Theory of Stochastic Processes
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M. V. TANTSIURA

ON STRONG SOLUTIONS TO COUNTABLE SYSTEMS OF SDES WITH INTERACTION AND NON-LIPSCHITZ DRIFT

A countable system of stochastic differential equations is considered. A theorem on existence and uniqueness of a strong solution is proved if drift and diffusion coefficients satisfy finite interaction radius condition.

1. Introduction

Consider an infinite system of stochastic differential equations in $\mathbb{R}$

$$(1)\qquad \begin{cases} dX_k(t) = a(X_k(t), \mu(t))dt + b(X_k(t), \mu(t))dw_k(t),\ k \in \mathbb{Z}, t \in [0, T], \\ \mu(t) = \sum_{k\in\mathbb{Z}} \delta_{X_k(t)}, \\ X_k(0) = u_k,\ k \in \mathbb{Z}. \end{cases}$$

Equation (1) can be considered as an equation that describes motion of an infinite system of interacting particles in a random medium. We can interpret $X_k(t)$ as a position of the k-th particle at a time instant t, the measure $\mu(t)$ as the distribution of all particles' mass at a time instant t. The functions a and b are interaction functions, u_k is an initial position of the k-th particle. We will suppose that $\{u_k | k \in \mathbb{Z}\}$ is a nondecreasing sequence such that $\lim_{k\to+\infty} u_k = +\infty$, $\lim_{k\to-\infty} u_k = -\infty$.

The aim of this work is to prove the existence and uniqueness of a strong solution to equation (1) with non-Lipschitz drift coefficient.

The existence and uniqueness of a strong solution of a stochastic differential equation is well studied in a finite-dimensional case. Zvonkin [11] proved the existence and uniqueness of a strong solution of a one-dimensional stochastic differential equation with non-Lipschitz coefficients. Veretennikov [10] proved the similar result for the multidimensional case. For example, it follows from [10] that if a is a measurable function that satisfies the linear growth condition, diffusion coefficient b is Lipschitz continuous and uniformly elliptic, then there exists a unique strong solution of the equation

$$dX(t) = a(X(t))dt + b(X(t))dw(t), X(0) = x_0.$$

In [1] Veretennikov's result is generalized for the infinite-dimensional case, but their assumptions are not satisfied for equation (1).

The existence and uniqueness of a strong solution to equation (1) was proved in [7] for a bounded measurable drift coefficient $a(\cdot, \cdot)$ that satisfies the finite interaction radius condition and $b(\cdot, \cdot) \equiv 1$. The main idea of the proof was to divide the system of particles into a countable number of finite subsystems that do not interact. Then we apply Veretennikov's theorem with some additional argumentation because the division into subsystems is anticipating. The idea to divide a system of particles into finite clusters that do not interact was implemented in [8] to build Hamiltonian dynamics for a countable systems of interacting particles. Also the similar idea was used in [5] for the construction of a countable system of sticky Brownian particles. However, proofs of cited

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papers are unapplicable directly to equation (1). For instance, if $b(\cdot,\cdot)$ is not constant then stochastic integrals

$$\int_0^t b(X_k(s),\mu(s))dw_k(s),\ k\geq 1,$$

are not independent. Thats why we need some additional argumentation.

In section 2 we formulate the main results of the paper. In section 3 we prove the existence of a weak solution. In section 4 we prove the pathwise uniqueness of a solution to (1). Then the existence and uniqueness of a strong solution follow from the Yamada-Watanabe theorem.

2. Main results

Denote by $\mathfrak{M}$ the space of all locally finite measures on $\mathbb{R}$ with a vague topology τ defined by

$$\nu_n \xrightarrow{\tau} \nu \Leftrightarrow \forall f\in C_c(\mathbb{R}): \int_{\mathbb{R}} f d\nu_n \to \int_{\mathbb{R}} f d\nu, n\to\infty,$$

where $C_c(\mathbb{R})$ is the set of all continuous functions with compact support. Denote

$$p_w(t,x) = P(\sup_{s\in[0,t]} w(s)\geq x) = 2\int_{x\vee 0}^{\infty}\frac{1}{\sqrt{2\pi t}}\exp(-y^2/2t)dy,\ x\in\mathbb{R}, \tag{2}$$

where w is a Wiener process.

The following theorem is the main result of this paper.

