Theory of Stochastic Processes
Vol. 21 (37), no. 1, 2016, pp. 17–30

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ASYMPTOTIC NORMALITY OF LINEAR REGRESSION PARAMETER ESTIMATOR IN THE CASE OF RANDOM REGRESSORS

Sufficient conditions of asymptotic normality of the least squares estimator of linear regression model parameter in the case of discrete time and weak or long-range dependent random regressors and noise are obtained in the paper.

1. Introduction

Linear regression model with random regressors is considered in the paper. The least squares estimator (LSE) is chosen for parameter estimation as one of the most important and much used regression model parameter estimator.

The LSE asymptotic properties of linear regression model parameter with time independent trends in weak dependent random regressors were considered in the book by A. Ya. Dorogovstev [1]. Consistency and asymptotic normality of parameter LSE of the model with continuous time and weak or long-range dependent errors in regressors and random noise were studied in the papers by L. P. Golubovska, A. V. Ivanov, I. V. Orlovsky [2], A. V. Ivanov, I. V. Orlovsky [3, 4]. Strong consistency of parameter LSE of the model with discrete time and weak or long-range dependent random noise and errors in regressors are considered in the paper by I. V. Orlovsky [5].

The aim of the paper is to obtain results similar to [2, 3, 4] for models with discrete time which are widely used in applications. It is not a trivial transfer and requires specific property of sums that contain slowly varying functions (Lemma 3.2). This property is needed to cover the case of long-range dependent errors in regressors and/or random noise. In addition, one of the key points of asymptotic normality proof is the use of a discrete analogue of the Hölder-Young-Brascamp-Lieb inequality (Lemma 3.4).

2. Model and estimator

Consider a regression model

$$X_j = \sum_{i=1}^{q} \theta_i z_{ij} + \varepsilon_j,\ j = \overline{1,N},\quad z_{ij} = a_{ij} + y_{ij},\ i = \overline{1,q}, \tag{1}$$

where $\theta^* = (\theta_1, ..., \theta_q) \in \mathbb{R}^q$ is a vector of unknown parameters (* means transposition), $\{a_{ij},\ j \in \mathbb{N}\} \subset \mathbb{R},\ i = \overline{1,q}$, are some non random sequences and

A1. $y_{ij},\ j \in \mathbb{Z},\ i = \overline{1,q}$, are independent centered stationary Gaussian sequences with covariance functions (c.f.) $B_i(k),\ k \in \mathbb{Z}, B_i(0) = \sigma_i^2$.

A2. Random noise $\varepsilon_j,\ j \in \mathbb{Z}$, is centered stationary Gaussian sequence independent of $y_{ij},\ j \in \mathbb{Z},\ i = \overline{1,q}$, with c.f. $B(k),\ k \in \mathbb{Z}, B(0) = \sigma^2$.

2000 *Mathematics Subject Classification.* Primary 62J02; Secondary 62J99.

Key words and phrases. asymptotic normality, least squares estimator, linear regression, random regressors, weak dependence, long-range dependence.

Definition 2.1. Any random vector $\widehat{\theta}_N = \widehat{\theta}_N\left(X_j, z_{ij},\ i = \overline{1,q},\ j = \overline{1,N}\right)$, having the property

$$S_N(\widehat{\theta}_N) = \min_{\tau \in \mathbb{R}^q} S_N(\tau), \quad S_N(\tau) = \sum_{j=1}^{N} \left[X_j - \sum_{i=1}^{q} \tau_i z_{ij} \right]^2,$$

is said to be the LSE of unknown parameter θ obtained by the observations $\left\{X_j, z_{ij},\ i = \overline{1,q},\ j = \overline{1,N}\right\}$ of the form (1).

Introduce the following notation:

$$A_j^* = (a_{1j}, ..., a_{qj})\,,\ Y_j^* = (y_{1j}, .., y_{qj})\,,\ Z_j = A_j + Y_j,$$
$$f_1 \underset{N\to\infty}{\sim} f_2 \text{ means } \frac{f_1(N)}{f_2(N)} \underset{N\to\infty}{\longrightarrow} 1.$$

Then, formally,

$$\widehat{\theta}_N = \Lambda_N^{-1} N^{-1} \sum_{j=1}^{N} Z_j X_j = \theta + \Lambda_N^{-1} N^{-1} \sum_{j=1}^{N} Z_j \varepsilon_j, \tag{2}$$

where

$$\Lambda_N = \left(\Lambda_N^{il}\right)_{i,\,l=1}^{q} = N^{-1} \sum_{j=1}^{N} Z_j Z_j^*.$$

3. Auxiliary assertions

Assume that random sequences y_{ij} satisfy condition

A3. $B_i \in l_1$, i.e. $c_i = \sum\limits_{n=-\infty}^{\infty} |B_i(n)| < \infty$, $i = \overline{1,q}$.

Write

$$J_N = \left(J_N^{il}\right)_{i,\,l=1}^{q}\,,\ J_N^{il} = N^{-1} \sum_{j=1}^{N} a_{ij} a_{lj}.$$

B1. **(i)** $\{a_{ij},\ j \in \mathbb{N}\},\ i = \overline{1,q}$, are bounded sequences: $\sup\limits_{j \in \mathbb{N}} |a_{ij}| = \widetilde{k}_i < \infty$, $i = \overline{1,q}$.

(ii) $\lim\limits_{N\to\infty} J_N = J$, where $J = \left(J^{il}\right)_{i=1}^{q}$ is some positive definite matrix.

Denote by $\Lambda = \operatorname{diag}\left(\sigma_i^2\right)_{i=1}^{q} + J$.

Lemma 3.1. *If conditions* **A1, A3** *and* **B1** *hold, then* $\Lambda_N \underset{N\to\infty}{\longrightarrow} \Lambda$ *a.s.*

Proof. For fixed $i,\ l$ consider general element of the matrix Λ_N:

$$\begin{aligned} \Lambda_N^{il} = & N^{-1} \sum_{j=1}^{N} y_{ij} y_{lj} + N^{-1} \sum_{j=1}^{N} a_{ij} y_{lj} + N^{-1} \sum_{j=1}^{N} y_{ij} a_{lj} + \\ & + N^{-1} \sum_{j=1}^{N} a_{ij} a_{lj} = \Delta^{il}(N) + \Delta_i^l(N) + \Delta_l^i(N) + J_N^{il}. \end{aligned} \tag{3}$$

Let $i \neq l$. As $B_i B_l \in l_1$, then

$$\begin{aligned} E\left(\Delta^{il}(N)\right)^2 =& N^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N} B_i(j-k)B_l(j-k) = \\ =& \sigma_i^2\sigma_l^2 N^{-1} + 2N^{-2}\sum_{k=1}^{N-1}(N-k)B_i(k)B_l(k) \leq \\ \leq& 2N^{-1}\sum_{k=0}^{N-1} B_i(k)B_l(k) \leq 2N^{-1}\sum_{k=0}^{\infty}|B_i(k)B_l(k)| = 2b_{il}N^{-1}. \end{aligned} \tag{4}$$

Set $N_n = n^2$. Then $\sum\limits_{n=0}^{\infty} E\left(\Delta^{il}(N_n)\right)^2 < \infty$ and consequently $\Delta^{il}(N_n) \underset{n\to\infty}{\longrightarrow} 0$ a.s.

