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INTERVAL ESTIMATION OF THE FRACTIONAL BROWNIAN MOTION PARAMETER IN A MODEL WITH MEASUREMENT ERROR

In this article we show how to use Baxter statistics for the construction of the non–asymptotic confidence intervals for the Hurst index associated with a fractional Brownian motion within one errors–in–variables model.

1. Introduction

Let $\xi = \{\xi(t), t \in \mathbb{R}\}$ be a fractional Brownian motion with the Hurst index $H \in (0,1)$. This means that ξ is a centered Gaussian random process with a covariance function

$$r(t,s) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{1}$$

The fractional Brownian motion is widely used in such diverse areas as hydrology, geophysics, medicine, genetics, meteorology, financial mathematics. The estimation problem for the Hurst parameter of a fractional Brownian motion, which sometimes is also called self-similarity index, was studied by T. Higuchi [1], J.–M. Poggi and M.–C. Viano [2], J.–F. Coeurjolly [3], B. L. S. Prakasa Rao [4] and others.

It is known that

$$S_n(w) = \sum_{k=1}^{2^n}\left(w\left(\frac{k}{2^n}\right) - w\left(\frac{k-1}{2^n}\right)\right)^2 \to 1,\ n \geq 1$$

with probability one as $n \to \infty$, where $\{w(t), t \in \mathbb{R}\}$ is a standard Brownian motion. This result was discovered by the famous French mathematician P. Levy [5]. Later on, H. Baxter [6] generalized this result for a particular class of Gaussian stochastic processes. The sums $S_n(\xi)$, where $\{\xi(t), t \in [0,1]\}$ is a random process, are named as the Baxter sums.

The Baxter sums method allows us to build the strongly consistent estimators for the Hurst parameter of the fractional Brownian motion. The Baxter type theorems are often used for the parameter estimation of the covariance function of stochastic processes. In particular, J.-M. Bardet [7] used the Baxter approach in order to estimate the Hurst parameter for a fractional Brownian motion. In 2002 O.O. Kurchenko [8] used the Baxter type theorem to construct one strongly consistent estimator for Hurst parameter of a fractional Brownian motion. J.–C. Breton, I. Nourdin and G. Peccati [9] had constructed the confidence intervals for the Hurst index of the one–dimensional fractional Brownian motion using the Baxter statistics and applying the concentration inequality.

Recently, in many papers on statistics of stochastic processes, the various models with measurement errors were used. Such books as G. Shneyevays and G. Mittag [10], R. J. Caroll et al. [11] are devoted to the studying of the linear and nonlinear regression models, K.-L. Cheng and J. Van Ness [12] investigate the linear and polynomial models,

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etc. The monograph by S.V. Masuk, A.G. Kukush et al. [13] describes the statistical methods for the risk estimation at the presence of errors in a regressor.

2. Problem Statement

Consider the following model with errors–in–variables. Suppose that we observe the values $X(0), X\left(\frac{1}{n}\right), \ldots, X(1)$, which differ from the true values of the fractional Brownian motion $\{\xi(t), t \in [0,1]\}$ at the points

$$\left\{ \frac{k}{n} \middle| 0 \le k \le n,\, n \ge 1 \right\}.$$

These differences are the measurement errors $\{\delta_{k,n} \big| 0 \le k \le n\}$, which do not depend on the fractional Brownian motion values $\{\xi\left(\frac{k}{n}\right) \big| 0 \le k \le n\}$. More exactly

$$X\left(\frac{k}{n}\right) = \xi\left(\frac{k}{n}\right) + \delta_{k,n}. \tag{2}$$

Suppose that $\delta_{k,n}$ are i.i.d. Gaussian random variables such that $\delta_{k,n} \simeq N(0, \sigma^2)$ and σ^2 is fixed.

Observing the stochastic process $\{X\left(\frac{k}{n}\right) \big| 0 \le k \le n\}$ in model (2), we need to construct non–asymptotic confidence intervals for the unknown Hurst parameter $H \in (0, H^*]$ with known $H^* < 1$.