Theorem 2.1. *Suppose that:*

1) function a is continuous and bounded:

$$\|a\|_\infty := \sup_{x\in\mathbb{R}}\sup_{\nu\in\mathfrak{M}}|a(x,\nu)|<\infty;$$

2) function b is continuous, bounded and separated from zero:

$$\|b\|_\infty := \sup_{x\in\mathbb{R}}\sup_{\nu\in\mathfrak{M}}|b(x,\nu)|<\infty,\quad \inf_{x\in\mathbb{R}}\inf_{\nu\in\mathfrak{M}}|b(x,\nu)|>0;$$

3) for every $n\in\mathbb{N}$ there exists a constant $C_{b,n}$ such that for every x, y, $x_1,...,x_n$, $y_1,...,y_n$:

$$\big|b\big(x,\sum_{k=1}^n\delta_{x_k}\big)-b\big(y,\sum_{k=1}^n\delta_{y_k}\big)\big|\leq C_{b,n}|x-y|+C_{b,n}\sum_{k=1}^n|x_k-y_k|;$$

4) the functions a and b satisfy the finite interaction radius condition:

$$\exists d>0\ \forall x\in\mathbb{R}\ \forall\nu\in\mathfrak{M}: a(x,\nu)=a(x,\nu I_{(x-d,x+d)}),\ b(x,\nu)=b(x,\nu I_{(x-d,x+d)}),$$

where $(\nu I_B)(A)=\nu(A\cap B)$, $A,B\in B(\mathbb{R})$;

5) measure μ satisfies the following condition: there exists a deterministic increasing sequence $\{z_n|n\in\mathbb{Z}\}$ such that

$$\lim_{n\to\infty} z_n=+\infty,\quad \lim_{n\to-\infty} z_n=-\infty$$

and

$$\exists r>0\ \forall n\in\mathbb{Z}: \prod_{i\in\mathbb{Z}}\big(1-2p_w(T\|b\|_\infty^2,|z_n-u_i|-\|a\|_\infty T-d/2)\big)>r. \tag{3}$$

Then there exists a unique strong solution of equation (1).

Remark 2.1. Condition 4 means that if distance between two particles is greater then d, then they do not interact.

The interaction functions a and b can be the following

$$a(x,\nu) = g\left(\int_{\mathbb{R}} f(x-u)\nu(du)\right) = g\left(\sum_{k\in\mathbb{Z}} f(x-u_k)\right),$$

where $\nu = \sum_{k\in\mathbb{Z}} \delta_{u_k}$, function f is a continuous function such that $\operatorname{supp} f \subset (-d,d)$, function g is a continuous bounded function.

Condition 5 in Theorem 2.1 is the hardest condition to check. Following examples give a wide class of measures $\mu(0)$ that satisfy this condition.

Example 2.1. For a locally finite measure ν denote

$$\Lambda(\nu) := \limsup_{n\to\infty} \frac{\nu([-n,n])}{2n}.$$

If $2\Lambda(\mu(0))d < 1$, then measure $\mu(0)$ satisfies condition 5 of Theorem 2.1 (see [7]).

Example 2.2. Let m be a locally finite measure on $\mathbb{R}$ such that

$$\exists C_m > 0 \ \forall [a,b] \subset \mathbb{R} : m([a,b]) \le C_m(b-a+1),$$

$\mu(0)$ be a Poisson point measure with intensity m. Suppose that measure $\mu(0)$ and $\{w_k,\ k\in\mathbb{Z}\}$ are independent. Then measure $\mu(0)$ satisfies condition 5 of Theorem 2.1 almost surely (see [7]).

The idea of proof of Theorem 2.1 is to verify the existence of a weak solution and pathwise uniqueness. Then the existence and uniqueness of a strong solution follow from the Yamada-Watanabe theorem.

The existence of a weak solution is proved in section 3 for bounded continuous coefficients. The proof is rather standard. Weak compactness of a sequence of solution approximations is proved and then Skorokhod's representation theorem is used.

The pathwise uniqueness is proved in section 4 in the following way. Using assumption 5 of Theorem 2.1 we prove that for any two strong solutions (1) the system of particles almost surely can be divided into a countable number of the same finite subsystems so that distance between any two subsystems is greater than d for every $t \in [0,T]$. Then it follows from condition 4 that these subsystems do not interact. Hence the infinite system of stochastic differential equations (1) can be divided into a countable number of finite systems of stochastic differential equations. Generally speaking, the division into subsystems is random and anticipating, so the pathwise uniqueness doesn't follow directly from Veretennikov's theorem. The required argumentation can be done similarly to the work [7].

The fact that $\mathbb{R}$ is linearly ordered is used in this paper to prove that the system of particles can be divided into finite subsystems that do not interact. On the other side, the idea to divide the system into a countable number of clusters that do not interact may work also in $\mathbb{R}^d$. But some other methods are needed to prove existence of such clusters. The multidimensional case will be considered in future works.

3. Existence of the weak solution

Theorem 3.1. *Suppose that a and b are bounded and continuous and there exists a constant $L > 0$ such that*

$$\limsup_{m\to\infty} \mu(0,[-m,m])/m^L < \infty. \tag{4}$$

Then there exists a weak solution of equation(1).

The idea of the proof is to approximate the countable system (1) by a sequence of finite systems, select a convergent subsequence and pass to a limit.

For any segment $[\alpha,\beta] \subset \mathbb{R}$ and number $x \in \mathbb{R}$ we will denote by $d([\alpha,\beta],x)$ the Euclidian distance from the point x to the segment $[\alpha,\beta]$.

