defined everywhere in the ball $\mathcal{B}^m(\mathbf{0}, ct)$, while the function

$$p_{ac}(\mathbf{x}, t) = \sum_{n=1}^{\infty} p_n(\mathbf{x}, t) \tag{4.2}$$

forms its absolutely continuous part concentrated in the interior $int\,\mathcal{B}^m(\mathbf{0}, ct)$ of the ball. Therefore, the series (4.2) converges uniformly everywhere in the closed ball $\mathcal{B}^m(\mathbf{0}, ct-\varepsilon)$ for arbitrary small $\varepsilon > 0$.

In the following theorem we present an integral equation for the density (4.1).

Theorem 2. *The transition probability density $p(\mathbf{x}, t)$ of the Markov random flight $\mathbf{X}(t)$ satisfies the integral equation:*

$$p(\mathbf{x}, t) = p_0(\mathbf{x}, t) + \lambda \int_0^t \left[p_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*}\ p(\mathbf{x}, \tau) \right] d\tau, \qquad \mathbf{x} \in \mathcal{B}^m(\mathbf{0}, ct), \quad t > 0. \tag{4.3}$$

In the class of functions with compact support, integral equation (4.3) has the unique solution given by the series

$$p(\mathbf{x}, t) = \sum_{n=0}^{\infty} \lambda^n \left[p_0(\mathbf{x}, t) \right]^{\overset{\mathbf{x}}{*}\overset{t}{*}(n+1)}, \tag{4.4}$$

where the symbol $\overset{\mathbf{x}}{*}\overset{t}{*}(n+1)$ means the $(n+1)$-multiple double convolution with respect to spatial and time variables defined by (3.8), that is,

$$\left[p_0(\mathbf{x}, t) \right]^{\overset{\mathbf{x}}{*}\overset{t}{*}(n+1)} = \underbrace{p_0(\mathbf{x}, t) \overset{\mathbf{x}}{*}\overset{t}{*} p_0(\mathbf{x}, t) \overset{\mathbf{x}}{*}\overset{t}{*} \dots \overset{\mathbf{x}}{*}\overset{t}{*} p_0(\mathbf{x}, t)}_{(n+1)\ terms}.$$

The series (4.4) is convergent everywhere in the open ball $int\,\mathcal{B}^m(\mathbf{0}, ct)$. For any small $\varepsilon > 0$, the series (4.4) converges uniformly (in $\mathbf{x}$ for any fixed $t > 0$) in the closed ball $\mathcal{B}^m(\mathbf{0}, ct-\varepsilon)$ and, therefore, it determines the density $p(\mathbf{x}, t)$ which is continuous and bounded in this ball.

Proof. Applying Theorem 1 and taking into account the uniform convergence of the series (4.2) and of the integral in the formula (3.3), we have:

$$\begin{aligned}
p(\mathbf{x}, t) &= \sum_{n=0}^{\infty} p_n(\mathbf{x}, t) \\
&= p_0(\mathbf{x}, t) + \sum_{n=1}^{\infty} p_n(\mathbf{x}, t) \\
&= p_0(\mathbf{x}, t) + \lambda \sum_{n=1}^{\infty} \int_0^t \left[p_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*}\ p_{n-1}(\mathbf{x}, \tau) \right] d\tau \\
&= p_0(\mathbf{x}, t) + \lambda \int_0^t \sum_{n=1}^{\infty} \left[p_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*}\ p_{n-1}(\mathbf{x}, \tau) \right] d\tau
\end{aligned}$$

$$= p_0(\mathbf{x}, t) + \lambda \int_0^t \left[p_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*} \left\{ \sum_{n=1}^{\infty} p_{n-1}(\mathbf{x}, \tau) \right\} \right] \, d\tau$$

$$= p_0(\mathbf{x}, t) + \lambda \int_0^t \left[p_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*} \left\{ \sum_{n=0}^{\infty} p_{n}(\mathbf{x}, \tau) \right\} \right] \, d\tau$$

$$= p_0(\mathbf{x}, t) + \lambda \int_0^t \left[p_0(\mathbf{x}, t-\tau) \overset{\mathbf{x}}{*} \; p(\mathbf{x}, \tau) \right] \, d\tau,$$

proving (4.3).

Another way of proving the theorem is to apply the Fourier transformation to both sides of (4.3). After justifying the change of the order of integrals in the same way as it was done in (3.4), we arrive at Volterra integral equation (2.16) for Fourier transforms.