3. Main results

Let us introduce the following notations:

$$\Delta\xi_{k,n} = \xi\left(\frac{k+1}{n}\right) - \xi\left(\frac{k}{n}\right),\ \Delta\delta_{k,n} = \delta_{k+1,n} - \delta_{k,n},$$

$$\Delta X_{k,n} = X\left(\frac{k+1}{n}\right) - X\left(\frac{k}{n}\right), 0 \le k \le n-1.$$

Consider the sequences of related Baxter sums:

$$S_n(X) = \sum_{k=0}^{n-1} (\Delta X_{k,n})^2 - 2n\sigma^2,\ \hat{S}_n(X) = n^{2H-1} S_n(X), n \ge 1.$$

It can be shown (see [8]) that for the fractional Brownian motion $\{\xi(t), t \in [0,1]\}$ as $n \to \infty$ and for all $H \in (0,1)$, the convergence in the square mean takes place

$$n^{2H-1} \sum_{k=0}^{n-1} (\Delta\xi_{k,n})^2 \to 1.$$

Lemma 3.1. *Let $H \in (0, H^*] \subset (0,1)$. Then:*

$$\sup_{H \in (0, H^*]} Var\hat{S}_n(X) \le D\left(H^*, n\right), \tag{3}$$

where

$$D(H^*, n) = \frac{10}{n} + 8n^{2H^*-1}\sigma^2 + 8n^{4H^*-1}\left(1 - \frac{1}{n}\right)\sigma^4 +$$
$$+ \begin{cases} \frac{2}{n}\ \zeta(4 - 4H^*), & H^* \in (0, \frac{3}{4}); \\ \frac{2}{n}(1 + \ln n), & H^* = \frac{3}{4}; \\ \frac{2}{n}\left(1 + \frac{n^{4H^*-3}}{4H^*-3}\right), & H^* \in \left(\frac{3}{4}, 1\right), \end{cases} \tag{4}$$

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, s > 1.$

Proof. Let us calculate the mean and variance of random variable $\hat{S}_n(X), n \geq 1$. Since the fractional Brownian motion $\{\xi(t), t \in [0,1]\}$ under the model (2) is a stochastic process with zero mean, covariance function (1) and homogeneous increments, we get for the expectation of $\hat{S}_n(X)$:

$$E\hat{S}_n(X) = n^{2H-1}ES_n(X) = n^{2H-1}E\left(\sum_{k=0}^{n-1}(\Delta X_{k,n})^2 - 2n\sigma^2\right) =$$
$$= n^{2H-1}\left(\sum_{k=0}^{n-1}E\left(X^2\left(\frac{k+1}{n}\right) - 2X\left(\frac{k+1}{n}\right)X\left(\frac{k}{n}\right) + \right.\right.$$
$$\left.\left. +X^2\left(\frac{k}{n}\right)\right) - 2n\sigma^2\right).$$

Using that $\{\delta_{k,n} \big| 0 \leq k \leq n\}$ and variables $\{\xi\left(\frac{k}{n}\right), \delta_{k,n} \big| 0 \leq k \leq n\}$ are stochastically independent and taking into account the covariance function form (1) from the previous equality, we can conclude that

$$E\hat{S}_n(X) = n^{2H-1}\left(\sum_{k=0}^{n-1}\left(E\xi^2\left(\frac{1}{n}\right) + E\delta_{k+1,n}^2 + E\delta_{k,n}^2 - 2E\delta_{k,n}\delta_{k+1,n}\right) - 2n\sigma^2\right) =$$
$$= n^{2H-1}\left(\sum_{k=0}^{n-1}\left(\frac{1}{n^{2H}} + 2\sigma^2\right) - 2n\sigma^2\right) = 1.$$

Note that for the Gaussian random variables $\eta_1, \eta_2, \eta_3, \eta_4$ with zero mean the Isserlis formula can be applied [14, p. 29] as follows:

$$E\left(\eta_1\eta_2\eta_3\eta_4\right) = E\eta_1\eta_2E\eta_3\eta_4 + E\eta_1\eta_3E\eta_2\eta_4 + E\eta_1\eta_4E\eta_2\eta_3. \tag{5}$$