The following lemma gives an apriori estimate of the probability that $\{X_k(t), t \in [0,T]\}$ visits a fixed interval.

Lemma 3.1. *If $\{X_k(\cdot),\ k \in \mathbb{Z},\ \mu(\cdot)\}$ is a solution of equation (1) then for every $i \in \mathbb{Z}$*

$$P(\exists t \in [0,T] : X_i(t) \in [\alpha,\beta]) \le p_w(T\|b\|_\infty^2, d([\alpha,\beta],u_i) - \|a\|_\infty T).$$

Proof of Lemma 3.1. It follows from (1) that

$$M_i(t) := X_i(t) - u_i - \int_0^t a(X_i(s),\mu(s))ds = \int_0^t b(X_i(s),\mu(s))dw_i(s), t \in [0,T],$$

is a continuous martingale. Therefore there exists a Wiener process $B_i(\cdot)$ such that

$$\forall t \in [0,T]\ M_i(t) = B_i(\langle M_i\rangle(t)). \tag{5}$$

For every $t \in [0,T]$,

$$\left|\int_0^t a(X_i(s),\mu(s))ds\right| \le \|a\|_\infty t \le \|a\|_\infty T.$$

Hence

$$P(\exists t \in [0,T] : X_i(t) \in [\alpha,\beta]) \le P(\sup_{t\in[0,T\|b\|_\infty^2]} B_i(t) \ge d([\alpha-\|a\|_\infty T, \beta+\|a\|_\infty T], u_i)) =$$

$$= p_w(T\|b\|_\infty^2, d([\alpha,\beta],u_i) - \|a\|_\infty T).$$

The lemma is proved.

For any $n \ge 1$ consider an equation

$$\begin{cases} dX_k^n(t) = a(X_k^n(t),\mu^n(t))dt + b(X_k^n(t),\mu^n(t))dw_k(t),\ k = \overline{-n,n}, t \in [0,T], \\ \mu_t^n = \sum_{i=-n}^n \delta_{X_i^n(t)}, \\ X_k^n(0) = u_k,\ k = \overline{-n,n}, \\ X_k^n(t) = u_k,\ |k| > n,\ t \in [0,T]. \end{cases} \tag{6}$$

For $|k| \le n$ the system (6) is a usual finite system of stochastic differential equations. Expressions $a(X_k^n(t), \sum_{i=-n}^n \delta_{X_i^n(t)})$ and $b(X_k^n(t), \sum_{i=-n}^n \delta_{X_i^n(t)})$ are continuous functions as functions of the vector

$$(X_{-n}^n(t), X_{-n+1}^n(t), ..., X_{n-1}^n(t), X_n^n(t)).$$

It follows from [9, section 3.3] that there exists a weak solution of equation (6) for $|k| \le n$. So, there exists a weak solution of equation (6).

We need the following lemma about weak relative compactness.

Lemma 3.2. *Suppose that the assumptions of Theorem 3.1 are satisfied. Then for every integer i the sequence of distributions of $\{X_i^n(\cdot), n \ge 1\}$ is relatively compact as a sequence of random elements in $C([0,T])$.*

Proof of Lemma 3.2. It is enough to verify (see [6, Theorem 1.4.7]) that there exists a constant $C > 0$ such that:

$$\forall n \ge 1\ \forall t_0, t_1 \in [0,T] : E(X_i^n(t_1) - X_i^n(t_0))^4 \le C|t_1 - t_0|^{3/2} \tag{7}$$

and

$$\forall n \ge 1\ \forall t \in [0,T] : E(X_i^n(t))^4 \le C. \tag{8}$$

These inequalities can be checked in a standard way because the coefficients a and b are bounded.

Lemma 3.3. *Suppose that the assumptions of Theorem 3.1 are satisfied. Let f be a continuously differentiable function with compact support. Then the sequence $\{\langle f, \mu^n(t)\rangle\}$ is weakly relatively compact as a sequence of random elements in $C([0,T])$.*

Proof of Lemma 3.3. It is enough to check two conditions:

$$\forall \varepsilon > 0\ \exists C > 0\ \forall k \geq 1\ P(|\langle f, \mu^n(0)\rangle| > C) < \varepsilon, \tag{9}$$

$$\forall \varepsilon > 0\ \exists \delta \in (0,1)\ \forall k \geq 1\ P(\omega_{\langle f,\mu^n(\cdot)\rangle}(\delta) \geq \varepsilon) \leq \varepsilon, \tag{10}$$

where $\omega_g(\delta) = \sup_{|s-t|<\delta} |g(t) - g(s)|$.

The function f is a continuous function with compact support. So

$$\langle f, \mu^n(0)\rangle = \sum_{|i|\leq n} f(X_i^n(0)) = \sum_{|i|\leq n} f(u_i) \to \sum_{i\in\mathbb{Z}} f(u_i),\ n \to \infty.$$

Therefore the sequence $\{\langle f, \mu^n(0)\rangle,\ n \geq 1\}$ is bounded and the condition (9) is satisfied.