Suppose that $N_n \leq N \leq N_{n+1}$. Then

$$|\Delta^{il}(N)| \leq \max_{N_n \leq N \leq N_{n+1}} |\Delta^{il}(N) - \Delta^{il}(N_n)| + |\Delta^{il}(N_n)|.$$

Let us show that

$$\max_{N_n \leq N \leq N_{n+1}} |\Delta^{il}(N) - \Delta^{il}(N_n)| \underset{n\to\infty}{\longrightarrow} 0 \text{ a.s.} \tag{5}$$

Obviously

$$\begin{aligned} \Delta^{il}(N) - \Delta^{il}(N_n) =& N^{-1}\sum_{j=1}^{N} y_{ij}y_{lj} - N_n^{-1}\sum_{j=1}^{N_n} y_{ij}y_{lj} = \\ =& \left(N^{-1} - N_n^{-1}\right)\sum_{j=1}^{N_n} y_{ij}y_{lj} + N^{-1}\sum_{j=N_n}^{N} y_{ij}y_{lj} = S_1 + S_2, \end{aligned} \tag{6}$$

$$|S_1| \leq \frac{N_{n+1} - N_n}{N_n} \cdot \left|\Delta^{il}(N_n)\right| \underset{n\to\infty}{\sim} 2n^{-1}\left|\Delta^{il}(N_n)\right|.$$

Consider the second term in (6):

$$ES_2^2 \leq N_n^{-2}\sum_{j=N_n}^{N_{n+1}}\sum_{k=N_n}^{N_{n+1}} |B_i(j-k)B_l(j-k)| \leq$$

$$\leq 2N_n^{-2}\sum_{k=0}^{N_{n+1}-N_n}(N_{n+1} - N_n - k)|B_i(k)B_l(k)| \leq \frac{2b_{il}(N_{n+1} - N_n)}{N_n^2} \underset{n\to\infty}{\sim} 4b_{il}n^{-3}.$$

Thus $S_2 \underset{n\to\infty}{\longrightarrow} 0$ a.s., and

$$\Delta^{il}(N) \underset{N\to\infty}{\longrightarrow} 0 \text{ a.s.} \tag{7}$$

Let us prove

$$\Delta_i^l(N) \underset{N\to\infty}{\longrightarrow} 0 \text{ a.s., } i, l = \overline{1, q}. \tag{8}$$

Evidently $E\Delta_i^l(N) = 0$ and

$$\begin{aligned} E\left(\Delta_i^l(N)\right)^2 =& N^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N} a_{ij}a_{ik}B_l(j-k) \leq \\ \leq& \widetilde{k}_i^2 N^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N} |B_l(j-k)| \leq \widetilde{k}_i^2 c_l N^{-1}. \end{aligned} \tag{9}$$

Set $N_n = n^2$. Then $\sum\limits_{n=1}^{\infty} E\left(\Delta_i^l(N_n)\right)^2 < \infty$ and $\Delta_i^l(N_n) \underset{n\to\infty}{\longrightarrow} 0$ a.s. Subsequent proof of (8) is similar to the proof of (7).

From (7),(8) and condition **B1(ii)** it follows that for $i \neq l$, $\Lambda_N^{il} \underset{N\to\infty}{\longrightarrow} J^{il}$ a.s.

Now let $i = l$. Then (3) can be rewritten in the form

$$\Lambda_N^{ii} = \Delta^{ii}(N) + 2\Delta_i^i(N) + J_N^{ii}.$$

Similarly to the proof of (7) one can get

$$\Delta^{ii}(N) \underset{N\to\infty}{\longrightarrow} \sigma_i^2 \text{ a.s.} \tag{10}$$

Indeed, $E\Delta^{ii}(T) = \sigma_i^2$, and due to Isserlis formula (see, for example [7], p. 30),

$$E[\Delta^{ii}(T) - \sigma_i^2]^2 = 2N^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N} B_i^2(j-k) \le 2c_i\sigma_i^2 N^{-1}.$$

Further proof of (10) is similar to (7).

Then from (10), (8) and condition **B1(ii)** it follows that $\Lambda_N^{ii} \underset{N\to\infty}{\longrightarrow} \sigma_i^2 + J^{ii}$ a.s., and Lemma 3.1 is proved. □

Corollary 3.1. *If conditions of Lemma 3.1 hold, then for almost all $\omega \in \Omega$ there exists such $N_0 = N_0(\omega)$ that for any $N > N_0$ LSE $\widehat{\theta}_N$ given by (2) is defined.*

Next statement can be understood as a discrete analogue of the Theorem 2.7 (p. 65-66) of the book [6], on integrals containing slowly varying, at infinity, functions (s.v.f.).

Lemma 3.2. *Let for some $\eta \ge 0$ and function $a(t)$, $t \in (0,1]$, for any $\varepsilon \in (0,1]$*

$$\limsup_{n\to\infty} n^{-1}\sum_{k=1}^{[\varepsilon n]}\left|a\left(\frac{k}{n}\right)\right|\left(\frac{k}{n}\right)^{-\eta} \le k_a(\varepsilon), \tag{11}$$

and, moreover, $\lim\limits_{\varepsilon\downarrow 0} k_a(\varepsilon) = 0$, $k_a(1) < \infty$. Let L be a s.v.f. bounded on every finite interval from $(0,\infty)$. Then for $\eta > 0$

$$n^{-1}\sum_{k=1}^{n}\frac{L(k)}{L(n)}a\left(\frac{k}{n}\right) - n^{-1}\sum_{k=1}^{n} a\left(\frac{k}{n}\right) \to 0,\ n\to\infty. \tag{12}$$

When $\eta = 0$ this relation is valid when the function L is nondecreasing on $(0,\infty)$.