Then the formula (5) and the variance properties imply:

$$Var\hat{S}_n(X) = n^{4H-2}VarS_n(X) =$$
$$= n^{4H-2}E\left(\sum_{k=0}^{n-1}(\Delta X_{k,n})^2 - \sum_{k=0}^{n-1}E\left(\Delta X_{k,n}\right)^2\right)^2 =$$
$$= 2n^{4H-2}\sum_{k,j=0}^{n-1}\left(E\Delta X_{k,n}\Delta X_{j,n}\right)^2 = 2n^{4H-2}\sum_{k=0}^{n-1}\left(E\left(\Delta X_{k,n}\right)^2\right)^2 +$$
$$+4n^{4H-2}\sum_{\substack{k,j=0,\\ j<k}}^{n-1}\left(E\left(\Delta\xi_{k,n} + \Delta\delta_{k,n}\right)\left(\Delta\xi_{j,n} + \Delta\delta_{j,n}\right)\right)^2. \tag{6}$$

Since $\{\xi(t), t \in [0,1]\}$ is the fractional Brownian motion by definition, we obtain:

$$E\left(\Delta X_{k,n}\right)^2 = E\left(X^2\left(\frac{k+1}{n}\right) - 2X\left(\frac{k+1}{n}\right)X\left(\frac{k}{n}\right) + X^2\left(\frac{k}{n}\right)\right) =$$
$$= E\xi^2\left(\frac{k+1}{n}\right) + E\delta_{k+1,n}^2 + E\xi^2\left(\frac{k}{n}\right) + E\delta_{k,n}^2 -$$
$$-2E\left(\xi\left(\frac{k+1}{n}\right) + \delta_{k+1,n}\right)\left(\xi\left(\frac{k}{n}\right) + \delta_{k,n}\right) = n^{-2H} + 2\sigma^2. \tag{7}$$

It follows from the stochastic independency of random variables $\Delta\xi_{k,n}$ and $\Delta\delta_{k,n}, 0 \leq k \leq n$ that

$$E\left(\Delta\xi_{k,n}+\Delta\delta_{k,n}\right)\left(\Delta\xi_{j,n}+\Delta\delta_{j,n}\right)=E\Delta\xi_{k,n}\Delta\xi_{j,n}+E\Delta\delta_{k,n}\Delta\delta_{j,n}, k\neq j.$$

Further, from formula (1) we have

$$E\Delta\xi_{k,n}\Delta\xi_{j,n}=\left(\xi\left(\frac{k+1}{n}\right)-\xi\left(\frac{k}{n}\right)\right)\left(\xi\left(\frac{j+1}{n}\right)-\xi\left(\frac{j}{n}\right)\right)=$$

$$=\frac{1}{2}\left|\frac{(k-j)+1}{n}\right|^{2H}-\left|\frac{k-j}{n}\right|^{2H}+\frac{1}{2}\left|\frac{(k-j)-1}{n}\right|^{2H}. \tag{8}$$

Then from the equality (8) we get:

$$E\Delta X_{k,n}\Delta X_{j,n}=\frac{1}{2}n^{-2H}v_{k-j}+E\Delta\delta_{k,n}\Delta\delta_{j,n},$$

where

$$v_{k-j}=|(k-j)+1|^{2H}-2|k-j|^{2H}+|(k-j)-1|^{2H},\ 1\leq k,j\leq n-1.$$

Let us calculate the expectation of the product of variables $\Delta\delta_{k,n}$ and $\Delta\delta_{j,n}$

$$E\Delta\delta_{k,n}\Delta\delta_{j,n}=E\Big(\delta_{k+1,n}\delta_{j+1,n}-\delta_{k+1,n}\delta_{j,n}-\delta_{k,n}\delta_{j+1,n}+$$

$$+\delta_{k,n}\delta_{j,n}\Big)=\begin{cases}2\sigma^2, & \text{if } k=j;\\ -\sigma^2, & \text{if } k=j-1;\\ 0, & \text{if } |j-k|>1.\end{cases} \tag{9}$$

Thus, based on the relations (7) – (9), the inequality $(a+b)^2\leq 2\left(a^2+b^2\right), a,b\in\mathbb{R}$ and the equality (6) we receive that:

$$Var\hat{S}_n(X)=2n^{4H-2}\left(\sum_{k=0}^{n-1}\left(n^{-2H}+2\sigma^2\right)^2+2\sum_{\substack{k,j=0,\\ j<k}}^{n-1}\left(\frac{1}{2}n^{-2H}v_{k-j}+\right.\right.$$

$$\left.\left.+E\Delta\delta_{k,n}\Delta\delta_{j,n}\right)^2\right)\leq 2n^{4H-2}\left(\sum_{k=0}^{n-1}\left(n^{-4H}+4n^{-2H}\sigma^2+4\sigma^4\right)+\right.$$