Let us check the condition (10). It is easy to see that for every $M \in \mathbb{N}$

$$P(\omega_{\langle f,\mu^n(\cdot)\rangle}(\delta) \geq \varepsilon) \leq P(\exists |i| > M\ \exists t \in [0,T]:\ X_i^n(t) \in supp f) + \\ + \sum_{|i|\leq M} P\left(\omega_{X_i}(\delta) \geq \frac{\varepsilon}{\|f'\|_\infty (2M+1)}\right).$$

Let m be an integer such that $supp\ f \subset\ [-m,m]$. Then

$$P(\exists |i| > M\ \exists t \in [0,T] : X_i^n(t) \in [-m,m]) \leq \\ \leq \sum_{|i|>M} P(\exists t \in [0,T] : X_i^n(t) \in [-m,m]) \leq \\ \leq \sum_{i:i>M} p_w(T\|b\|_\infty^2, u_i - \|a\|_\infty T - m) + \sum_{i:i<-M} p_w(T\|b\|_\infty^2, -m - u_i - \|a\|_\infty T).$$

Here the last inequality follows from Lemma 3.1. Denote by S_M the right hand side of the last inequality. The inequality (4) implies that $S_M < \infty$. It follows from Lemma 3.2 (see [3, Theorem 3.7.2]) that

$$\forall \varepsilon > 0\quad \exists \delta_1 = \delta_1(\varepsilon) \in (0,1)\quad \forall n \geq 1 : P(|\omega_{X_n^i}(\delta_1)| \geq \varepsilon) \leq \varepsilon.$$

Let $\varepsilon > 0$ be a fixed positive number. Let us choose $M \in \mathbb{N}$ such that $S_M < \varepsilon/2$. Now for $\delta = \delta_1(\varepsilon/(2M+1)\|f'\|_\infty)$ we have

$$P(\omega_{\langle f,\mu^n(\cdot)\rangle}(\delta) > \varepsilon) \leq P(\exists |i| \geq M\ \exists t \in [0,T] : X_i^n(t) \in [-m,m]) + \\ + \sum_{|i|\leq M} P(\omega_{X_i^n}(\delta) > \varepsilon/(2M+1)\|f'\|_\infty) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

The lemma is proved.

Lemma 3.4. *Suppose that the assumptions of Theorem 3.1 are satisfied. Then the sequence of distributions of $\{\mu^n(\cdot), n \geq 1\} \subset C([0,T], \mathfrak{M})$ is relatively compact.*

Proof of Lemma 3.4. Let $\{f_n, n \in \mathbb{N}\}$ be a sequence of continuously differentiable functions with compact support. If $\{f_n, n \in \mathbb{N}\}$ is dense in $C_c(\mathbb{R})$, then the topology in $\mathfrak{M}$ is induced by the metric

$$\rho(\mu,\nu) = \sum_{n\in\mathbb{N}} 1/2^n \left(\left|\int_{\mathbb{R}} f_n d\mu - \int_{\mathbb{R}} f_n d\nu\right| \wedge 1\right) \tag{11}$$

and $(\mathfrak{M}, \rho)$ is a complete separable metric space [2].

Denote

$$M(C_m, m \geq 1) = \{\mu \mid \forall m \geq 1 \ \ \mu((-m,m)) \leq C_m\}.$$

It is easy to prove that for every sequence $\{C_m, m \in \mathbb{N}\}$ the set $M(C_m, m \geq 1)$ is compact in $\mathfrak{M}$. To prove the lemma it is enough to check two conditions [3, Theorem 3.7.2]:

1) for every $\varepsilon > 0$ and $t \in [0,T] \cap \mathbb{Q}$ there exists a compact $\Gamma_{\varepsilon,t} \subset \mathfrak{M}$ such that

$$\sup_n P(\mu^n(t) \in \Gamma_{\varepsilon,t}) \geq 1 - \varepsilon;$$

2) $\forall \varepsilon > 0 \ \ \exists \delta > 0 : \sup_n P(\omega_{\mu^n(\cdot)}(\delta) \geq \varepsilon) \leq \varepsilon$.

Let us check condition 1. First note that

$$(12) \quad E \sum_{i \in \mathbb{Z}} 1\!\!1_{\exists t \in [0,T]: X_i^n(t) \in [-m,m]} \leq |\{i : u_i \in [-m - \|a\|_\infty T, m + \|a\|_\infty T]\}| + \\ + \sum_{i: u_i < -m - \|a\|_\infty T} p_w(T\|b\|_\infty^2, -m - \|a\|_\infty T - u_i) + \\ + \sum_{i: u_i > m + \|a\|_\infty T} p_w(T\|b\|_\infty^2, u_i - m - \|a\|_\infty T).$$

Denote by C_m the right hand side of the inequality (12). It follows from (4) that

$$\sum_{i: u_i > m + \|a\|_\infty T} p_w(T\|b\|_\infty^2, u_i - m - \|a\|_\infty T) \leq \\ \leq \sum_{k \geq m + \|a\|_\infty T} \big|\{i : |u_i| < k\}\big| p_w(T\|b\|_\infty^2, k - m - \|a\|_\infty T) < \infty.$$

We can check similarly that the other sum in (12) is also finite. Hence, $C_m < \infty$.

