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ON FELLER SEMIGROUPS ASSOCIATED WITH ONE-DIMENSIONAL DIFFUSION PROCESSES WITH MEMBRANES

By analytical method we obtain the integral representation of a two-parameter semigroup of operators associated with Feller process on a line which is a result of pasting together given diffusions at finite number of fixed points. The behavior of this process at each point of pasting together is described by Feller-Wentzell conjugation condition containing the integral term (nonlocal component).

1. Introduction

The paper deals with the problem of construction of diffusion process in a domain by given differential operators with supplementation of boundary conditions which determine the behavior of the process at the boundary points of the domain. Recall that the general form of the boundary conditions for one-dimensional diffusion processes was established in works of W. Feller [1] and A. D. Wentzell [2] (see also [3] where the multidimensional case was considered). There were considered the assertions from which it follows that if the ordinary operator of the second order which is given on a closed interval $\Delta = [r_1, r_2]$ and acts on $C^2(\Delta)$ is the generator of the one-parameter Feller semigroup then its domain of definition consists of the functions which satisfy mentioned boundary conditions.

Notice that boundary conditions of Feller-Wentzell mentioned above have nonlocal nature, that is, they except the value of the function and its derivatives at the boundary points of the domain Δ contain also the integral of this function over the set Δ with respect to some nonnegative and, generally speaking, infinite measure μ which corresponds to the possibility of jump-like exit of the process from the boundary of the domain. Many publications (see, for instance, works [1-9] and the references given there) are devoted to the questions on construction of diffusion processes by given boundary conditions.

The generalization of the problem mentioned above is the so-called problem of pasting together diffusion processes on a line (see, for instance, [10-14]). Precisely this problem is the object of research in the present paper. The problem is to construct the two-parameter Feller semigroup which corresponds to inhomogeneous Markov process (not necessarily continuous) on a line $\mathbb{R}$ separated into intervals by some finite number of points $r_1,\ r_2, \dots, r_n,\ n \in \mathbb{N}$, such that its parts at interior points of the corresponding intervals coincide with the diffusion processes given there and its behavior at points $r_i,\ i = \overline{1, n}$, is described by Feller-Wentzell conjugation conditions given at these points. Another n conjugation conditions given at points $r_1,\ r_2, \dots, r_n$, respectively, reflect Feller property of the required process. The study of the problem is performed by analytical methods. With such an approach the question on construction of the required semigroup in fact is being reduced to the investigation of the corresponding problem of conjugation for a linear parabolic equation of the second order with discontinuous coefficients. The classical solvability of the last problem is established by the boundary integral equations

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method with the use of the ordinary simple-layer parabolic potentials. Perhaps in the statement proposed here the problem of pasting together diffusion processes and also the corresponding to it parabolic conjugation problem are considered for the first time.

Notice that in mentioned works [11-13] the described problem was already considered in case $n=1$ for both homogeneous and inhomogeneous diffusion processes. Concerning the work [14] the general form of the conjugation condition at point of pasting together two diffusion processes by analogy with the work [2] was established there.

2. Problem statement

Let $r_1,\ r_2,\dots,r_n,\ n\in\mathbb{N}$, be the fixed points on a line $\mathbb{R}$ which separate it into $n+1$ domains $D_1=(-\infty,r_1),\ D_2=(r_1,r_2),\dots,\ D_n=(r_{n-1},r_n),\ D_{n+1}=(r_n,+\infty)$. In the sequel, symbols $-\infty$ and $+\infty$ in expressions for domains D_1 and D_{n+1} will be denoted by r_0 and r_{n+1} respectively. Denote by $\overline{D}_i$ the closure of the domain $D_i,\ i=\overline{1,n+1}$, and by φ_i the restriction of any function φ defined on $\mathbb{R}$ to $\overline{D}_i$. If Γ is the set $D_i,\ \overline{D}_i$ or $\mathbb{R}$, then $C_b(\Gamma)$ is the Banach space of all real-valued bounded and continuous on Γ functions with the norm $\|\varphi\|=\sup_{x\in\Gamma}|\varphi(x)|$ and $C_2(\Gamma)$ is the space of all functions φ, bounded and uniformly continuous on Γ together with their first- and second-order derivatives.