$$\left.+4\sum_{\substack{k,j=0,\\ j<k}}^{n-1}\left(\frac{1}{4}n^{-4H}v_{k-j}^2+\left(E\Delta\delta_{k,n}\Delta\delta_{j,n}\right)^2\right)\right)=\frac{2}{n}+8n^{2H-1}\sigma^2+8n^{4H-1}\sigma^4+$$

$$+8n^{4H-2}\sum_{k-j=1}^{n-1}\left(\frac{1}{4}n^{-4H}v_{k-j}^2+\sigma^4\right)+8n^{4H-2}\sum_{k-j>1}^{n-1}\left(\frac{1}{4}n^{-4H}v_{k-j}^2\right)=$$

$$=\frac{2}{n}+8n^{2H-1}\sigma^2+8n^{4H-1}\sigma^4+8n^{4H-2}(n-1)\sigma^2+\frac{4}{n^2}\sum_{k-j>1}^{n-1}v_{k-j}^2.$$

Denote $k-j=l, j<k$ and let:

$$v_l=(l+1)^{2H}-2l^{2H}+(l-1)^{2H},\ 1\leq l\leq n-1.$$

Then from the previous inequality we obtain:

$$Var\hat{S}_n(X)\leq\frac{2}{n}+8n^{2H-1}\sigma^2+8n^{4H-1}\left(1-\frac{1}{n}\right)\sigma^4+$$

$$+\frac{2}{n^2}\sum_{l=1}^{n-1}(n-l)v_l^2\leq\frac{2}{n}+8n^{2H-1}\sigma^2+8n^{4H-1}\left(1-\frac{1}{n}\right)\sigma^4+\frac{2}{n}\sum_{l=1}^{n-1}v_l^2.$$

Since $v_1^2 = \left(2^{2H} - 2\right)^2 \le 4$ for $H \in (0,1)$, we have

$$Var\hat{S}_n(X) \le \frac{2}{n} + 8n^{2H-1}\sigma^2 + 8n^{4H-1}\left(1 - \frac{1}{n}\right)\sigma^4 + \frac{2}{n}\left(4 + \sum_{l=2}^{n-1} v_l^2\right).$$

The variable v_l is an increment of the second order of the function $f(x) = x^{2H}, x \ge 1$, $H \in (0, H^*]$, $H^* < 1$, corresponding to the interval $[l-1, l+1]$. Thus, the representation obtained in [15, p. 244] for the increments of an arbitrary order implies that for $n = 2$

$$v_l = f''(\theta_l) \cdot 1^2 = 2H(2H-1)\theta_l^{2H-2}, \theta_l \in (l-1, l+1).$$

Note that

$$\sup_{H\in(0,1)} |2H(2H-1)| = 2, l-1 < \theta_l, l \ge 2.$$

So, the following inequality takes place

$$v_l^2 \le \frac{4}{(l-1)^{4-4H}}, \, l \ge 2.$$

Hence, for the variance of the random variable $\hat{S}_n(X)$ we get the following upper estimate:

$$\sup_{H\in(0,H^*]} Var\hat{S}_n(X) \le \frac{10}{n} + 8n^{2H^*-1}\sigma^2 +$$

$$+8n^{4H^*-1}\left(1 - \frac{1}{n}\right)\sigma^4 + \frac{2}{n}\sum_{l=2}^{n-1} \frac{1}{(l-1)^{4-4H^*}}. \tag{10}$$

For $H^* \in (0, \frac{3}{4})$ we get

$$\sum_{l=1}^{n-1} \frac{1}{l^{4-4H^*}} \le \zeta(4 - 4H^*).$$

At $H^* = \frac{3}{4}$ we obtain

$$\sum_{l=1}^{n-1} \frac{1}{l} \le 1 + \int_1^n \frac{dx}{x} = 1 + \ln n,$$

and for $H^* \in \left(\frac{3}{4}, 1\right)$:

$$\sum_{l=1}^{n-1} \frac{1}{l^{4-4H^*}} \le 1 + \int_1^n \frac{dx}{x^{4-4H^*}} = 1 + \frac{n^{4H^*-3}}{4H^*-3}.$$

Therefore, the latter considerations and inequality (10), for all $H \in (0, H^*], H^* < 1$ imply the inequality

$$\sup_{H\in(0,H^*]} Var\hat{S}_n(X) \le D(H^*, n),$$

where $D(H^*, n)$ is determined by (4). The Lemma is proved. $\square$

4. Confidence intervals

Let us proceed to the confidence intervals construction.