Using notation (3.8), the equation (4.3) can be represented in the convolutional form

$$p(\mathbf{x}, t) = p_0(\mathbf{x}, t) + \lambda \big[p_0(\mathbf{x}, t) \overset{\mathbf{x}}{*} \overset{t}{*} p(\mathbf{x}, t) \big], \qquad \mathbf{x} \in \mathcal{B}^m(\mathbf{0}, ct), \quad t > 0. \tag{4.5}$$

Let us check that the series (4.4) satisfies the equation (4.5). Substituting (4.4) into the right-hand side of (4.5), we have:

$$\begin{aligned} p_0(\mathbf{x}, t) + \lambda \left[p_0(\mathbf{x}, t) \overset{\mathbf{x}}{*} \overset{t}{*} \left(\sum_{n=0}^{\infty} \lambda^n \left[p_0(\mathbf{x}, t) \right]^{\overset{\mathbf{x}}{*}\overset{t}{*}(n+1)} \right) \right] &= p_0(\mathbf{x}, t) + \sum_{n=0}^{\infty} \lambda^{n+1} \left[p_0(\mathbf{x}, t) \right]^{\overset{\mathbf{x}}{*}\overset{t}{*}(n+2)} \\ &= p_0(\mathbf{x}, t) + \sum_{n=1}^{\infty} \lambda^{n} \left[p_0(\mathbf{x}, t) \right]^{\overset{\mathbf{x}}{*}\overset{t}{*}(n+1)} \\ &= \sum_{n=0}^{\infty} \lambda^{n} \left[p_0(\mathbf{x}, t) \right]^{\overset{\mathbf{x}}{*}\overset{t}{*}(n+1)} \\ &= p(\mathbf{x}, t) \end{aligned}$$

and, therefore, the series (4.4) is the solution to the equation (4.5) indeed.

Note that by applying Fourier transformation to (4.3) and (4.4) and taking into account (2.2), we arrive at the known results (2.17) and (2.18), respectively. The uniqueness of solution (4.4) in the class of functions with compact support follows from the uniqueness of the solution of Volterra integral equation (2.16) for its Fourier transform (2.18) (i.e. characteristic function) in the class of continuous functions.

Since the transition density $p(\mathbf{x}, t)$ is absolutely continuous in the open ball $int\, \mathcal{B}^m(\mathbf{0}, ct)$, then, for any $\varepsilon > 0$, it is continuous and uniformly bounded in the closed ball $\mathcal{B}^m(\mathbf{0}, ct-\varepsilon)$. From this fact and taking into account the uniqueness of the solution of integral equation (4.3) in the class of functions with compact support, we can conclude that series (4.4) converges uniformly in $\mathcal{B}^m(\mathbf{0}, ct-\varepsilon)$ for any small $\varepsilon > 0$. This completes the proof. □

5. Some particular cases

In this section we consider two important particular cases of general Markov random flights described in Section 2 when the dissipation function has the uniform distribution on the unit sphere $S^m(\mathbf{0}, 1)$ or the circular Gaussian law on the unit circle $S^2(\mathbf{0}, 1)$.

5.1. **Symmetric random flights.** Suppose that the initial and every new direction are chosen according to the uniform distribution on the unit sphere $S^m(\mathbf{0}, 1)$. Such processes in the Euclidean spaces $\mathbb{R}^m$ of different dimensions $m \geq 2$, which are referred to as symmetric Markov random flights, have become the subject of a series of works [3–8], [14], [19,20].

In this symmetric case the function $\varrho(\mathbf{x}, t)$ is the density of the uniform distribution on the surface of the sphere $S^m(\mathbf{0}, ct)$ and, therefore, it does not depend on spatial variable

x. Then, according to (2.2), the singular part of the transition density of process $\mathbf{X}(t)$ takes the form:

$$p^{(s)}(\mathbf{x},t) = e^{-\lambda t}\frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{m/2}\,(ct)^{m-1}}\,\delta(c^2t^2-\|\mathbf{x}\|^2), \qquad m\geq 2, \quad t>0. \tag{5.1}$$

Therefore, according to Theorem 1, for arbitrary dimension $m \geq 2$, the joint probability densities $f_n(\mathbf{x},t),\ n\geq 1$, of a symmetric Markov random flight are connected with each other by the following recurrent relation:

$$f_{n+1}(\mathbf{x},t) = \frac{\lambda\,\Gamma\left(\frac{m}{2}\right)}{2\pi^{m/2}\,c^{m-1}}\int_0^t \frac{e^{-\lambda(t-\tau)}}{(t-\tau)^{m-1}}\left\{\int\limits_{M(\mathbf{x},\tau)} f_n(\boldsymbol{\xi},\tau)\,d\boldsymbol{\xi}\right\}d\tau, \tag{5.2}$$

$$\mathbf{x}=(x_1,\dots,x_m)\in int\,\mathcal{B}^m(\mathbf{0},ct), \quad m\geq 2, \quad n\geq 1, \quad t>0,$$

where the integration area $M(\mathbf{x},\tau)$ is given by (3.6).