Assume that the inhomogeneous diffusion process is given in each domain $D_i,\ i=\overline{1,n+1}$, and it is generated by the second-order differential operator $(A_s^{(i)},C_2(\overline{D}_i)),\ s\in[0,T]$ $(T>0$ fixed)

$$A_s^{(i)}\varphi_i(x)=\frac{1}{2}b_i(s,x)\frac{d^2\varphi_i(x)}{dx^2}+a_i(s,x)\frac{d\varphi_i(x)}{dx}, \tag{1}$$

where the diffusion coefficient $b_i(s,x)$ and the drift coefficient $a_i(s,x)$ have the following properties:

1) there exist constants b and B such that $0<b<b_i(s,x)<B$ for all $(s,x)\in[0,T]\times\overline{D}_i$;
2) function $a_i(s,x)$ is bounded on $[0,T]\times\overline{D}_i$;
3) for all $s,s'\in[0,T],\ x,x'\in\overline{D}_i$ the next inequalities hold:
$$|b_i(s,x)-b_i(s',x')|\le C(|s-s'|^{\frac{\alpha}{2}}+|x-x'|^{\alpha}),$$
$$|a_i(s,x)-a_i(s',x')|\le C(|s-s'|^{\frac{\alpha}{2}}+|x-x'|^{\alpha}),$$
where C and α are positive constants, $0<\alpha<1$.

Consider the differential operator $A_s,\ s\in[0,T]$, which acts on the set $\vartheta(A_s)=\{\varphi\in C_b(\mathbb{R}):\ \varphi_i\in\vartheta(A_s^{(i)}),\ i=\overline{1,n+1}\ \wedge\ A_s^{(j)}\varphi_j(r_j)=A_s^{(j+1)}\varphi_{j+1}(r_j),\ j=\overline{1,n}\}$ by the following rule:

$$A_s\varphi(x)=A_s^{(i)}\varphi_i(x),\quad x\in\overline{D}_i,\ i=\overline{1,n+1}.$$

Assume that at points $r_1,\ r_2,\dots,r_n$ the Feller-Wentzell conjugation conditions are given (see [1, 2])

$$L_s^{(i)}\varphi(r_i)\equiv q_{i,i}(s)\varphi'(r_i-)-q_{i,i+1}(s)\varphi'(r_i+)+\int\limits_{D_i\cup D_{i+1}}(\varphi(r_i)-\varphi(y))\mu_i(s,dy)=0, \tag{2}$$

where functions $q_{i,i},\ q_{i,i+1}$ and measures $\mu_i,\ i=\overline{1,n}$, satisfy the conditions:

a) $q_{i,j}\in C([0,T]),\ q_{i,j}(s)\ge0,\ \sum\limits_j q_{i,j}(s)>0,\ s\in[0,T],\ j=i,i+1$;
b) $\mu_i(s,\cdot)$ is nonnegative measure on $D_i\cup D_{i+1}$ such that for any function $f\in C_b(\mathbb{R})$ and any number $\delta>0$ the integrals
$$\int_{D_j^\delta(r_i)}|y-r_i|f(y)\mu_i(s,dy),\quad \int_{D_j\setminus D_j^\delta(r_i)}f(y)\mu_i(s,dy),\quad j=i,i+1,$$

are continuous on $[0,T]$ as functions of variable s; $D_j^\delta(r_i) = \{y \in D_j : \ |y - r_i| < \delta\}$.