Proof. For fixed $\varepsilon \in (0,1)$ consider

$$\begin{aligned}\left|n^{-1}\sum_{k=1}^{n}\frac{L(k)}{L(n)}a\left(\frac{k}{n}\right) - n^{-1}\sum_{k=1}^{n} a\left(\frac{k}{n}\right)\right| \le n^{-1}\left|\sum_{k=1}^{[\varepsilon n]}\frac{L(k)}{L(n)}a\left(\frac{k}{n}\right)\right| + \\ + n^{-1}\left|\sum_{k=1}^{[\varepsilon n]} a\left(\frac{k}{n}\right)\right| + \left|n^{-1}\sum_{k=[\varepsilon n]}^{n}\left(\frac{L(k)}{L(n)} - 1\right)a\left(\frac{k}{n}\right)\right| = S_3 + S_4 + S_5.\end{aligned} \tag{13}$$

Let $\eta > 0$. Then

$$\begin{aligned}S_3 &\le \frac{n^{-\eta}\sup\limits_{0\le u\le[\varepsilon n]} u^\eta L(u)}{L(n)}\cdot n^{-1}\sum_{k=1}^{[\varepsilon n]}\left|a\left(\frac{k}{n}\right)\right|\left(\frac{k}{n}\right)^{-\eta} \le \\ &\le \frac{n^{-\eta}\sup\limits_{0\le u\le n} u^\eta L(u)}{L(n)}\cdot n^{-1}\sum_{k=1}^{[\varepsilon n]}\left|a\left(\frac{k}{n}\right)\right|\left(\frac{k}{n}\right)^{-\eta} \underset{n\to\infty}{\sim} n^{-1}\sum_{k=1}^{[\varepsilon n]}\left|a\left(\frac{k}{n}\right)\right|\left(\frac{k}{n}\right)^{-\eta},\end{aligned}$$

and from (11) it follows the existence of such a constant d_1 that for sufficiently large n

$$S_3 \le d_1 k_a(\varepsilon). \tag{14}$$

Similarly to (14) one can get

$$S_4 \le n^{-1} \sum_{k=1}^{[\varepsilon n]} \left| a\left(\frac{k}{n}\right) \right| \le n^{-1} \sum_{k=1}^{[\varepsilon n]} \left| a\left(\frac{k}{n}\right) \right| \left(\frac{k}{n}\right)^{-\eta} \le d_2 k_a(\varepsilon), \tag{15}$$

where d_2 is some constant.

On the other hand,

$$\begin{aligned} S_5 &\le n^{-1} \sum_{k=[\varepsilon n]}^{n} \left| \frac{L(k)}{L(n)} - 1 \right| \left| a\left(\frac{k}{n}\right) \right| \le \sup_{\varepsilon \le u \le 1} \left| \frac{L(nu)}{L(n)} - 1 \right| n^{-1} \sum_{k=[\varepsilon n]}^{n} \left| a\left(\frac{k}{n}\right) \right| \le \\ &\le \sup_{\varepsilon \le u \le 1} \left| \frac{L(nu)}{L(n)} - 1 \right| n^{-1} \sum_{k=1}^{n} \left| a\left(\frac{k}{n}\right) \right| \le d_3 k_a(1) \sup_{\varepsilon \le u \le 1} \left| \frac{L(nu)}{L(n)} - 1 \right|, \end{aligned} \tag{16}$$

where d_3 is some constant.

Now we firstly pass to the upper limit as $n \to \infty$ in the righthand side of (13), and then to the limit as $\varepsilon \to 0$. The needed result follows from (14)-(16) and the next property of s.v.f.:

$$\sup_{\varepsilon \le u \le 1} \left| \frac{L(nu)}{L(n)} - 1 \right| \underset{n\to\infty}{\longrightarrow} 0.$$

If $\eta = 0$ then due to nondecrease of L

$$n^{-1} \left| \sum_{k=1}^{[\varepsilon n]} L(k) a\left(\frac{k}{n}\right) \right| \le L(n) n^{-1} \sum_{k=1}^{[\varepsilon n]} \left| a\left(\frac{k}{n}\right) \right|,$$

i.e relation (14) holds with $\eta = 0$. Further proof is obvious. □

A4. Random sequences $y_{ij},\ j \in \mathbb{Z},\ i = \overline{1,q}$, satisfy long-range dependence condition with seasonal effects, i.e. their c.f. are of the form $B_i(n) = \cos \varkappa_i n \cdot L_i(|n|)|n|^{-\alpha_i},\ n \in \mathbb{Z}\backslash\{0\}$, where $L_i(t),\ t > 0$, are s.v.f. bounded on every finite interval from $(0, \infty)$, $\alpha_i \in (0,1)$, $\varkappa_i \in [0, \pi)$, $i = \overline{1,q}$.

Note that in the case $\varkappa_i = 0$ sequence $y_{ij},\ j \in \mathbb{Z}$, satisfies standard long-range dependent condition.

Lemma 3.3. *If conditions* **A1, A4** *and* **B1** *hold, then* $\Lambda_N \underset{N\to\infty}{\longrightarrow} \Lambda$ *a.s.*

Proof. Similarly to the proof of Lema 3.1, consider the behaviour of terms in (3).

Let $i \ne l$. We shall prove

$$\Delta^{il}(N) \underset{N\to\infty}{\longrightarrow} 0 \text{ a.s.} \tag{17}$$

Obviously

$$\begin{aligned} E\left(\Delta^{il}(N)\right)^2 =& N^{-2} \sum_{j=1}^{N} \sum_{k=1}^{N} B_i(j-k) B_l(j-k) = \\ =& \sigma_i^2 \sigma_l^2 N^{-1} + 2N^{-2} \sum_{k=1}^{N-1} (N-k) B_i(k) B_l(k) \le \\ \le& \sigma_i^2 \sigma_l^2 N^{-1} + 2N^{-1} \sum_{k=1}^{N-1} L_i(k) L_l(k) k^{-(\alpha_i+\alpha_l)}. \end{aligned} \tag{18}$$

If $\alpha_i + \alpha_l > 1$, then

$$E\left(\Delta^{il}(N)\right)^2 \le \sigma_i^2\sigma_l^2 N^{-1} + 2N^{-1}\sum_{k=1}^{N-1} L_i(k)L_l(k)k^{-(\alpha_i+\alpha_l)} \le \tag{19}$$
$$\le N^{-1}\left(\sigma_i^2\sigma_l^2 + 2\sum_{k=1}^{\infty} L_i(k)L_l(k)k^{-(\alpha_i+\alpha_l)}\right) = K_{il}N^{-1}.$$

Further proof of (17) is similar to the proof of (7) in Lemma 3.1.