It is known (see [5, formula (7)]) that, in arbitrary dimension $m\geq 2$, the joint density of the symmetric Markov random flight $\mathbf{X}(t)$ and of the single change of direction is given by the formula

$$f_1(\mathbf{x},t) = \lambda e^{-\lambda t}\,\frac{2^{m-3}\Gamma\left(\frac{m}{2}\right)}{\pi^{m/2}c^m t^{m-1}}\,F\left(\frac{m-1}{2}, -\frac{m}{2}+2;\ \frac{m}{2};\ \frac{\|\mathbf{x}\|^2}{c^2t^2}\right), \tag{5.3}$$

$$\mathbf{x}=(x_1,\dots,x_m)\in int\,\mathcal{B}^m(\mathbf{0},ct), \quad m\geq 2, \quad t>0,$$

where

$$F(\alpha,\beta;\gamma;z) = \sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\,\frac{z^k}{k!}$$

is the Gauss hypergeometric function.

Then, by substituting (5.3) into (5.2) (for $n=1$), we obtain the following formula for the joint density of process $\mathbf{X}(t)$ and of two changes of direction:

$$f_2(\mathbf{x},t) = \lambda^2 e^{-\lambda t}\,\frac{2^{m-4}\left[\Gamma\left(\frac{m}{2}\right)\right]^2}{\pi^m\,c^{2m-1}}$$
$$\times\int_0^t\left\{\int\limits_{M(\mathbf{x},\tau)} F\left(\frac{m-1}{2}, -\frac{m}{2}+2;\ \frac{m}{2};\ \frac{\|\boldsymbol{\xi}\|^2}{c^2\tau^2}\right)d\boldsymbol{\xi}\right\}\frac{d\tau}{(\tau(t-\tau))^{m-1}},$$

$$\mathbf{x}=(x_1,\dots,x_m)\in int\,\mathcal{B}^m(\mathbf{0},ct),\ m\geq 2,\ t>0. \tag{5.4}$$

In the three-dimensional Euclidean space $\mathbb{R}^3$, the joint density (5.3) was computed explicitly by different methods and it has the form (see [6, formula (25)] or [19, the second term of formulas (1.3) and (4.21)]):

$$f_1(\mathbf{x},t) = \frac{\lambda e^{-\lambda t}}{4\pi c^2 t\|\mathbf{x}\|}\,\ln\left(\frac{ct+\|\mathbf{x}\|}{ct-\|\mathbf{x}\|}\right), \tag{5.5}$$

$$\mathbf{x}=(x_1,x_2,x_3)\in int\,\mathcal{B}^3(\mathbf{0},ct), \quad \|\mathbf{x}\|=\sqrt{x_1^2+x_2^2+x_3^2}, \quad t>0.$$

By substituting this joint density into (5.2) (for $n=1,\ m=3$), we arrive at the formula:

$$f_2(\mathbf{x},t) = \frac{\lambda^2 e^{-\lambda t}}{16\pi^2c^4}\int_0^t\left\{\int\limits_{M(\mathbf{x},\tau)}\ln\left(\frac{c\tau+\|\boldsymbol{\xi}\|}{c\tau-\|\boldsymbol{\xi}\|}\right)\frac{d\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right\}\frac{d\tau}{\tau(t-\tau)^2}, \tag{5.6}$$

$$\mathbf{x}=(x_1,x_2,x_3)\in int\,\mathcal{B}^3(\mathbf{0},ct), \quad t>0.$$

Formula (5.6) can also be obtained by setting $m=3$ in (5.4).