It is known ([14]) that the conjugation conditions (2) restrict the differential operator A_s to the generator of some Feller semigroup in the space $C_b(\mathbb{R})$. Such a semigroup is constructed in the present paper. Thus, our problem is to construct the semigroup of operators $T_{s,t}, \ 0 \le s < t \le T$, which describes inhomogeneous Feller process on $\mathbb{R}$ such that in each domain $D_i, \ i = \overline{1, n+1}$ it coincides with the diffusion process generated by the operator $A_s^{(i)}$ and its behavior at points $r_1, \ r_2, \ldots, \ r_n$ is determined by Feller-Wentzell conjugation conditions (2).

The problem formulated above is also called the problem of pasting together n diffusion processes on a line or the problem of mathematical modeling of the diffusion phenomenon on a line with membranes placed at fixed points $r_1, \ r_2, \ldots, \ r_n$ that separates different (by their diffusion characteristics) media. In considered case the membrane at point r_i is characterized by the functions $q_{i,i}, \ q_{i,i+1}$ and the measure μ_i. If the diffusion particle reaches the point r_i from the domain D_i, then function $q_{i,i}$ corresponds to its reflection back to D_i, and the function $q_{i,i+1}$ corresponds to its penetration to the domain D_{i+1}, and conversely, the function $q_{i,i}$ corresponds to penetration of the particle to D_i, and $q_{i,i+1}$ corresponds to its reflection to D_{i+1}, when reaching the membrane is from the domain D_{i+1}. The measure μ_i which, generally speaking, can be infinite in a neighborhood of r_i, corresponds to the jump of the diffusion particle into the domain D_i or D_{i+1}.

Notice that (see [14]) the general Feller-Wentzell conjugation condition, except terms which are in the condition (2), also contains terms which characterize the delay capability and disappearing capability of the process after it reaches the boundary of the domain. Process with these additional properties (see [12]) can be constructed by similar considerations to those leading to the required process in the present paper.

According to the analytical approach to study of the formulated problem, the required family of operators $T_{s,t}, \ 0 \le s < t \le T$ will be defined by solution $u(s,x,t)$ of the following parabolic conjugation problem:

$$\frac{\partial u(s,x,t)}{\partial s} + A_s^{(i)} u(s,x,t) = 0, \quad 0 \le s < t \le T, \ x \in D_i, \quad i = \overline{1, n+1}, \tag{3}$$

$$\lim_{s \uparrow t} u(s,x,t) = \varphi(x), \quad x \in \mathbb{R}, \tag{4}$$

$$u(s, r_i-, t) = u(s, r_i+, t), \quad 0 \le s < t \le T, \quad i = \overline{1, n}, \tag{5}$$

$$L_s^{(i)} u(s, r_i, t) = 0, \quad 0 \le s < t \le T, \quad i = \overline{1, n}, \tag{6}$$

where $\varphi \in C_b(\mathbb{R})$ is the given function. Notice that in problem (3)-(6) the conjugation conditions (5) reflect the Feller property of the required semigroup $T_{s,t}$ and equalities (6) correspond to Feller-Wentzell conjugation conditions (2).

3. Solution of the problem

We establish the classical solvability of the problem (3)-(6) by the boundary integral equations method. For this purpose, without loss of generality, we suppose that the functions $a_i(s,x), \ b_i(s,x), \ i = \overline{1, n+1}$, are defined on $[0,T] \times \mathbb{R}$ and satisfy conditions 1)-3) in this domain.

These conditions ensure the existence of the fundamental solution in the domain $[0,T] \times \mathbb{R}$ for each equation in (3), that is existence of such a function $G_i(s,x,t,y)$ defined for $0 \le s < t \le T, \ x, y \in \mathbb{R}$ which (see [10, Ch.II, §2], [17, Ch.IV, §11])

(i) is continuous in the aggregate of variables;
(ii) satisfies the equation (3) for fixed $t \in (0,T], y \in \mathbb{R}$ as function of $(s,x) \in [0,t) \times \mathbb{R}$;

(iii) satisfies the condition

$$\lim_{s\uparrow t}\int_R G_i(s,x,t,y)\varphi(y)dy=\varphi(x)$$

for any function $\varphi\in C_b(\mathbb{R})$ and any $t\in(0,T], x\in\mathbb{R}$.