If $0 < \alpha_i + \alpha_l < 1$, then

$$E\left(\Delta^{il}(N)\right)^2 \le \sigma_i^2\sigma_l^2 N^{-1} + 2N^{-1}\sum_{k=1}^{N-1} L_i(k)L_l(k)k^{-(\alpha_i+\alpha_l)} = \sigma_i^2\sigma_l^2 N^{-1} +$$
$$+ 2B_i(N)B_l(N)\left(N^{-1}\sum_{k=1}^{N-1}\frac{L_i(k)L_l(k)}{L_i(N)L_l(N)}\left(\frac{k}{N}\right)^{-(\alpha_i+\alpha_l)} - N^{-1}\sum_{k=1}^{N-1}\left(\frac{k}{N}\right)^{-(\alpha_i+\alpha_l)}\right) +$$
$$+ 2B_i(N)B_l(N)N^{-1}\sum_{k=1}^{N-1}\left(\frac{k}{N}\right)^{-(\alpha_i+\alpha_l)} = \sigma_i^2\sigma_l^2 N^{-1} + S_6 + S_7.$$

Function $a(t) = t^{-(\alpha_i+\alpha_l)},\ t \in (0,1]$, satisfies conditions of Lemma 3.2, so

$$N^{-1}\sum_{k=1}^{N-1}\frac{L_i(k)L_l(k)}{L_i(N)L_l(N)}\left(\frac{k}{N}\right)^{-(\alpha_i+\alpha_l)} - N^{-1}\sum_{k=1}^{N-1}\left(\frac{k}{N}\right)^{-(\alpha_i+\alpha_l)} \to 0,\ N \to \infty.$$

It means that

$$S_6 = o\Big(B_i(N)B_l(N)\Big),\ N \to \infty. \tag{20}$$

Since

$$S_7 \sim 2B_i(N)B_l(N)\int_0^1 t^{-(\alpha_i+\alpha_l)}dt = \frac{2B_i(N)B_l(N)}{1-\alpha_i-\alpha_l},$$

then, taking into account (20),

$$E\left(\Delta^{il}(N)\right)^2 \le \sigma_i^2\sigma_l^2 N^{-1} + 2N^{-1}\sum_{k=1}^{N-1} L_i(k)L_l(k)k^{-(\alpha_i+\alpha_l)} \underset{N\to\infty}{\sim} \frac{2B_i(N)B_l(N)}{1-\alpha_i-\alpha_l}. \tag{21}$$

Note that in the case $\alpha_i + \alpha_l = 1$, one can get for $0 < \delta < 1$

$$E\left(\Delta^{il}(N)\right)^2 \le \sigma_i^2\sigma_l^2 N^{-1} + 2N^{-1}\sum_{k=1}^{N-1} L_i(k)L_l(k)k^{-1} \le \tag{22}$$
$$\le \sigma_i^2\sigma_l^2 N^{-1} + 2N^{-1+\delta}\sum_{k=1}^{N-1} L_i(k)L_l(k)k^{-1-\delta} \underset{N\to\infty}{\sim} K_\delta N^{-1+\delta},$$

where $K_\delta = 2\sum\limits_{k=1}^{\infty} L_i(k)L_l(k)k^{-1-\delta} < \infty$. Thus, this case can be reduced to (21).

Set $N_n = n^{\left[\frac{1}{2\min\{\alpha_i,\alpha_l\}}\right]+1}$. Then $\sum\limits_{n=0}^{\infty} E\left(\Delta^{il}(N_n)\right)^2 < \infty$ and consequently

$$\Delta^{il}(N_n) \underset{n\to\infty}{\longrightarrow} 0 \text{ a.s.}$$

Suppose that $N_n \le N \le N_{n+1}$. Then

$$|\Delta^{il}(N)| \le \sup_{N_n \le N \le N_{n+1}} |\Delta^{il}(N) - \Delta^{il}(N_n)| + |\Delta^{il}(N_n)|.$$

Similarly to S_1 from Lemma 3.1 (see (5) and (6)),

$$|S_1| \leq \frac{N_{n+1}-N_n}{N_n} \cdot \left| N_n^{-1} \sum_{j=1}^{N_n} y_{ij} y_{lj} \right| \underset{n\to\infty}{\sim} \left(\left[\frac{1}{2\min\{\alpha_i, \alpha_l\}} \right] + 1 \right) n^{-1} \cdot |\Delta^{il}(N_n)|,$$

and consequently $S_1 \underset{n\to\infty}{\longrightarrow} 0$ a.s.

Consider the second term in (6)

$$|S_2| \leq N_n^{-1} \sum_{j=N_n}^{N_{n+1}} |y_{ij} y_{lj}| \leq \frac{1}{2} \left(N_n^{-1} \sum_{j=N_n}^{N_{n+1}} y_{ij}^2 + N_n^{-1} \sum_{j=N_n}^{N_{n+1}} y_{lj}^2 \right) \leq$$

$$\leq \frac{1}{2} \left(\frac{N_{n+1}-N_n}{N_n} \cdot \left(\sigma_i^2 + \sigma_l^2\right) + N_n^{-1} \sum_{j=N_n}^{N_{n+1}} \left(y_{ij}^2 - \sigma_i^2\right) + \right.$$

$$\left. + N_n^{-1} \sum_{j=N_n}^{N_{n+1}} \left(y_{lj}^2 - \sigma_l^2\right) \right) = \frac{1}{2} \left(S_8(n) + S_9^i(n) + S_9^l(n) \right).$$

It is obvious that $S_8(n) \underset{n\to\infty}{\longrightarrow} 0$. On the other hand, for $i = \overline{1,q}$,

$$S_9^i(n) = \frac{N_{n+1}}{N_n} S_{10}^i(n+1) - S_{10}^i(n), \quad S_{10}^i(n) = N_n^{-1} \sum_{j=1}^{N_n} \left(y_{ij}^2 - \sigma_i^2\right).$$

We shall prove that

$$S_{10}^i(n) \underset{n\to\infty}{\longrightarrow} 0 \text{ a.s.} \tag{23}$$

If $\alpha_i \in \left(\frac{1}{2}, 1\right)$, then due to Isserlis formula and similarly to (19)

$$\begin{aligned} E\left(S_{10}^i(n)\right)^2 &= 2N_n^{-2} \sum_{j=1}^{N_n} \sum_{k=1}^{N_n} B_i^2(j-k) \leq \\ &\leq 2N_n^{-1} \left(\sigma_i^4 + 2 \sum_{k=1}^{N_n-1} L_i^2(k) k^{-2\alpha_i} \right) \leq 2K_{ii} N_n^{-1}. \end{aligned} \tag{24}$$