It follows from (12) and the Chebyshev inequality that

$$P(\mu^n(t) \notin M(2^m C_m/\varepsilon, m \geq 1)) \leq \\ \leq \sum_{m \in \mathbb{N}} P(\mu^n(t)([-m,m]) \geq 2^m C_m/\varepsilon) \leq \sum_{m \in \mathbb{N}} \varepsilon/2^m = \varepsilon.$$

Hence, condition 1 is satisfied. Let us check condition 2.

$$\rho(\mu^n(t_1), \mu^n(t_2)) = \sum_{m \in \mathbb{N}} 1/2^m \left(|\langle f_m, \mu^n(t_1)\rangle - \langle f_m, \mu^n(t_2)\rangle| \wedge 1\right) \leq \\ \leq \sum_{m \in \mathbb{N}} 1/2^m (\omega_{\langle f_m, \mu^n(\cdot)\rangle}(|t_1 - t_2|) \wedge 1).$$

Now condition 2 follows from (10). The lemma is proved.

Combining Lemma 3.3, Lemma 3.4 and using Kantor's diagonal method we can prove that there exists a subsequence $\{n_k, k \geq 1\}$ such that the sequence $(X_i^{n_k}(\cdot), i \in \mathbb{Z}, \mu^{n_k}(\cdot))$, $k \geq 1$, is weakly convergent as a sequence of random elements in $C([0,T], \mathbb{R}^\infty) \times C([0,T], \mathfrak{M})$. Therefore, Skorokhod's representation theorem implies that there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ and random processes $\{\widetilde{X}_i^{n_k}(\cdot), \widetilde{w}_i^{n_k}(\cdot), \widetilde{\mu}^{n_k}(\cdot)\}_{i \in \mathbb{Z},\, k \geq 1}$ and $\widetilde{X}_i(\cdot), \widetilde{w}_i(\cdot)$, $i \in \mathbb{Z}, \widetilde{\mu}(\cdot)$ defined on this space such that $\widetilde{X}_i^{n_k} \overset{a.s.}{\to} \widetilde{X}_i$, $\widetilde{w}_i^{n_k} \overset{a.s.}{\to} \widetilde{w}_i$ as random elements in $C([0,T])$, and $\widetilde{\mu}^{n_k}(\cdot) \to \widetilde{\mu}(\cdot)$, $k \to \infty$, as random elements in $C([0,T], \mathfrak{M})$. Moreover,

$$\forall k \geq 1 \ (\widetilde{X}_i^{n_k}(\cdot), \widetilde{w}_i^{n_k}(\cdot), \widetilde{\mu}^{n_k}(\cdot))_{i \in \mathbb{Z}} \overset{d}{=} (X_i^{n_k}(\cdot), w_i^{n_k}(\cdot), \mu^{n_k}(\cdot))_{i \in \mathbb{Z}}.$$

For simplicity of notation we will assume that

$$\forall i \ X_i^n(\cdot) \overset{a.s.}{\to} X_i(\cdot), \ w_i^n(\cdot) \overset{a.s.}{\to} w_i(\cdot) \text{ and } \mu^n(\cdot) \overset{a.s.}{\to} \mu(\cdot) \text{ as } n \to \infty.$$

Lemma 3.5. *Suppose that assumptions of Theorem 3.1 are satisfied. Then*

$$\forall f \in C_c(\mathbb{R}) : E \sup_{t\in[0,T]} \sum_{i\in\mathbb{Z}} |f(X_i^n(t)) - f(X_i(t))| \to 0,\ n \to \infty.$$

Proof of Lemma 3.5. Consider any $f \in C_c(\mathbb{R})$. Let $l \in \mathbb{N}$ be such that *supp* $f \subset (-l,l)$.