Besides, functions $G_i(s,x,t,y)$ can be represented as

$$G_i(s,x,t,y)=Z_i(s,x,t,y)+Z_i^{'}(s,x,t,y),\quad 0\le s<t\le T,\quad x,y\in\mathbb{R}, \tag{7}$$

where

$$Z_i(s,x,t,y)=[2\pi b_i(t,y)(t-s)]^{-\frac{1}{2}}\,e^{-\frac{(y-x)^2}{2b_i(t,y)(t-s)}}$$

and $Z_i^{'}$ is written in the form of integral operator with kernel Z_i and density Q_i which is determined from some integral equation.

We mention, among other properties of fundamental solutions $G_i(s,x,t,y)$, the following estimations $(0\le s<t\le T,\ x,y\in\mathbb{R})$:

$$\begin{aligned}|D_s^rD_x^pG_i(s,x,t,y)|&\le C(t-s)^{-\frac{1+2r+p}{2}}e^{-h\frac{(y-x)^2}{t-s}},\\ |D_s^rD_x^pZ_i^{'}(s,x,t,y)|&\le C(t-s)^{-\frac{1+2r+p-\alpha}{2}}e^{-h\frac{(y-x)^2}{t-s}},\end{aligned} \tag{8}$$

where r and p are the nonnegative integers for which $2r+p\le 2$, C, h are positive constants, α is the constant in 3), D_s^r is the partial derivative with respect to s of order r, D_x^p is the partial derivative with respect to x of order p.

We find a solution of problem (3)-(6) of the form

$$u(s,x,t)=u_0^{(i)}(s,x,t)+u^{(i)}(s,x,t),\quad 0\le s<t\le T,\quad x\in\overline{D}_i,\quad i=\overline{1,n+1}, \tag{9}$$

where

$$u_0^{(i)}(s,x,t)=\int_{\mathbb{R}}G_i(s,x,t,y)\varphi(y)dy,\quad i=\overline{1,n+1},$$

$$u^{(i)}(s,x,t)=\int_s^t\big[G_i(s,x,\tau,r_{i-1})V_{n+i-1}(\tau,t)+G_i(s,x,\tau,r_i)V_i(\tau,t)\big]d\tau,\quad i=\overline{1,n+1},$$

$V_1,\dots,\ V_{2n}$ are the unknown functions to be determined. Here and in the sequel kernels $G_1(s,x,t,r_0)$ and $G_{n+1}(s,x,t,r_{n+1})$ are assumed to be equal to zero.

Notice that in theory of parabolic equations functions $u_0^{(i)}(s,x,t)$ are called the Poisson potentials and integrals by which the functions $u^{(i)}(s,x,t)$ are expressed are called the parabolic simple-layer potentials.

From mentioned properties of fundamental solutions of uniformly parabolic operators it follows that if $\varphi\in C_b(\mathbb{R})$, then each function $u_0^{(i)}(s,x,t)$ is continuous for $s\in[0,t),\ x\in\mathbb{R}$, continuously differentiable with respect to s, twice continuously differentiable with respect to x and for it the inequality

$$|D_s^rD_x^pu_0(s,x,t)|\le C\|\varphi\|(t-s)^{-\frac{2r+p}{2}}, \tag{10}$$

holds (when $2r+p\le 2$ and C is some constant) in each domain of the form $0\le s<t\le T,\ x\in\mathbb{R}$. Further, the function $u_0^{(i)}(s,x,t)$ satisfies equation (3) in domain $(s,x)\in[0,t)\times D_i,\ i=1,\dots,n+1$, and the initial condition (4). Concerning the function $u^{(i)}(s,x,t)$ under corresponding assumptions on densities V_i and V_{n+i-1} it satisfies equation (3) and the initial condition $\lim\limits_{s\uparrow t}u^{(i)}(s,x,t)=0$.