According to Theorem 2 and (5.1), the transition density of the m-dimensional symmetric Markov random flight solves the integral equation

$$p(\mathbf{x},t) = \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{m/2}\, c^{m-1}} \Biggl\{ \frac{e^{-\lambda t}}{t^{m-1}}\, \delta(c^2t^2 - \|\mathbf{x}\|^2) + \lambda \int_0^t \left[\left(\frac{e^{-\lambda(t-\tau)}}{(t-\tau)^{m-1}}\, \delta(c^2(t-\tau)^2 - \|\mathbf{x}\|^2) \right) \overset{\mathbf{x}}{*}\, p(\mathbf{x},\tau) \right] d\tau \Biggr\}, \tag{5.7}$$

$$\mathbf{x} = (x_1, \dots, x_m) \in \mathcal{B}^m(\mathbf{0}, ct), \quad t > 0.$$

In the class of functions with compact support, the equation (5.7) has the unique solution given by the series

$$p(\mathbf{x},t) = \sum_{n=0}^{\infty} \lambda^n \left(\frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{m/2}\, c^{m-1}} \right)^{n+1} \left[\frac{e^{-\lambda t}}{t^{m-1}}\, \delta(c^2t^2 - \|\mathbf{x}\|^2) \right]^{\overset{\mathbf{x}\,t}{**}(n+1)}. \tag{5.8}$$

5.2. **The circular Gaussian law on a circle.** Consider now the case of a non-symmetric planar random flight when the initial and each new direction are chosen according to the distribution on the unit circle $S^2(\mathbf{0},1)$ with the two-dimensional density

$$\chi_k(\mathbf{x}) = \frac{1}{2\pi\, I_0(k)}\, \exp\left(\frac{kx_1}{\|\mathbf{x}\|} \right) \delta(1 - \|\mathbf{x}\|^2), \tag{5.9}$$

$$\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \qquad \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2} \qquad k \in \mathbb{R},$$

where $I_0(z)$ is the modified Bessel function of order 0. Formula (5.9) determines the one-parametric family of densities $\{\chi_k(\mathbf{x}),\ k \in \mathbb{R}\}$, and for any fixed real $k \in \mathbb{R}$ the density $\chi_k(\mathbf{x})$ is absolutely continuous and uniformly bounded on $S^2(\mathbf{0},1)$. If $k = 0$, then formula (5.9) yields the density of the uniform distribution on the unit circle $S^2(\mathbf{0},1)$, while for $k \neq 0$ it produces non-uniform densities.

In the unit polar coordinates $x_1 = \cos\theta,\ x_2 = \sin\theta$, the two-dimensional density (5.9) takes the form of the circular Gaussian law (also called the von Mises distribution):

$$\chi_k(\theta) = \frac{e^{k\cos\theta}}{2\pi\, I_0(k)}, \qquad \theta \in [-\pi, \pi), \quad k \in \mathbb{R}. \tag{5.10}$$

For arbitrary real $k \in \mathbb{R}$, density (5.9) on the unit circle $S^2(\mathbf{0},1)$ generates the density

$$p^{(s)}(\mathbf{x},t) = \frac{e^{-\lambda t}}{2\pi ct\, I_0(k)}\, \exp\left(\frac{kx_1}{\|\mathbf{x}\|} \right) \delta(c^2t^2 - \|\mathbf{x}\|^2), \tag{5.11}$$

$$\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \quad \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}, \quad t > 0, \quad k \in \mathbb{R},$$

concentrated on the circle $S^2(\mathbf{0}, ct)$ of radius ct. Then, according to Theorem 1, the joint densities are connected with each other by the recurrent relation

$$f_{n+1}(\mathbf{x},t) = \frac{\lambda}{2\pi c I_0(k)} \times \int_0^t \left\{ \int_{M(\mathbf{x},\tau)} \exp\left(\frac{k(x_1 - \xi_1)}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}} \right) f_n(\xi_1, \xi_2, \tau)\, d\xi_1 d\xi_2 \right\} \frac{e^{-\lambda(t-\tau)}}{t-\tau}\, d\tau,$$

$$\mathbf{x} = (x_1, x_2) \in int\, \mathcal{B}^2(\mathbf{0}, ct),\ n \geq 1,\ t > 0,\ k \in \mathbb{R}. \tag{5.12}$$

