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ON A LIMIT BEHAVIOR OF A ONE-DIMENSIONAL RANDOM WALK WITH NON-INTEGRABLE IMPURITY

We consider the limit behavior of a one-dimensional symmetric random walk that is perturbed at zero. For the natural scaling of time and space the invariance principle is proved. The limit process is a skew Brownian motion.

1. Introduction

Consider a Markov chain $\{X(n), n \geq 0\}$ on $\mathbb{Z}$ with transition probabilities

$$p_{ij} = 1/2 \text{ if } |i-j| = 1 \text{ and } i \neq 0;$$

$$p_{0j} = P(\xi = j), \tag{1}$$

where ξ is some integer-valued random variable.

Define a stochastic process

$$X_n(t) := \frac{X([nt])}{\sqrt{n}}, t \geq 0. \tag{2}$$

Under the condition $E|\xi| < \infty$, Harrison and Shepp [3] showed that the limiting process of the sequence $\{X_n\}$ is a skew Brownian motion with parameter $\gamma = \frac{E\xi}{E|\xi|}$, that is a continuous Markovian stochastic process with transition density

$$p_t(x, y) = \varphi_t(x-y) + \gamma \operatorname{sgn}(y)\varphi_t(|x| + |y|), \ x, y \in \mathbb{R},$$

where $\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$.

The skew Brownian motion can also be obtained as a limit of normalized random walks that are homogeneous everywhere except on some finite set $A \subset \mathbb{Z}$, and that have zero mathematical expectation of the jump and the finite variance of the jump outside of A. Note that on A it is enough to require only boundedness of the mathematical expectation of the jump, see different approaches to this issue [6, 9, 7, 4].

We remark that if the tails of the random variable (1) have the form $|x|^{-\alpha}l(x), \ x \to \infty$, where $\alpha \in (0, 1)$, and l is a slowly varying at infinity function, then the limit process for $\{X_n\}$ could be discontinuous at the instances of reaching 0, see [8], where the limit process is a Brownian motion satisfying Feller-Wentzell boundary condition at the origin.

The goal of this paper is to establish the invariance principle for the sequence of the random processes $\{X_n\}$ in the critical case when $\alpha = 1$, but the mathematical expectation of ξ does not exist. As it would be seen from the proof, apparently the existence of continuous limit is facilitated not by existence of the mathematical expectation of ξ, but mainly by the asymptotical behavior of its distribution function at infinity. The proof could have been easily generalized to the case when the symmetry of the random walk is violated in the finite number of states (as it was done in [7] when the mathematical expectation is finite), however it would have made the presentation too heavy with many

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technical details. Finally, let us note that if $\alpha < 1$, then any limit process cannot be a continuous process (see Remark 3.2).

The case of multi-dimensional random walk with impurities was considered in the work of Szasz and Telcs [10]. Contrary to the one-dimensional case under rather wide assumptions on the impurity, the limit of the normalized perturbed random walks turns out to be the usual multidimensional Brownian motion. In case when the dimension is greater than or equals 3, this could be explained through transience of the random walk. For the case of dimension 2, the proof is more complicated since one has to investigate the number of times the random walk visits the set where the transition probability is perturbed.

2. Main result

Consider the following generating functions

$$f_+(z) := \sum_{j=0}^{\infty} p_j z^j,\ f_-(z) := \sum_{j=-\infty}^{0} p_j z^j,\ z \in [0,1],$$

where $p_j = P(\xi = j)$.

Theorem 2.1. *Suppose there exists a slowly varying at infinity function l such that*

$$f_+(1) - f_+(1-z) \sim \beta_+ z l(1/z), z \downarrow 0,$$

$$f_-(1) - f_-(1-z) \sim \beta_- z l(1/z), z \downarrow 0,$$

where $\beta_\pm$ are some positive finite constants. Then the sequence of random processes $\{X_n\}$, defined in (2), converges weakly in $D([0,\infty))$ to a skew Brownian motion with parameter $\gamma = \frac{\beta_+ - \beta_-}{\beta_+ + \beta_-}$, starting at 0.