Then

$$E \sup_{t\in[0,T]} \sum_{i\in\mathbb{Z}} |f(X_i^n(t)) - f(X_i(t))| \le E \sup_{t\in[0,T]} \sum_{|i|\le N} |f(X_i^n(t)) - f(X_i(t))| + \tag{13}$$
$$+ \|f\|_\infty \sum_{|i|>N} (P(\exists t \in [0,T] : X_i^n(t) \in (-l,l)) + P(\exists t \in [0,T] : X_i(t) \in (-l,l)))$$

Similarly to estimation of (12), we can prove that Lemma 3.1 (applied to X_i^n) and condition (4) imply the following

$$\lim_{N\to\infty} \sup_{n\in\mathbb{N}} \sum_{|i|>N} P(\exists t \in [0,T] : X_i^n(t) \in (-l,l)) = 0.$$

Let $\varepsilon > 0$ be any fixed positive number. Select $N \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} \sum_{|i|>N} P(\exists t \in [0,T] : X_i^n(t) \in (-l,l)) < \varepsilon.$$

It follows from Fatou's lemma that

$$\underline{\lim}_{n\to\infty} \sum_{|i|>N} P(\exists t \in [0,T] : |X_i^n(t)| > l) \ge \sum_{|i|>N} P(\exists t \in [0,T] : |X_i(t)| > l).$$

Thus, the right hand side of (13) is less than or equal to 3ε for all $N \ge N_0$. The lemma is proved.

Lemma 3.5 implies that

$$\forall j \in \mathbb{N} \sup_{t\in[0,T]} \sum_{i\in\mathbb{Z}} |f_j(X_i^n(t)) - f_j(X_i(t))| \xrightarrow{P} 0,\ n \to \infty,$$

where functions f_j are from the definition of the metric (11). Therefore, $\mu^n(\cdot) \to \mu(\cdot)$ in probability. Passing if necessary to a subsequence, without loss of generality we may assume that $\mu^n(\cdot) \to \mu(\cdot),\ n \to \infty$ almost surely. Now passing to the limit as $n \to \infty$ in the system

$$\begin{cases} dX_k^n(t) = a(X_k^n(t), \mu^n(t))dt + b(X_k^n(t), \mu^n(t))dw_k^n(t),\ k \in \mathbb{Z}, t \in [0,T], \\ \mu^n(t) = \sum_{i=-n}^n \delta_{X_i^n(t)}, \\ X_k^n(0) = u_k,\ k \in \mathbb{Z}, \end{cases} \tag{14}$$

we obtain that $(X_i(\cdot),\ i \in \mathbb{Z}, \mu(\cdot))$ is a weak solution of equation (1). The convergence of Lebesgue integrals follows from Lebesgue's dominated convergence theorem. The convergence of stochastic integrals can be checked similarly to the proof of the Skorokhod theorem on weak existence of a solution of a stochastic differential equation(see [9, section 3.3]). Theorem 3.1 is proved.

4. Proof of Theorem 2.1

It is enough to prove pathwise uniqueness of the solution of equation (1). Then Theorem 2.1 will follow from Theorem 3.1 and the Yamada-Watanabe theorem (in our case the theorem can be proved similarly to [4, paragraph 4.1]).

Lemma 4.1. *Suppose that* $(X_k^1(t), \mu^1(t),\ k \in \mathbb{Z},\ t \in [0,T])$ *and* $(X_k^2(t), \mu^2(t),\ k \in \mathbb{Z},\ t \in [0,T])$ *are solutions of equation (1). Then there exists an infinite number of* $p \in \mathbb{Z}$ *such that*

$$\forall q \in \{1,2\}\ \forall t \in [0,T]\ \forall i < p\ \forall j \geq p\ |X_i^q(t) - X_j^q(t)| \geq d.$$

Lemma 4.1 and condition 5 of Theorem 2.1 imply that for any two strong solutions of equation (1) the system of particles can be divided into the same finite subsystems that do not interact. If we prove Lemma 4.1, then the pathwise uniqueness can be verified similarly to the proof of Theorem 1.1 in [7] using the Veretennikov theorem.

For every $k \in \mathbb{Z},\ q \in \{1,2\}$ the function

$$M_k^q(t) = \int_0^t b(X_k^q(s), \mu^q(s)) dw_k(s)$$

is a martingale. Moreover, for any $k, l\ \in \mathbb{Z}, k \neq l$ and $q, r \in \{1,2\}$ we have $\langle M_k^q, M_l^r \rangle = 0$. Hence there exists a sequence [4, Theorem 2.7.3] of independent pairs of Wiener processes $\{B_k^1(\cdot),\ B_k^2(\cdot) | k \in \mathbb{Z}\}$ (inside the pair the processes B_k^1, B_k^2 might be dependent) such that

$$\forall t \in [0,T]\ \forall k \in \mathbb{Z}\ \forall i \in \{1,2\}:\ M_k^i(t) = B_k^i(\langle M_k^i \rangle(t)). \tag{15}$$

Denote

$$\xi_k^i = u_k + \|a\|_\infty T + \sup_{t \in [0, T\|b\|_\infty^2]} B_k^i(t),\ \eta_k^i = u_k - \|a\|_\infty T + \inf_{t \in [0, T\|b\|_\infty^2]} B_k^i(t), \tag{16}$$

$$A_k = \{\forall q \in \{0,1\} \sup_{i: u_i < z_k} \xi_i^q \leq z_k - d/2 - \|a\|_\infty T, \inf_{i: u_i > z_k} \eta_i^q \geq z_k + d/2 + \|a\|_\infty T\}. \tag{17}$$

Assume that A_k occurs, and $u_i < z_k \leq u_j$. Then

$$\forall q \in \{1,2\}\ \forall t \in [0,T]\quad |X_i^q(t) - X_j^q(t)| \geq d,$$

where d is an interaction radius. So, the proof of Lemma 4.1 follows from the next lemma.