If $\alpha_i \in \left(0, \frac{1}{2}\right)$ (If $\alpha_i = \frac{1}{2}$, then the proof will be similar, see (22)), then

$$\begin{aligned} E\left(S_{10}^i(n)\right)^2 &= N_n^{-2} \sum_{j=1}^{N_n} \sum_{k=1}^{N_n} B_i^2(j-k) \leq \\ &\leq 2 \left(\sigma_i^4 N_n^{-1} + 2N_n^{-1} \sum_{k=1}^{N_n-1} L_i^2(k) k^{-2\alpha_i} \right) \underset{n\to\infty}{\sim} \frac{4B_i^2(N_n)}{1-2\alpha_i}. \end{aligned} \tag{25}$$

Since $N_n = n^{\left[\frac{1}{2\min\{\alpha_i,\alpha_l\}}\right]+1}$, then from (24) and (25) it follows that $\sum_{n=1}^{\infty} E\left(S_{10}^i(n)\right)^2 < \infty$ and (23) is fulfilled.

Collecting the relations obtained above it is easy to get $S_2 \underset{n\to\infty}{\longrightarrow} 0$ a.s., and, so, (17) is valid.

Let us show that

$$\Delta_i^l(N) \underset{N\to\infty}{\longrightarrow} 0 \text{ a.s.}, \ i, l = \overline{1,q}. \tag{26}$$

Evidently $E\Delta_i^l(N)=0$ and similarly to (21)

$$\begin{aligned} E\left(\Delta_i^l(N)\right)^2 =& N^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N} a_{ij}a_{ik}B_l(j-k) \le \widetilde{k}_i^2 N^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N}|B_l(j-k)| \\ \le& \widetilde{k}_i^2\left(\sigma_l^2 N^{-1} + 2N^{-1}\sum_{k=1}^{N-1}|L_l(k)|k^{-\alpha_l}\right) \underset{N\to\infty}{\sim} \frac{2\widetilde{k}_i^2 B_l(N)}{1-\alpha_l}. \end{aligned} \tag{27}$$

Set $N_n = n^{[\frac{1}{\alpha_l}]+1}$. Then $\sum\limits_{n=1}^{\infty} E\left(\Delta_i^l(N_n)\right)^2 < \infty$ and $\Delta_i^l(N_n) \underset{n\to\infty}{\longrightarrow} 0$ a.s. Further reasoning is similar to the proof of (17).

From (17), (26) and condition **B1(ii)** it follows that for $i \ne l$ $\Lambda_N^{il} \underset{N\to\infty}{\longrightarrow} J^{il}$ a.s.

If $i=l$, then we have to prove only

$$\Delta^{ii}(N) \underset{N\to\infty}{\longrightarrow} \sigma_i^2 \text{ a.s.} \tag{28}$$

Indeed, $E\Delta^{ii}(T)=\sigma_i^2$ and due to Isserlis formula

$$E[\Delta^{ii}(T)-\sigma_i^2]^2 = 2N^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N} B_i^2(j-k) \le 2N^{-1}\left(\sigma_i^4 + 2\sum_{k=1}^{N-1} L_i^2(k)k^{-2\alpha_i}\right),$$

and proof of (28) repeats the proof of (17). Then from (26), (28) and **B1(ii)** it follows that $\Lambda_T^{ii} \underset{T\to\infty}{\longrightarrow} \sigma_i^2 + J^{ii}$ a.s. and Lemma 3.3 is proved. □

Corollary 3.2. *If conditions of Lemma 3.3 hold, then for almost all $\omega\in\Omega$ there exists such $N_0=N_0(\omega)$ that for any $N>N_0$ LSE $\widehat{\theta}_N$ given by (2) is defined.*

Below we formulate homogeneous Hölder-Young-Brascamp-Lieb (HYBL) inequality for $\mathbb{Z}$ (see [9, 8] for details). Denote by $r(A)$ rank of a matrix A.

Lemma 3.4. *Let $l_i(J) = J^*\beta_i$, $i=\overline{1,k}$, be the linear functionals $l_i : \mathbb{Z}^m \to \mathbb{Z}$, $\beta_i\in\mathbb{Z}^m$, $i=\overline{1,k}$, M denotes the matrix with columns β_i, $i=\overline{1,k}$.*

Let functions $f_i\in l_{p_i}$, $i=\overline{1,k}$, $1\le p_i\le\infty$, and for values $z_i=\frac{1}{p_i}$, $i=\overline{1,k}$, one of next two conditions holds:

(i) $\sum\limits_{i=1}^{k} z_i = m$ *and for arbitrary $1\le d\le k$, and $\{s_1,...,\ s_d\}\subset\{1,...,k\}$*

$$\sum_{l=1}^{d} z_{s_l} \le r(A),$$

where $A=(\beta_{s_1}...\beta_{s_d})$;

(ii) $\sum\limits_{i=1}^{k} z_i = m$ *and for arbitrary $1\le d\le k$, and $\{s_1,\ ,...,\ s_d\}\subset\{1,...,k\}$*

$$\sum_{l=1}^{d} z_{s_l} \ge r(M) - r(A^c),$$

where $A=(\beta_{s_1}...\beta_{s_d})$, A^c is a matrix that contains only those columns of M which do not belong to A.

Then

$$\left|\sum_{J\in\mathbb{Z}^m}\prod_{i=1}^{k} f_i(l_i(J))\right| \le K\prod_{i=1}^{k}\|f_i\|_{p_i},$$

where $K=K(z_1,...,z_k)$ is some constant which depends on values $(z_1,...,z_k)$ only (determination of K can be found in [8]), $\|\cdot\|_{p_i}$ is norm in l_{p_i}.

4. Asymptotic normality of LSE

Let

$$d_N^2 = \operatorname{diag}\left(d_{iN}^2\right)_{i=1}^q, \ d_{iN}^2 = \sum_{j=1}^N a_{ij}^2, \ i = \overline{1,q}\,.$$

Introduce additional assumptions on random noise $\varepsilon_j, \ j \in \mathbb{Z}$.

A5. Random sequence $\varepsilon_j, \ j \in \mathbb{Z}$, has c.f. $B \in l_1$, i.e. $c = \sum\limits_{n=-\infty}^{\infty} |B(n)| < \infty$.

Note that from condition **A5** it follows that sequence $\varepsilon_j, \ j \in \mathbb{Z}$, has continuous and bounded spectral density (s.d.) f.