Remark 2.1. There are several equivalent ways to define a skew Brownian motion; see e.g. [5]. For instance, a skew Brownian motion with parameter $\gamma \in [-1,1]$ can be obtained from a Wiener process $w(t), t \geq 0$, by directing the excursion of w into the upper half-plane with the probability $p = (1+\gamma)/2$ and into the lower half-plane with the probability $q = (1-\gamma)/2$ (independently from each other and from the Wiener process itself).

The solution of the following stochastic equation

$$dX(t) = dw(t) + \gamma dL_X^0(t),$$

is also a skew Brownian motion with parameter γ, where $|\gamma| \leq 1$, see e.g. [3], and does not have any solution if $|\gamma| > 1$. Here $L_X^0(t) = \lim_{\varepsilon \to 0+} (2\varepsilon)^{-1} \int_0^t \mathbb{1}_{|X(s)|<\varepsilon} ds$ is the symmetric local time at zero of the unknown process X.

Let us discuss the connection between asymptotics of the probability-generating functions $f_\pm$ at one and asymptotics of the sequence $\{p_j\}$ at infinity. To this end let us recall the result about the connection between the behavior of the Laplace transform at zero of a nonnegative random variable η and the asymptotical properties of its distribution function $F(x) = P(\eta \leq x)$ at infinity.

Theorem 2.2. [1, Corollary 8.1.7] *Let l be a slowly varying at infinity function. Then the following conditions are equivalent:*

$$1 - Ee^{-s\eta} \sim s\, l(1/s),\ s \downarrow 0 \tag{3}$$

and

$$\begin{cases} \int_{[0,x]} t dF(t) \sim l(x),\ x \to +\infty, \\ \int_0^x (1-F(t)) dt \sim l(x),\ x \to +\infty. \end{cases} \tag{4}$$

Remark 2.2. One of the conditions in (4) can be substituted by

$$x(1-F(x))=o(l(x)),\ x\to+\infty.$$

Indeed, this follows from the sequence of the equalities below

$$l(x)+o(l(x))=\int_0^x(1-F(t))dt=x(1-F(x))+\int_{[0,x]}tdF(t)=$$
$$=x(1-F(x))+l(x)+o(l(x)),\ x\to\infty.$$

Remark 2.3. Using the previous remark, the conditions of Theorem 2.1 are equivalent to the following

$$\begin{cases}\sum_{0\le j\le N}jp_j\sim\beta_+l(N),\ \sum_{-N\le j\le 0}jp_j\sim-\beta_-l(N),\ N\to+\infty;\\ N\sum_{|j|\ge N}p_j=o(l(N)),\ N\to+\infty\end{cases}$$

Remark 2.4. Applying classical Tauberian theorems, it follows that the condition $1-Ee^{-s\eta}\sim s^\delta\, l(1/s),\ s\downarrow 0$, where $\delta\in[0,1)$, is equivalent to [1, Corollary 8.1.7]

$$1-F(x)\sim\frac{l(x)}{x^\delta\Gamma(1-\delta)},x\to+\infty,$$

where Γ is the standard Gamma function. Note that the condition $1-F(x)\sim\frac{l(x)}{x},x\to+\infty$, is a stronger assumption than (4); see [1, p. 335].

Example 2.1.
- If $E|\xi|<\infty$, then $l(x)\equiv 1,\ \beta_+=E\xi\vee 0,\ \beta_-=E((-\xi)\vee 0)$. In this case $\gamma=E\xi/E|\xi|$, which coincides with the result of Harrison and Shepp [3].
- Let $p_N\sim\beta_\pm N^{-2},N\to\pm\infty$. Then $l(x)=\ln x,x>0$. Indeed, by the Stolz-Cesaro theorem

$$\lim_{N\to+\infty}\frac{\sum_{0\le j\le N}jp_j}{\ln N}=\lim_{N\to+\infty}\frac{Np_N}{\ln N-\ln(N-1)}$$
$$=\lim_{N\to+\infty}\frac{Np_N}{\ln(1-\frac{1}{N-1})}=\lim_{N\to+\infty}\frac{Np_N}{\frac{1}{N-1}}=\beta_+.$$

- Similarly to the previous, if $p_N\sim\beta_\pm\alpha N^{-2}(\ln|N|)^{\alpha-1},N\to\pm\infty$, where $\alpha>0$, then $l(x)=(\ln x)^\alpha,x>0$.