Lemma 4.2. *Events* $\{A_k,\ k \geq 1\}$ *and events* $\{A_k,\ k \leq -1\}$ *occur infinitely often.*

Remark 4.1. We will prove that the events $\{A_k,\ k \geq 1\}$ occur infinitely often. The fact that the events $\{A_k,\ k \leq -1\}$ occur infinitely often can be proved similarly.

Remark 4.2. The events A_k are dependent, so the second Borel-Cantelli lemma can't be directly applied. The idea of the proof is to approximate events from some subsequence A_{k_n} by independent events A'_{k_n}. The events A'_{k_n} will be defined in a similar way as A_{k_n} but the supremum and the infimum are taken over a finite set of indices. If for different n these sets of indices have an empty intersection, then the events A'_{k_n} will be independent.

Proof of Lemma 4.2. First let us prove that

$$\forall k \geq 1\ \ P(\sup_{i: u_i < z_k} \xi_k^i < \infty) = 1. \tag{18}$$

Fix any $k \geq 1$. Condition 5 of Theorem 2.1 implies that

$$\sum_{i: u_i < z_k} p_w(T\|b\|_\infty^2, |z_k - u_i| - \|a\|_\infty T - d/2) < +\infty. \tag{19}$$

The maximum of the random process $(X_k^i(t),\ t \in [0,T])$ can be estimated by the maximum of the Wiener process $(B_k^i(t), t \in [0, T\|b\|_\infty^2])$ similarly to the proof of Lemma 3.1 (see (5)). Hence for every i such that $u_i < z_k$ we have an inequality

$$P\big(\xi_i^q \geq z_k + \|a\|_\infty T + d/2\big) \leq p_w(T\|b\|_\infty^2, |z_k - u_i| - \|a\|_\infty T - d/2).$$

So, it follows from (3) that

$$\forall q \in \{1,2\} \sum_{i: u_i < z_k} P\big(\xi_i^q \geq z_k - \|a\|_\infty T - d/2\big) < +\infty. \tag{20}$$

Now the Borel-Cantelli lemma implies that for every $q \in \{1,2\}$ there exists only a finite number of integers i such that $u_i < z_k$ and $\xi_i^q(t) \geq z_k - \|a\|_\infty T - d/2$. We have proved (18).

For the same reason,

$$\forall q \in \{1,2\} \forall k \geq 1 \quad P(\inf_{i:u_i>z_k} \eta_i^q > -\infty) = 1.$$

Similarly to the proof of (18) we can check that for any integer k

$$\Big|\big\{i \in \mathbb{Z} | u_i > z_k, \forall q \in \{1,2\}\ \eta_i^q \leq z_k - \|a\|_\infty T - d/2\big\}\Big| < \infty \ a.s.$$

Hence,

$$\forall q \in \{1,2\} \quad \eta_n^q \overset{a.s.}{\to} +\infty,\ n \to +\infty,$$

and

$$\forall q \in \{1,2\} \quad \xi_n^q \overset{a.s.}{\to} +\infty,\ n \to +\infty.$$

Therefore, for any $n \in \mathbb{Z}$ and $\varepsilon > 0$ there exists $l(n,\varepsilon)$ such that

$$\forall l \geq l(n,\varepsilon) : P\Big(\forall q \in \{1,2\} \sup_{n<k\leq n+l} \xi_k^q \neq \sup_{k\leq n+l} \xi_k^q\Big) < \varepsilon,$$
$$P\Big(\forall q \in \{1,2\} \inf_{k>n} \eta_k^q \neq \inf_{n+l\geq k>n} \eta_k^q\Big) < \varepsilon. \tag{21}$$

Let $\{m_k | k \in \mathbb{N}\} \subset \mathbb{Z}$ be an increasing sequence. Denote

$$i(k) = \max\{i | u_i \leq z_{m_k}\}.$$

Let us construct a sequence $\{m_k\}$ such that

$$\forall k \geq 1 : \quad i(k+1) - i(k) \geq \max\{l(i(k), 1/2^k),\ 2\}. \tag{22}$$