Introduce matrix measure $\mu_N(dx)$ on $\Big([-\pi;\pi), \ \mathcal{B}\left([-\pi;\pi)\right)\Big)$ with density matrix

$$\left(\mu_N^{k\,l}(x)\right)_{k,\,l=1}^q = \left(a_N^k(x)\overline{a_N^l(x)} \left(\int_{-\pi}^{\pi} |a_N^k(x)|^2 dx \int_{-\pi}^{\pi} |a_N^l(x)|^2 dx \right)^{-\frac{1}{2}} \right)_{k,\,l=1}^q,$$

$$a_N^k(x) = \sum_{j=1}^N e^{ixj} a_{kj}, \ \ k,l = \overline{1,q}.$$

Note that $d_{kN}^2 = (2\pi)^{-1} \int\limits_{-\pi}^{\pi} |a_N^k(x)|^2 dx, \ k = \overline{1,q}$.

B2. Family of measures $\mu_N(\cdot)$ converges weakly, as $N \to \infty$, to a measure $\mu(\cdot)$ and $\int\limits_{-\pi}^{\pi} f(x)\mu(dx)$ is some positive definite matrix.

Definition 4.1. Matrix measure $\mu(\cdot)$ is said to be spectral measure of regression function $\sum\limits_{i=1}^q \theta_i a_{ij}$ (See [10], [11] for details).

Denote by $\Gamma = \operatorname{diag}\left(k_i\right)_{i=1}^q$, where k_i are defined in **B1**; $b_i = \sum\limits_{j=-\infty}^{\infty} B_i(j)B(j)$, $i = \overline{1,q}$;

$$\sigma = 2\pi \cdot \Gamma^{\frac{1}{2}} \left(\int_{-\pi}^{\pi} f(x)\mu(dx) \right) \Gamma^{\frac{1}{2}} + \operatorname{diag}\left(b_i\right)_{i=1}^q.$$

Theorem 4.1. *If conditions* **A1** − **A3, A5, B1** *and* **B2** *hold, then the distribution of the normed LSE* $N^{1/2}\left(\widehat{\theta}_N - \theta\right)$ *tends, as* $N \to \infty$, *to normal distribution* $N\left(0, \ \Lambda^{-1}\sigma\Lambda^{-1}\right)$.

Proof. Since $N^{1/2}\left(\widehat{\theta}_N - \theta\right) = \Lambda_N^{-1}\Psi_N$, where $\Psi_N = N^{-1/2}\sum\limits_{j=1}^N Z_j\varepsilon_j$, due to Lemma 3.1 it is sufficient to determine the asymptotic distribution of the vector Ψ_N.

Let $\lambda^* = (\lambda_1, ..., \lambda_q) \in \mathbb{R}^q$ be an arbitrary fixed vector, $\mathcal{F}$ is σ-algebra generated by $\{\varepsilon_j, \ j \in \mathbb{N}\}$. Conditional distribution relatively to $\mathcal{F}$ of random variable $\lambda^*\Psi_N$ is Gaussian with expectation $E\left\{\lambda^*\Psi_N | \mathcal{F}\right\} = 0$ and variance

$$E\left\{ \left(\lambda^*\Psi_N\right)^2 | \mathcal{F} \right\} = \lambda^* \left(N^{-1} \sum_{j=1}^N \sum_{k=1}^N Z_j Z_k^* B(j-k) \right) \lambda = \lambda^* \sigma_N \lambda,$$

where equalities are valid a.s. [1]. Then for characteristic function of the vector Ψ_N we have

$$\varphi_N(\lambda) = Ee^{i\lambda^*\Psi_N} = E\left\{ E\left(e^{i\lambda^*\Psi_N} | \mathcal{F} \right) \right\} = Ee^{-\frac{1}{2}\lambda^*\sigma_N\lambda}.$$

Consider behaviour of elements of matrix σ_N as $N \to \infty$. For i-th diagonal element of matrix σ_N

$$\sigma_N^{ii} = N^{-1}\sum_{j=1}^{N}\sum_{k=1}^{N} B(j-k)a_{ij}a_{ik} + 2N^{-1}\sum_{j=1}^{N}\sum_{k=1}^{N} B(j-k)a_{ij}y_{ik} + \\ + N^{-1}\sum_{j=1}^{N}\sum_{k=1}^{N} B(j-k)y_{ij}y_{ik} = I_{11} + I_{12} + I_{13}. \tag{29}$$

For the first term it is easy to get, using conditions **B1** and **B2**, that

$$I_{11} \underset{N\to\infty}{\sim} k_i d_{iN}^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N} B(j-k)a_{ij}a_{ik} = \\ = 2\pi k_i \int_{-\pi}^{\pi} f(x)\mu_N^{i\,i}(dx) \underset{N\to\infty}{\longrightarrow} 2\pi k_i \int_{-\pi}^{\pi} f(x)\mu^{i\,i}(dx;\theta). \tag{30}$$

Furthermore,

$$EI_{12}^2 = 4N^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} B(j-k)B(l-m)B_i(k-m)a_{ij}a_{il} \le$$
$$= 4N^{-2}\sum_{j,k,l,m=1}^{\infty} f_1(j-k)f_2(l-m)f_3(k-m)f_4(j)f_5(l)f_6(k)f_7(m),$$

where $f_1(j)=f_2(j)=B(j),\ j\in\mathbb{Z},\ f_3(j)=B_i(j),\ j\in\mathbb{Z},\ f_4(j)=f_5(j)=a_{ij}\chi_N(j),\ j\in\mathbb{Z},\ f_6(j)=f_7(j)=\chi_N(j),\ j\in\mathbb{Z},\ \chi_N(j),\ j\in\mathbb{Z}$, is indicator of the set $\{1,2,\dots,N\}$. We will use for the last sum HYBL inequality (Lemma 3.4). In this case

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we put $p_1 = p_2 = p_3 = 1$, $p_4 = p_5 = \infty$, $p_6 = p_7 = 2$, then one can show that assumptions of Lemma 3.4 are fulfilled and so

$$EI_{12}^2 \le 4K_1c^2c_iN^{-2}\cdot\|a_{i\cdot}\cdot\chi_N\|_\infty^2\cdot\|\chi_N\|_2^2 \le 4K_1c^2c_i\widetilde{k}_i^2N^{-1}, \tag{31}$$

where $K_1 = K_1\left(1,\ 1,\ 1,\ 0,\ 0,\ \frac{1}{2},\ \frac{1}{2}\right)$, c and c_i are constants from conditions **A3** and **A5**, respectively.