Let $1-p \in (0,1)$ be a given confidence level and the inequality

$$|\hat{S}_n(X) - 1| = |n^{2H-1} S_n(X) - 1| < \varepsilon$$

be true with high probability $\varepsilon > 0$. This implies that

$$P\left\{|\hat{S}_n(X) - 1| > \varepsilon\right\} \le p \tag{11}$$

with a low probability $p \in (0,1)$.

Under our assumptions and inequality (11) we have the following double inequality:

$$1 - \varepsilon < n^{2H-1} S_n(X) < 1 + \varepsilon.$$

Solving this inequality with regard to the unknown parameter $H \in (0, H^*]$, we obtain an estimate for it, i.e. we find an interval containing the unknown parameter:

$$\frac{1}{2}\left(1 + \frac{\ln(1-\varepsilon) - \ln S_n(X)}{\ln n}\right) < H < \frac{1}{2}\left(1 + \frac{\ln(1+\varepsilon) - \ln S_n(X)}{\ln n}\right). \tag{12}$$

Let us find an estimate for the value ε. Using Chebyshev inequality, from (12) we obtain the estimates:

$$P\left\{|\hat{S}_n(X) - 1| > \varepsilon\right\} \le \frac{E\left(\hat{S}_n(X) - 1\right)^2}{\varepsilon^2} \le p. \tag{13}$$

Further, let us apply the upper estimate for a random variable $Var\hat{S}_n(X)$, obtained in Lemma 3.1. Then, from the relations (3), (4) and inequality (13) we have

$$P\left\{|\hat{S}_n(X) - 1| > \varepsilon\right\} \le \frac{E\left(\hat{S}_n(X) - 1\right)^2}{\varepsilon^2} \le \frac{D(H^*, n)}{\varepsilon^2} \le p.$$

The inequality above implies that

$$\varepsilon \ge \sqrt{\frac{D(H^*, n)}{p}}. \tag{14}$$

For the corresponding values of $H^* \subset (0,1)$, σ and n, the number of observations, we obtain the optimal value of variance $D(H^*, n)$. Then from the inequality (14) we find an estimate for variable ε. Using the above reasoning again we construct the non–asymptotic confidence intervals for Hurst parameter H of a fractional Brownian motion.

Thus, the following theorem is true.

Theorem 4.1. *Let $H \in (0, H^*]$, where $H^* < 1$ is known. Then the interval $(I_l(n), I_r(n)) \cap (0,1)$ is a confidence interval for the Hurst parameter H of a fractional Brownian motion $\{\xi(t), t \in [0,1]\}$ with the confidence probability $1-p \in (0,1)$, where*

$$I_l(n) = \frac{1}{2}\left(1 + \frac{\ln(1-\varepsilon) - \ln S_n(X)}{\ln n}\right), I_r(n) = \frac{1}{2}\left(1 + \frac{\ln(1+\varepsilon) - \ln S_n(X)}{\ln n}\right),$$

where ε satisfies (14) and $D(H^*, n)$ is determined by the equality (4).

Example 4.1. Consider the given observation model with measurement error of the fractional Brownian motion values $\{\xi(t), t \in [0,1]\}$ in the form of (2).

Let $1 - p = 0.9$ be a given confidence level.

Table 1 shows the length of the confidence intervals $\Delta I(n) := I_r(n) - I_l(n)$ for the Hurst parameter H at the corresponding values of parameter $H^* \subset (0,1)$, variables σ and ε using Theorem 4.1.

σ	$H^*=0.5$		$H^*=0.7$		$H^*=0.8$	
	ε	$\Delta I(n)$	ε	$\Delta I(n)$	ε	$\Delta I(n)$
$10^{-2.5}$	$n=128900$		$n=4008$		$n=1727$	
	0.053	0.0045	0.315	0.0395	0.707	0.118

Table 1. Lengths of the confidence intervals $(I_l(n), I_r(n))$.

5. Conclusion

In this article we obtained the non–asymptotic confidence intervals for Hurst parameter H of the fractional Brownian motion in one measurement error model using the Baxter statistics.

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