3. Proof of the main result

We are going to apply Theorem 2, remark 8 in [7]. It is enough to check the following conditions.

A1. $\forall T>0\ \ \lim_{\varepsilon\downarrow0}\sup_{n\ge1}E\int_0^T\mathbb{1}_{|X_n(s)|\le\varepsilon}ds=0$;

A2. $\forall T>0\ \forall\varepsilon>0\ \exists\delta>0\ \exists n_0\ \forall n\ge n_0:$

$$P(\omega^T_{X_n}(\delta)\ge\varepsilon)\le\varepsilon,$$

where $\omega^T_f(\delta):=\sup_{s,t\in[0,T];\ |s-t|\le\delta}|f(s)-f(t)|$ is the modulus of continuity of function f;

A3. $\forall\alpha>0\ \forall\alpha_1,\ |\alpha_1|<\alpha\ \forall\{z_n\}\subset\mathbb{Z},\ \lim_{n\to\infty}z_n/\sqrt{n}=\alpha_1$

$$\lim_{n\to\infty}P_{z_n}(X(\sigma_{[-\sqrt{n}\alpha,\sqrt{n}\alpha]})\ge\sqrt{n}\alpha)=\frac{s(\alpha_1)-s(-\alpha)}{s(\alpha)-s(-\alpha)},$$

where $\sigma_{[-\sqrt{n}\alpha,\sqrt{n}\alpha]}=\inf\{k\ge0\ :\ |X(k)|>\sqrt{n}\alpha\}$ is the exit time from $[-\sqrt{n}\alpha,\sqrt{n}\alpha]$, and s is the scale of the skew Brownian motion (see, e.g. [5])

$$s(x)=\begin{cases}x/p, & x\ge0,\\ x/q, & x<0,\end{cases}$$

$p = \frac{\beta_+}{\beta_+ + \beta_-}$, $q = \frac{\beta_-}{\beta_+ + \beta_-}$.

A4. $\forall \alpha\ \forall \alpha_1 \in (0, |\alpha|)\ \forall \{z_n\} \subset \mathbb{Z},\ \lim_{n\to\infty} z_n/\sqrt{n} = \alpha$ the conditional distribution of the couple $(X_n(\cdot \wedge \tau_{[-\alpha_1,\alpha_1]}), \tau_{[-\alpha_1,\alpha_1]})$ under condition $X_n(0) = z_n/\sqrt{n}$, converges weakly to the conditional distribution of $(W_\gamma(\cdot \wedge \tau_{[-\alpha_1,\alpha_1]}), \tau_{[-\alpha_1,\alpha_1]})$ under the condition $W_\gamma(0) = \alpha$, where W_γ is the skew Brownian motion, and $\tau_{[-\alpha_1,\alpha_1]}$ is the entrance time of the corresponding process into the interval $[-\alpha_1, \alpha_1]$.

The proof of A1 is done similarly to [4] (see also [7] §4.3 and the proof of lemma 2.2). The idea rests on the comparison of the time spent in the interval $[-\sqrt{n}\varepsilon, \sqrt{n}\varepsilon]$ by the sequence $\{X(k), k = 1, \dots, [nT]\}$ and by the unperturbed random walk for which A1 obviously holds.

Since the distributions of the skew Brownian motion and the usual Brownian motion coincide before hitting 0, the condition A4 is well-known.

Let us check condition A3. Denote

$$x_i^{(n)} = P_i(X(\sigma_{[-\sqrt{n}\alpha, \sqrt{n}\alpha]}) \ge \alpha\sqrt{n}),$$

where $x_i^{(n)}$ is the probability that the Markov chain X to reach the set $[\alpha\sqrt{n}, \infty)$ before it reaches $(-\infty, -\alpha\sqrt{n}]$ when it starts from i.