The important property of simple-layer potential is represented by well-known theorem on the jump of the conormal derivative of this function (see [16, Ch.V, §2], [17, Ch.IV, §15], [10, Ch.II, §3]. Here this property is applied to compute the right- and left-side derivatives of function $u^{(i)}(s,x,t)$ at points $x=r_{i-1}$ and $x=r_i$ respectively. In our case the formulas for mentioned derivatives take the form

$$D_x^1 u^{(i)}(s,r_{i-1}+,t) = -\frac{V_{n+i-1}(s,t)}{b_i(s,r_{i-1})} + \int\limits_s^t \big[D_x^1 G_i(s,x,\tau,r_{i-1})|_{x=r_{i-1}} \cdot V_{n+i-1}(\tau,t) + \\ + D_x^1 G_i(s,x,\tau,r_i)|_{x=r_{i-1}} \cdot V_i(\tau,t)\big] d\tau, \quad i=\overline{2,n+1},$$

$$D_x^1 u^{(i)}(s,r_i-,t) = \frac{V_i(s,t)}{b_i(s,r_i)} + \int\limits_s^t \big[D_x^1 G_i(s,x,\tau,r_{i-1})|_{x=r_i} \cdot V_{n+i-1}(\tau,t) + \\ + D_x^1 G_i(s,x,\tau,r_i)|_{x=r_i} \cdot V_i(\tau,t)\big] d\tau, \quad i=\overline{1,n}, \tag{11}$$

Notice that the existence of integrals on the right side of (11) follows from the inequality $(0 \le s < t \le T)$

$$|D_x^1 G_i(s,r_j,\tau,r_k)| \le C(\tau-s)^{-1+\frac{\alpha}{2}}, \quad j=i-1,i, \quad k=i-1,i. \tag{12}$$

Thus, in order to solve the conjugation problem (3)-(6) it remains to determine the unknowns $V_i,\ i=\overline{1,2n}$. We find these functions from conjugation conditions (5), (6).

Substituting instead of function u its expression from the right side of (9) into (5), we get the following system of n Volterra integral equations of the first kind for V_i:

$$\int\limits_s^t \big[G_i(s,r_i,\tau,r_i)V_i(\tau,t) - G_{i+1}(s,r_i,\tau,r_i)V_{n+i}(\tau,t)\big] d\tau + \tag{13}$$

$$+\int\limits_s^t \big[G_i(s,r_i,\tau,r_{i-1})V_{n+i-1}(\tau,t) - G_{i+1}(s,r_i,\tau,r_{i+1})V_{i+1}(\tau,t)\big] d\tau = \Phi_i(s,t),\ i=\overline{1,n}.$$

where

$$\Phi_i(s,t) = u_0^{(i+1)}(s,r_i,t) - u_0^{(i)}(s,r_i,t).$$

Regularization of equations of system (13) can be performed by Holmgren transform ([15]). This transform is defined by integro-differential operator $\mathcal{E}$ which acts by the following rule

$$\mathcal{E}(s,t)f = \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int\limits_s^t (\tau-s)^{-\frac{1}{2}} f(\tau,t) d\tau, \quad 0 \le s < t \le T. \tag{14}$$

Application of the operator $\mathcal{E}$ to both sides of each equation of system (13) gives the equivalent system of Volterra integral equations of the second kind of the form

$$\frac{V_i(s,t)}{\sqrt{b_i(s,r_i)}} - \frac{V_{n+i}(s,t)}{\sqrt{b_{i+1}(s,r_i)}} + \\ + \sum_{j=i}^{i+1}\int\limits_s^t \big[M_{i,j}(s,\tau)V_j(\tau,t) - \hat{M}_{i,n+j-1}(s,\tau)V_{n+j-1}(\tau,t)\big] d\tau = \hat{\Phi}_i(s,t), \quad i=\overline{1,n}, \tag{15}$$