Then

$$P\Big(\forall M\ \exists k > M\ \forall q \in \{1,2\} \sup_{i\leq i(k)} \xi_i^q \leq z_{m_k} - d/2 - \|a\|_\infty T,$$
$$\inf_{i>i(k)} \eta_i^q \geq z_{m_k} + d/2 + \|a\|_\infty T\Big) \geq$$
$$\geq P\Big(\forall M\ \exists k > M\ \forall q \in \{1,2\} \sup_{i\leq i(k)} \xi_i^q = \sup_{i(k-1)<i\leq i(k)} \xi_i^q,$$
$$\inf_{i>i(k)} \eta_i^q = \inf_{i(k+1)\geq i>i(k)} \eta_i^q, \sup_{i(k-1)<i\leq i(k)} \xi_i^q \leq z_{m_k} - d/2 - \|a\|_\infty T,$$
$$\inf_{i(k+1)\geq i>i(k)} \eta_i^q \geq z_{m_k} + d/2 + \|a\|_\infty T\Big) \geq P(B_1 \cap B_2), \tag{23}$$

where

$$B_1 = \{\exists M\ \forall k > M\ \forall q \in \{1,2\} \sup_{i\leq i(k)} \xi_i^q = \sup_{i(k-1)<i\leq i(k)} \xi_i^q, \inf_{i>i(k)} \eta_i^q = \inf_{i(k+1)\geq i>i(k)} \eta_i^q\}, \tag{24}$$

$$B_2 = \{\forall M\ \exists k > M \forall q \in \{1,2\} \sup_{i(k-1)<i\leq i(k)} \xi_i^q \leq z_{m_k} - d/2 - \|a\|_\infty T,$$
$$\inf_{i(k+1)\geq i>i(k)} \eta_i^q \geq z_{m_k} + d/2 + \|a\|_\infty T\}. \tag{25}$$

The event B_1 means that all but a finite number of the events

$$\{\forall q \in \{1,2\} \sup_{i\leq i(k)} \xi_i^q = \sup_{i(k-1)<i\leq i(k)} \xi_i^q, \inf_{i>i(k)} \eta_i^q = \inf_{i(k+1)\geq i>i(k)} \eta_i^q\}$$

occur, the event B_2 means that the events

$$\{\forall q \in \{1,2\} \sup_{i(k-1)<i\leq i(k)} \xi_i^q \leq z_{m_k} - d/2 - \|a\|_\infty T, \inf_{i(k+1)\geq i>i(k)} \eta_i^q \geq z_{m_k} + d/2 + \|a\|_\infty T\}$$

occur infinitely often.

The inequalities (22) and (21) imply that

$$\sum_{k\geq 1} P\Big(\big\{\forall q\in\{1,2\}\ \sup_{i\leq i(k)}\xi_i^q\neq \sup_{i(k-1)<i\leq i(k)}\xi_i^q\big\}\cup$$
$$\cup\big\{\forall q\in\{1,2\}\ \inf_{i>i(k)}\eta_i^q\neq \inf_{i(k+1)\geq i>i(k)}\eta_i^q\big\}\Big)\leq \sum_{k\geq 1}(1/2^k+1/2^k)<+\infty.$$

Now it follows from the Borel-Cantelli lemma that $P(B_1)=1$.

Denote

$$C_k=\{\forall q\in\{1,2\}\ \sup_{i(2k-1)<i\leq i(2k)}\xi_i^q\leq z_{m_{2k}}-d/2-\|a\|_\infty T,$$
$$\inf_{i(2k+1)\geq i>i(2k)}\eta_i^q\geq z_{m_{2k}}+d/2+\|a\|_\infty T\}$$

Events $C_k, k\geq 1$ are independent. If we prove that

$$\sum_{k\geq 1}P(C_k)=+\infty, \tag{26}$$

then the Borel-Cantelli lemma will imply that $P(B_2)=1$.

Let us estimate the probability $P(C_k)$ from below:

$$P(C_k)=P\left(\bigcap_{i=i(2k-1)+1}^{i(2k)}\{\forall q\in\{1,2\}\ \xi_i^q\leq z_{m_k}-d/2-T\|a\|_\infty\}\cap\right.$$
$$\left.\cap\bigcap_{i=i(2k)+1}^{i(2k+1)}\{\forall q\in\{1,2\}\ \eta_i^q\geq z_{m_k}+d/2+T\|a\|_\infty\}\right)\geq$$
$$\geq\prod_{i=i(2k-1)+1}^{i(2k+1)}(1-2p_w(T,|z_{m_k}-u_i|-d/2-T\|a\|_\infty\}))\geq$$
$$\geq\prod_{i\in\mathbb{Z}}(1-2p_w(T,|z_{m_k}-u_i|-d/2-T\|a\|_\infty\}))\geq r>0,$$

where $r>0$ is from condition 5 of Theorem 2.1. Hence (26) is satisfied and, consequently, $P(B_2)=1$. Therefore, $P(B_1\cap B_2)=1$, and (23) implies that the events A_k occur infinitely often almost surely as $k\to+\infty$. The case $k\to-\infty$ can be considered similarly. This completes the proof of Lemma 4.2. Hence Lemma 4.1 and Theorem 2.1 are proved.

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Institute of Mathematics, National Academy of sciences of Ukraine
E-mail address: mtan@meta.ua