According to Theorem 2 and (5.11), the transition density of a planar Markov random flight with dissipation function (5.9) satisfies the integral equation

$$p(\mathbf{x},t) = \frac{e^{-\lambda t}}{2\pi ct\, I_0(k)}\, \exp\left(\frac{kx_1}{\|\mathbf{x}\|} \right) \delta(c^2t^2 - \|\mathbf{x}\|^2)$$

$$+ \frac{\lambda}{2\pi c\, I_0(k)} \int_0^t \left[\left(\frac{e^{-\lambda\tau}}{\tau} \exp\left(\frac{kx_1}{\|\mathbf{x}\|}\right) \delta(c^2\tau^2 - \|\mathbf{x}\|^2) \right) \overset{\mathbf{x}}{*} p(\mathbf{x},\tau) \right] d\tau,$$

$$\mathbf{x} = (x_1, x_2) \in \mathcal{B}^2(\mathbf{0}, ct), \; \|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}, \; t > 0, \; k \in \mathbb{R}. \quad (5.13)$$

In the class of functions with compact support, equation (5.13) has the unique solution given by the series

$$p(\mathbf{x},t) = \sum_{n=0}^{\infty} \lambda^n \left(\frac{1}{2\pi c\, I_0(k)} \right)^{n+1} \left[\frac{e^{-\lambda t}}{t} \exp\left(\frac{kx_1}{\|\mathbf{x}\|}\right) \delta(c^2t^2 - \|\mathbf{x}\|^2) \right]^{\overset{\mathbf{x}\,t}{**}(n+1)} . \quad (5.14)$$

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References

1. P. Becker-Kern, M. M. Meerschaert and H.-P. Scheffler, *Limit theorems for coupled continuous time random walks*, Ann. Probab. **32** (2004), 730–756.
2. A. Ghosh, R. Rastegar and A. Roitershtein, *On a directionally reinforced random walk*, Proc. Amer. Math. Soc. **142** (2014), 3269–3283.
3. A. D. Kolesnik, *The explicit probability distribution of a six-dimensional random flight*, Theory Stoch. Process. **15(31)** (2009), 33–39.
4. A. D. Kolesnik, *Random motions at finite speed in higher dimensions*, J. Statist. Phys. **131** (2008), 1039–1065.
5. A. D. Kolesnik, *Asymptotic relation for the density of a multidimensional random evolution with rare Poisson switchings*, Ukrain. Math. J. **60** (2008), 1915–1926.
6. A. D. Kolesnik, *A note on planar random motion at finite speed*, J. Appl. Probab. **44** (2007), 838–842.
7. A. D. Kolesnik, *A four-dimensional random motion at finite speed*, J. Appl. Probab. **43** (2006), 1107–1118.
8. A. D. Kolesnik and E. Orsingher, *A planar random motion with an infinite number of directions controlled by the damped wave equation*, J. Appl. Probab. **42** (2005), 1168–1182.
9. A. D. Kolesnik and M. A. Pinsky, *Random evolutions are driven by the hyperparabolic operators*, J. Statist. Phys. **142** (2011), 828–846.
10. G. Le Caër, *Two-step Dirichlet random walks* Physica A **430** (2015), 201–215.
11. G. Le Caër, *A new family of solvable Pearson-Dirichlet random walks*, J. Statist. Phys. **144** (2011), 23–45.
12. G. Le Caër, *A Pearson random walk with steps of uniform orientation and Dirichlet distributed lengths*, J. Statist. Phys. **140** (2010), 728–751.
13. G. Letac and M. Piccioni, *Dirichlet random walks*, J. Appl. Probab. **51** (2014), 1081–1099.
14. J. Masoliver, J. M. Porrá and G. H. Weiss, *Some two and three-dimensional persistent random walks*, Physica A **193** (1993), 469–482.
15. M. M. Meerschaert and P. Straka, *Semi-Markov approach to continuous-time random walk limit processes*, Ann. Probab. **42** (2014), 1699–1723.
16. M. M. Meerschaert and H.-P. Scheffler, *Limit theorems for continuous time random walks with infinite mean waiting times*, J. Appl. Probab. **41** (2004), 623–638.
17. A. A. Pogorui and R. M. Rodriguez-Dagnino, *Random motion with uniformly distributed directions and random velocity*, J. Statist. Phys. **147** (2012), 1216–1225.
18. A. A. Pogorui and R. M. Rodriguez-Dagnino, *Isotropic random motion at finite speed with K-Erlang distributed direction alternations*, J. Statist. Phys. **145** (2011), 102–112.
19. W. Stadje, *Exact probability distributions for non-correlated random walk models*, J. Statist. Phys. **56** (1989), 415–435.
20. W. Stadje, *The exact probability distribution of a two-dimensional random walk*, J. Statist. Phys. **46** (1987), 207–216.

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