where

$$\hat{\Phi}_i(s,t) = \mathcal{E}(s,t)\Phi_i = \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int\limits_s^t (\tau-s)^{-\frac{1}{2}} \Phi_i(\tau,t) d\tau,$$

$$M_{i,i}(s,\tau) = -\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int\limits_s^\tau (u-s)^{-\frac{1}{2}} Z_i'(u,r_i,\tau,r_i) du,$$

$$M_{i,i+1}(s,\tau) = \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int\limits_s^\tau (u-s)^{-\frac{1}{2}} G_i(u,r_i,\tau,r_{i+1})du,$$

and kernels $\hat{M}_{i,n+i}$ and $\hat{M}_{i,n+i-1}$ are defined analogously to $M_{i,i}$ and $M_{i,i+1}$ with functions $Z_i'(u,r_i,\tau,r_i)$ and $G_i(u,r_i,\tau,r_{i+1})$ replaced by $Z_{i+1}'(u,r_i,\tau,r_i)$ and $G_i(u,r_i,\tau,r_{i-1})$ respectively.

Let us consider right sides of equations of system (15). Since the function $\Phi_i(\tau,t)$ is differentiable with respect to variable τ when $0 \le \tau < t$, it is easy to get for $\hat{\Phi}_i(s,t)$ the following representation:

$$\hat{\Phi}_i(s,t) = -\frac{1}{\sqrt{2\pi}}\int\limits_s^t (\tau-s)^{-\frac{3}{2}}[\Phi_i(\tau,t)-\Phi_i(s,t)]d\tau + \\ + \sqrt{\frac{2}{\pi}}(t-s)^{-\frac{1}{2}}\Phi_i(s,t), \quad i=\overline{1,n}. \tag{16}$$

Splitting the integral on the right side of (16) into two $\int\limits_s^t = \int\limits_s^{\frac{s+t}{2}} + \int\limits_{\frac{s+t}{2}}^t$ and applying the inequality (10) with $r=0,\ p=0$ to the first one (as well as to the very function $\Phi_i(s,t)$) and mean value theorem and then inequality (10) with $r=1,\ p=0$ to the second one, we obtain

$$|\hat{\Phi}_i(s,t)| \le C\|\varphi\|(t-s)^{-\frac{1}{2}}, \quad i=\overline{1,n}, \tag{17}$$

where $0 \le s < t \le T$.

By the previous considerations and the inequalities (8), one can also estimate kernels $M_{i,j}(s,\tau)$ and $\hat{M}_{i,n+j-1}(s,\tau),\ i=\overline{1,n},\ j=i,i+1$, of the integrals on the left side of system of equation (15). Consequently we make sure that all of them satisfy estimation of the form (12).

We obtain one more system of n Volterra integral equations of the second kind for $V_i,\ i=\overline{1,2n}$, upon substituting the expressions from the right side of (9) into conjugation conditions (6) and using, at the same time, the relation (11). This system of equations is as follows:

$$\frac{q_{i,i}(s)}{b_i(s,r_i)}V_i(s,t) + \frac{q_{i,i+1}(s)}{b_{i+1}(s,r_i)}V_{n+i}(s,t) + \\ + \sum_{j=i}^{i+1}\int\limits_s^t \big[N_{i,j}(s,\tau)V_j(\tau,t) + \hat{N}_{i,n+j-1}(s,\tau)V_{n+j-1}(\tau,t)\big]d\tau = \Psi_i(s,t), \quad i=\overline{1,n}, \tag{18}$$

where

$$\Psi_i(s,t) = \sum_{j=i}^{i+1}\big[(-1)^{i+j}q_{i,j}(s)D_x^1u_0^{(j)}(s,r_i,t) - \int\limits_{D_j}(u_0^{(j)}(s,r_i,t) - \\ - u_0^{(j)}(s,y,t))\mu_i(s,dy)\big], \tag{19}$$