Now consider the behavior of the last term in (29). Since $B_iB \in l_1$, then

$$EI_{13} = N^{-1}\sum_{j=1}^{N}\sum_{k=1}^{N} B(j-k)B_i(j-k) = \sum_{j=-N+1}^{N-1}\left(1-\frac{|j|}{N}\right)B_i(j)B(j) \underset{N\to\infty}{\longrightarrow} b_i. \tag{32}$$

Using the normality of the sequences $y_{ij},\ j\in\mathbb{Z},\ i=\overline{1,q}$, one can obtain

$$E\left(I_{13}-EI_{13}\right)^2 = 2N^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} B(j-k)B(l-m)B_i(j-l)B_i(k-m) =$$
$$= 2N^{-2}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} f_1(j-k)f_2(l-m)f_3(j-l)f_4(k-m)f_5(j)f_6(k)f_7(l)f_8(m),$$

where $f_1(j) = f_2(j) = B(j),\ j \in \mathbb{Z},\ f_3(j) = f_4(j) = B_i(j),\ j \in \mathbb{Z},\ f_k(j) = \chi_N(j),\ j \in \mathbb{Z},\ k = 5,6,7,8$. We will use for the last sum HYBL inequality. In this case

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we put $p_1 = p_2 = 1,\ p_3 = p_4 = 2,\ p_k = 4,\ k = 5,6,7,8$, then one can show that assumptions of Lemma 3.4 are fulfilled and so

$$E\left(I_{13} - EI_{13}\right)^2 \le 2K_2 c^2 \widetilde{c}_i^2 N^{-1}, \tag{33}$$

where $K_2 = K_2\left(1,\ 1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)$, $\widetilde{c}_i = \left(\sum\limits_{n=-\infty}^{\infty} B_i^2(n)\right)^{\frac{1}{2}}$, $i = \overline{1,q}$.

From (30)–(33) we obtain

$$\sigma_T^{ii} \xrightarrow[N\to\infty]{\mathbf{P}} \sigma^{ii}. \tag{34}$$

Similarly one can show that for elements σ_N^{il} with $i \ne l$

$$\sigma_N^{il} \xrightarrow[N\to\infty]{\mathbf{P}} \sigma^{il}. \tag{35}$$

Thus, from (34) and (35) it follows that $\sigma_N \xrightarrow[N\to\infty]{\mathbf{P}} \sigma$. Note that $\lambda^* \sigma_N \lambda \ge 0$. Then

$$\lim_{T\to\infty} \varphi_N(\lambda) = \lim_{N\to\infty} E e^{-\frac{1}{2}\lambda^* \sigma_N \lambda} = e^{-\frac{1}{2}\lambda^* \sigma \lambda},$$

and Theorem 4.1 is proved. □

Theorem 4.2. *If conditions* **A1, A2, A4, A5, B1** *and* **B2** *hold, then the distribution of the normed LSE* $N^{1/2}\left(\widehat{\theta}_N - \theta\right)$ *tends, as* $N \to \infty$*, to normal distribution* $N\left(0,\ \Lambda^{-1}\sigma\Lambda^{-1}\right)$.

Proof. Proof of the Theorem repeats the proof of Theorem 4.1, except the next details.

As in (31), one can apply HYBL inequality to EI_{12}^2 , but with $f_3(j) = B_i(|j|)\chi_N(|j|)$, $j \in \mathbb{Z}$, and use reasoning similar for obtaining (21) to get

$$\begin{aligned} EI_{12}^2 \ &\le\ 4K_1 c^2 \widetilde{k}_i^2 N^{-1} \cdot \left(\sigma_i^2 + 2\sum_{j=1}^{N} \left|\cos(\varkappa_i j)\right| \left|L_i(j)\right| j^{-\alpha_i}\right) \le \\ &\le\ 4K_1 c^2 \widetilde{k}_i^2 N^{-1} \cdot \left(\sigma_i^2 + 2\sum_{j=1}^{N} \left|L_i(j)\right| j^{-\alpha_i}\right) \underset{N\to\infty}{\sim} \frac{8K_1 c^2 \widetilde{k}_i^2 \left|B_i(N)\right|}{1-\alpha_i}. \end{aligned} \tag{36}$$

Since $B_i B \in l_1$, then as in Theorem 4.1 (32) holds true. For $E\left(I_{13} - EI_{13}\right)^2$ one can use, as in (33), HYBL inequality, however with $f_3(j) = f_4(j) = B_i(|j|)\chi_N(|j|),\ j \in \mathbb{Z}$,

$$E\left(I_{13} - EI_{13}\right)^2 \le 2K_2 c^2 N^{-1} \left(\sigma_i^4 + 2\sum_{j=1}^{N} L_i^2(j) j^{-2\alpha_i}\right).$$

If $\alpha_i \in \left(0, \frac{1}{2}\right)$ (if $\alpha_i = \frac{1}{2}$, then the proof will be similar, see (22)), then, as in (21),

$$E\left(I_{13} - EI_{13}\right)^2 \ \le\ 2K_2 c^2 N^{-1} \left(\sigma_i^4 + 2\sum_{j=1}^{N} L_i^2(j) j^{-2\alpha_i}\right) \underset{N\to\infty}{\sim} \frac{4K_2 c^2 B_i^2(N)}{1-2\alpha_i}. \tag{37}$$

If $\alpha_i \in \left(\frac{1}{2}, 1\right)$, then similarly to (19)

$$E\left(I_{13} - EI_{13}\right)^2 \le 2K_2c^2N^{-1}\left(\sigma_i^4 + 2\sum_{j=1}^{N} L_i^2(j)j^{-2\alpha_i}\right) \le 2K_2c^2K_{ii}N^{-1}. \tag{38}$$

From (30), (32), (36), (37) and (38) it follows that $\sigma_T^{ii} \xrightarrow[T\to\infty]{\mathbf{P}} \sigma^{ii}$. Next steps of the proof repeat the proof of Theorem 4.1. □

Now let ε_j, $j \in \mathbb{Z}$, be a random sequence that satisfies long-range dependence condition with seasonal effects, i.e.

A6. ε_j, $j \in \mathbb{Z}$, has c.f. $B(n) = \cos\varkappa n \cdot L(|n|)|n|^{-\alpha}$, $n \in \mathbb{Z}\backslash\{0\}$, where $L(t)$, $t > 0$, is s.v.f. bounded on every finite interval from $(0, \infty)$, $\alpha \in \left(\frac{1}{2}, 1\right)$, $\varkappa \in (0, \pi)$.

From condition **A6** it follows that ε_j, $j \in \mathbb{Z}$, has s.d. f with two singularities at the points $\pm\varkappa$.

Introduce the notion of μ-admissability of s.d. $f(\lambda)$ (See [10], [12] for details).