Then $x_i^{(n)}$ satisfies the system of linear equations

$$x_i^{(n)} = \frac{x_{i+1}^{(n)} + x_{i-1}^{(n)}}{2},\ 0 < |i| < [\alpha\sqrt{n}], \tag{5}$$

$$x_0^{(n)} = \sum_j x_j^{(n)} p_j,\ \text{where } p_j = P(\xi = j), \tag{6}$$

$$x_i^{(n)} = 1,\ i > [\alpha\sqrt{n}], \tag{7}$$

$$x_i^{(n)} = 0,\ i < -[\alpha\sqrt{n}]. \tag{8}$$

It follows from (5) that $x_i^{(n)}$ is a linear function of i when $i < [\alpha\sqrt{n}]$ or $i > -[\alpha\sqrt{n}]$. This means that there exist constants $a_n^\pm, b_n^\pm$ such that

$$x_i^{(n)} = a_n^+ + i b_n^+,\ 0 \le i \le [\alpha\sqrt{n}], \tag{9}$$

$$x_i^{(n)} = a_n^- + i b_n^-,\ -[\alpha\sqrt{n}] \le i \le 0. \tag{10}$$

Conditions (9), (10) for $i = 0$ imply the assertion $a_n^+ = a_n^- =: a_n$.

Set $N := [\alpha\sqrt{n}]$. Then from the boundary conditions (7), (8) we obtain

$$\begin{cases} a_n + N b_n^+ = 1, \\ a_n - N b_n^- = 0. \end{cases}$$

Consequently,

$$b_n^+ = \frac{1 - a_n}{N},\ b_n^- = \frac{a_n}{N},$$

$$x_i^{(n)} = \begin{cases} a_n + i\frac{1-a_n}{N},\ 0 \le i \le N; \\ a_n + i\frac{a_n}{N},\ -N \le i \le -1. \end{cases} \tag{11}$$

From (6), (11) we have

$$a_n = x_0^{(n)} = \sum_{j=-N}^{-1} a_n(1 + \frac{j}{N})p_j + \sum_{j=0}^{N}(a_n + j\frac{1-a_n}{N})p_j + P(\xi > N).$$

Then

$$a_n \left(P(|\xi| > N) + N^{-1} \sum_{|j| \le N} |j| p_j \right) = N^{-1} \sum_{j=0}^{N} j p_j + P(\xi > N).$$

Denote

$$F_+(x) = P(\xi > x),\ F_-(x) = P(\xi < x);$$
$$G_+(x) = \sum_{0 \le j \le x} j p_j,\ G_-(x) = \sum_{-x \le j \le 0} j p_j.$$

Then

$$a_n = \frac{NF_+(N) + G_+(N)}{NF_+(N) + G_+(N) + NF_-(N) - G_-(N)},$$
$$x_i^{(n)} = \begin{cases} a_n(1 - \frac{i}{N}) + \frac{i}{N},\ 0 \le i \le N, \\ a_n(1 + \frac{i}{N}),\ -N \le i \le 0. \end{cases}$$

From remark 2.3 it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{G_+(N)}{G_+(N) - G_-(N)} = \frac{\beta_+}{\beta_+ + \beta_-},$$

which yields A3.

It remains to check condition A2 about the modulus of continuity.

Let $\{\varepsilon_k\}$, $\{\xi_k\}$ be independent sequences of jointly independent random variables with $P(\varepsilon_k = \pm 1) = 1/2,\ \xi_k =^d \xi$. Denote $S(n) := \sum_{k=1}^{n} \varepsilon_k$.

For simplicity we assume that $X(0) = 0$.

Define a random walk $\{\tilde{X}(n), n \ge 0\}$ in the following way:

$$\tilde{X}(n) := \sum_{k=1}^{n-r(n)} \varepsilon_k + \sum_{j=0}^{r(n)} \xi_j = S(n - r(n)) + \sum_{j=0}^{r(n)} \xi_j,$$

where $r(n) = \sum_{k=1}^{n-1} \mathbb{1}_{\tilde{X}(k)=0}$ is the number of times the sequence $\tilde{X}$ visits 0 before the moment n.

It is not difficult to see that the distributions of the random sequences $\tilde{X}$ and X coincide. Therefore it is enough to check condition A2 for the sequence $\tilde{X}$. Without loss of generality we assume below $T = 1$.

Note that if $\delta > 1/n$, then

$$\frac{\max_{0\le k\le r(n)} |\xi_k|}{\sqrt{n}} \le \omega_{\tilde{X}_n}(\delta) \le 2\omega_{S_n}(\delta) + \frac{2\max_{0\le k\le r(n)} |\xi_k|}{\sqrt{n}},$$

where $\tilde{X}_n(t) := n^{-1/2}\tilde{X}([nt]),\ S_n(t) := n^{-1/2} S([nt]),\ t \ge 0$.

Since for the usual random walk S the following condition is obviously satisfied

$$\forall \varepsilon > 0\ \exists \delta > 0\ \exists n_0\ \forall n \ge n_0\ \ P(\omega_{S_n}(\delta) \ge \varepsilon) \le \varepsilon,$$

then in order to establish A2 it is necessary and sufficient to check that

$$\frac{\max_{0\le k\le r(n)} |\xi_k|}{\sqrt{n}} \xrightarrow{P} 0, n \to \infty. \tag{12}$$

Let τ_k be the moment of the k-th return of the sequence $\tilde{X}$ into 0. The strong Markov property of a Markov chain implies that $\alpha_n := \tau_{n+1} - \tau_n, n \ge 1$ are independent identically distributed random variables.

Note that the distribution of α_n coincides with that of $\sum_{k=1}^{|\xi|} \eta_k$, where $\{\eta_k\}$ are jointly independent and independent of ξ,

$$\eta_k\ =^d\ \inf\{i \ge 0\ :\ S_i = -1\}.$$

Probability-generating function of α equals (see [2])

$$f_\alpha(s) = Es^\alpha = f_{|\xi|}(f_\eta(s)) = f_{|\xi|}\left(\frac{1-\sqrt{1-s^2}}{s}\right).$$

Therefore the Laplace transform of the distribution of α has the form

$$\hat{F}_\alpha(\lambda) = f_\alpha(e^{-\lambda}) = f_{|\xi|}(\frac{1-\sqrt{1-e^{-2\lambda}}}{e^{-\lambda}}) =$$
$$\hat{F}_{|\xi|}\left(-\ln\left(\frac{1-\sqrt{1-e^{-2\lambda}}}{e^{-\lambda}}\right)\right).$$

Assumptions on the distribution of ξ imply that

$$1-\hat{F}_{|\xi|}(\lambda) \sim \lambda(\beta_- + \beta_+)l\left(\frac{1}{\lambda}\right), \ \lambda \to 0+.$$

Therefore

$$1-\hat{F}_\alpha(\lambda) \sim (\beta_- + \beta_+)\sqrt{2\lambda}l\left(\frac{1}{\sqrt{\lambda}}\right), \lambda \to 0+.$$

Remark 3.1. From the fact that $l(x)$ is a slowly varying function it follows that $l(\sqrt{x})$ is also a slowly varying function.

Let $\{a_n\}$ be such a sequence that

$$\frac{n(\beta_- + \beta_+)\sqrt{2}l(\sqrt{a_n})}{\sqrt{a_n}} \to 1, \ n \to \infty.$$

Then the following weak convergence result holds in $D([0,\infty))$

$$\frac{\alpha_1 + \cdots + \alpha_{[n\cdot]}}{a_n} \Rightarrow U(\cdot), \ n \to \infty,$$

where $U(t) = U_{1/2}(t), t \geq 0$ is a non-decreasing stable process with index 1/2, $U(0) = 0$, $E\exp\{-\lambda U(t)\} = \exp\{-\lambda^{1/2}t\}$.