Definition 4.2. S.d. f is called μ-admissible, if f is μ-integrable, i.e. all elements of matrix $\int\limits_{-\pi}^{\pi} f(\lambda)\mu(d\lambda)$ are finite and

$$\int\limits_{-\pi}^{\pi} f(\lambda)\mu_N(d\lambda) \to \int\limits_{-\pi}^{\pi} f(\lambda)\mu(d\lambda), \quad N \to \infty.$$

Sufficient conditions of μ-admissability of stationary process s.d. can be found in [13]. The main condition of μ-admissability requires that the set of singularity points of f and the set of atoms of spectral measure μ do not intersect. Note that the spectral measure is atomic for all known at present examples of its existence.

A7. S.d. f of random sequence ε_j, $j \in \mathbb{Z}$, is μ-admissible.

Theorem 4.3. *If conditions* **A1-A3, A6, A7, B1** *and* **B2** *hold, then the distribution of the normed LSE* $N^{1/2}\left(\widehat{\theta}_N - \theta\right)$ *tends, as* $N \to \infty$*, to normal distribution* $N\left(0,\ \Lambda^{-1}\sigma\Lambda^{-1}\right)$.

Proof. Proof of the Theorem is similar to the proofs of Theorems 4.2. Consider just slight differences.

Relation (30) holds true due to condition **A7**. For EI_{12}^2 (similarly to (36)) one can use HYBL inequality, with $f_1(j) = f_2(j) = B(|j|)\chi_N(|j|)$, $j \in \mathbb{Z}$, $f_3(j) = B_i(j)$, $j \in \mathbb{Z}$, $f_4(j) = f_5(j) = a_{ij}\chi_N(j)$, $j \in \mathbb{Z}$, $f_6(j) = f_7(j) = \chi_N(j)$, $j \in \mathbb{Z}$, and the same p_j, $j = \overline{1,7}$, as in (31),

$$\begin{aligned} EI_{12}^2 &\le 4K_1c_i\widetilde{k}_i^2N^{-1}\cdot\left(\sigma^2 + 2\sum_{j=1}^{N} |\cos(\varkappa j)|\cdot|L(j)|j^{-\alpha}\right)^2 \le \\ &\le 4K_1c_i\widetilde{k}_i^2N^{-1}\cdot\left(\sigma^2 + 2\sum_{j=1}^{N} |L(j)|j^{-\alpha}\right)^2 \underset{N\to\infty}{\sim} \frac{16K_1c_i\widetilde{k}_i^2L^2(N)N^{1-2\alpha}}{(1-\alpha)^2}, \end{aligned} \tag{39}$$

where $1 - 2\alpha < 0$, as $\alpha \in \left(\frac{1}{2}, 1\right)$.

As in (33), one can apply HYBL inequality to $E\left(I_{13} - EI_{13}\right)^2$, with $f_1(j) = f_2(j) = B(|j|)\chi_N(|j|)$, $j \in \mathbb{Z}$, $f_3(j) = f_4(j) = B_i(j)$, $j \in \mathbb{Z}$, $f_k(j) = \chi_N(j)$, $j \in \mathbb{Z}$, $k = 5, 6, 7, 8$, and $p_1 = p_2 = 2$, $p_3 = p_4 = 1$, $p_k = 4$, $k = 5, 6, 7, 8$,

$$\begin{aligned} E\left(I_{13} - EI_{13}\right)^2 \; &\leq \; 2K_3 c_i^2 N^{-1} \left(\sigma^4 + 2\sum_{j=1}^{N} L^2(j) j^{-2\alpha}\right) \leq \\ &\leq 2K_3 c_i^2 N^{-1} \left(\sigma^4 + 2\sum_{j=1}^{\infty} L^2(j) j^{-2\alpha}\right) = 2K_3 c_i^2 K_\alpha N^{-1}, \end{aligned} \tag{40}$$

where $K_3 = K_3\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ 1,\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4}\right)$.

From (30), (32), (39) and (40) it follows that $\sigma_T^{ii} \underset{T\to\infty}{\overset{\mathbf{P}}{\longrightarrow}} \sigma^{ii}$. Next steps of the proof repeat the proof of Theorem 4.1. □

Theorem 4.4. *If conditions* **A1, A2, A4** *with* $\alpha_i \in \left(\frac{1}{2}, 1\right)$, **A6** *with* $\alpha \in \left(\frac{3}{4}, 1\right)$, **A7, B1** *and* **B2** *hold, then the distribution of the normed LSE* $N^{1/2}\left(\widehat{\theta}_N - \theta\right)$ *tends, as* $N \to \infty$, *to normal distribution* $N\left(0,\ \Lambda^{-1}\sigma\Lambda^{-1}\right)$.

Proof. Proof of the Theorem repeats the proof of Theorem 4.3, besides the next fragments.

For EI_{12}^2 we get

$$\begin{aligned} EI_{12}^2 \; &\leq \; 4K_1 \widetilde{k}_i^2 N^{-1} \cdot \left(\sigma_i^2 + 2\sum_{j=1}^{N} \left|L_i(j)\right| j^{-\alpha_i}\right)\left(\sigma^2 + 2\sum_{j=1}^{N} |L(j)| j^{-\alpha}\right)^2 \underset{N\to\infty}{\sim} \\ &\underset{N\to\infty}{\sim} \frac{32 K_1 \widetilde{k}_i^2 \left|L_i(N)\right| L^2(N) N^{2-2\alpha-\alpha_i}}{(1-\alpha_i)(1-\alpha)^2}, \end{aligned} \tag{41}$$

where $2 - 2\alpha - \alpha_i < 0$ under the conditions of the Theorem.

Since $\alpha + \alpha_i > 1$ under the conditions of the Theorem, then $B_i B \in l_1$, and so (32) holds true.

Similarly to (40), but with $f_3(j) = f_4(j) = B_i(|j|)\chi_N(|j|)$, $j \in \mathbb{Z}$, one can get

$$\begin{aligned} E\left(I_{13} - EI_{13}\right)^2 \; &\leq \; 2K_3 K_\alpha N^{-1} \cdot \left(\sigma_i^2 + 2\sum_{j=1}^{N} \left|L_i(j)\right| j^{-\alpha_i}\right)^2 \underset{N\to\infty}{\sim} \\ &\underset{N\to\infty}{\sim} \frac{8 K_3 K_\alpha L_i^2(N) N^{1-2\alpha_i}}{(1-\alpha_i)^2}, \end{aligned} \tag{42}$$

as $\alpha_i \in \left(\frac{1}{2}, 1\right)$ and $\alpha \in \left(\frac{3}{4}, 1\right)$. □

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