Let

$$b_n = \inf\{k \geq 1 \ : \ a_k \geq n\}. \tag{13}$$

Then

$$\frac{\alpha_1 + \cdots + \alpha_{[b_n\cdot]}}{n} \Rightarrow U(\cdot), n \to \infty. \tag{14}$$

Recall that we assume $X(0) = 0$. Then the following events are equal

$$\{r(n) \geq [b_n x]\} = \left\{\alpha_1 + \cdots + \alpha_{[b_n x]} \leq n\right\}.$$

From (14) it follows

$$\forall \varepsilon > 0 \ \exists x \ \exists n_0 \ \forall n \geq n_0 : \quad P\left(r(n) \geq [b_n x]\right) \leq \varepsilon.$$

Let $\delta > 0$ be arbitrary. Then

$$P\left(\frac{\max_{0\leq k\leq r(n)}|\xi_k|}{\sqrt{n}} > \delta\right) \leq P\left(r(n) \geq [b_n x]\right) + P\left(\frac{\max_{0\leq k\leq b_n x}|\xi_k|}{\sqrt{n}} > \delta\right) \leq$$
$$\varepsilon + 1 - P(|\xi| \leq \sqrt{n}\delta)^{b_n x} =$$
$$\varepsilon + 1 - \exp\{b_n x \ln(1 - P(|\xi| > \sqrt{n}\delta))\} =$$
$$\varepsilon + 1 - \exp\{-b_n x P(|\xi| > \sqrt{n}\delta)(1+o(1))\} =$$
$$\varepsilon + 1 - \exp\{-b_n x o(\frac{l(\sqrt{n}\delta)}{\sqrt{n}\delta})(1+o(1))\}, \ n \to \infty \tag{15}$$

In the last step we used Remark 2.2.

From the definition of b_n, see (13), it follows that

$$\frac{(\beta_- + \beta_+)\sqrt{2}b_n l(\sqrt{n})}{\sqrt{n}} \to 1, \ n \to \infty. \tag{16}$$

That is why the right-hand side of (15) converges to ε as $n \to \infty$. Since both $\varepsilon > 0$ and $\delta > 0$ can be taken arbitrarily we obtain the formula (12). The statement A2 and thus the Theorem are proved.

Remark 3.2. Suppose that ξ does not satisfy the condition of Theorem 2.1, but $1 - Ee^{-s|\xi|} \sim s^\delta \ l(1/s), \ s \downarrow 0$, where $\delta \in [0,1)$, see remark 2.4. Then the limit of the sequence $\{X_n\}$ cannot be a continuous process. To check this, similar to the proof of the Theorem, it is sufficiently to show that the sequence $\{\frac{\max_{0\le k\le r(n)}|\xi_k|}{\sqrt{n}}\}$ does not converge to 0 in probability. Similarly to (15) we have

$$P\left(\frac{\max_{0\le k\le r(n)}|\xi_k|}{\sqrt{n}} > c\right) \ge P\left(\frac{\max_{0\le k\le [b_n x]}|\xi_k|}{\sqrt{n}} > c\right) - P\left(r(n) \le [b_n x]\right) =$$

$$1 - \exp\{[b_n x]\ln(1 - P(|\xi| > \sqrt{n}c))\} - P\left(r(n) \le [b_n x]\right), \tag{17}$$

where $\sqrt{2}b_n l(\sqrt{n}) \sim \sqrt{n^\delta}, \ n \to \infty$.

Let $\varepsilon > 0$ be arbitrary. Applying (14) we choose $x > 0$ such that $P\left(r(n) \le [b_n x]\right) < \varepsilon, n \ge 0$. Remark 2.4 imply that

$$[b_n x]P(|\xi| > \sqrt{n}c) \sim \frac{b_n x l(\sqrt{n}c)}{(\sqrt{n}c)^\delta \Gamma(1-\delta)} \sim \frac{x}{\sqrt{2}c^\delta \Gamma(1-\delta)}, \ n \to \infty.$$

Thus, the right-hand side of (17) can be made arbitrary close to 1 by choosing appropriate x and c for all sufficiently large n, which implies the non-existence of the continuous limit processes.

In the work [8] this statement was established using a different approach. They solved a generalized Skorokhod's problem for the case of non-negative jumps of the initial Markov chain out of zero and directly obtained the exact form of the limit process.

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