$$N_{i,j}(s,\tau) = (-1)^{i+j}q_{i,j}(s)D_x^1G_j(s,r_i,\tau,r_j) + \int\limits_{D_j}\big(Z_j'(s,r_i,\tau,r_j) - Z_j'(s,y,\tau,r_j)\big)\mu_i(s,dy) + \\ + \int\limits_{D_j}\big(Z_j(s,r_i,\tau,r_j) - Z_j(s,y,\tau,r_j)\big)\mu_i(s,dy), \quad j=i,i+1,$$

and kernel $\hat{N}_{i,n+j-1}(s,\tau)$ is defined analogously to $N_{i,j}(s,\tau)$ with r_j replaced by r_{j-1}.

In the system of equations (18) the right sides $\Psi_i(s,t)$ are continuous functions in domain $0 \le s < t \le T$ and for them just as for functions $\hat{\Phi}_i(s,t)$ the inequality (17) holds with constant C depending on δ. This assertion for the first term in square brackets in the expression for $\Psi_i(s,t)$ is an easy consequence of properties of functions $q_{i,j}(s)$ and the inequality (10) with $r=0,\ p=1$. To estimate the integral term in the expression for $\Psi_i(s,t)$, write it in the form

$$\int\limits_{D_j} \left(u_0^{(j)}(s,r_i,t) - u_0^{(j)}(s,y,t)\right)\mu_i(s,dy) = \\ = \int\limits_{D_j^\delta(r_i)} \left(u_0^{(j)}(s,r_i,t) - u_0^{(j)}(s,y,t)\right)\mu_i(s,dy) + \\ + \int\limits_{D_j \setminus D_j^\delta(r_i)} \left(u_0^{(j)}(s,r_i,t) - u_0^{(j)}(s,y,t)\right)\mu_i(s,dy).$$

Then, estimating the integral over $D_j^\delta(r_i)$, taking into account the condition b), mean value theorem for difference $u_0^{(j)}(s,r_i,t) - u_0^{(j)}(s,y,t)$ and the inequality (10) with $r=0,\ p=1$, and the integral over $D_j \setminus D_j^\delta(r_i)$, using the condition b) and inequalities (10) with $r=0,\ p=1$ applied separately to each function $u_0^{(j)}(s,r_i,t)$ and $u_0^{(j)}(s,y,t)$, we get

$$\left|\int\limits_{D_j^\delta(r_i)} \left(u_0^{(j)}(s,r_i,t) - u_0^{(j)}(s,y,t)\right)\mu_i(s,dy)\right| = \\ = \left|\int\limits_{D_j^\delta(r_i)} D_y u_0^{(j)}(s,y+\theta(r_i-y),t)(r_i-y)\mu_i(s,dy)\right| \le \\ \le C\|\varphi\|(t-s)^{-\frac{1}{2}} \int\limits_{D_j^\delta(r_i)} |y-r_i|\mu_i(s,dy) \le C_1(\delta)\|\varphi\|(t-s)^{-\frac{1}{2}},$$

$$\left|\int\limits_{D_j \setminus D_j^\delta(r_i)} \left(u_0^{(j)}(s,r_i,t) - u_0^{(j)}(s,y,t)\right)\mu_i(s,dy)\right| \le \\ \le \int\limits_{D_j \setminus D_j^\delta(r_i)} \left(|u_0^{(j)}(s,r_i,t)| + |u_0^{(j)}(s,y,t)|\right)\mu_i(s,dy) \le \\ \le C\|\varphi\| \int\limits_{D_j \setminus D_j^\delta(r_i)} \mu_i(s,dy) \le C_2(\delta)\|\varphi\|.$$

The last two inequalities together with the inequality for $(-1)^{i+j}q_{i,j}(s)D_x^1 u_0^{(j)}(s,r_i,t)$ prove the estimation (17) for functions $\Psi_i(s,t),\ i=1,\dots,n$.

Concerning the kernels of integrals on the left sides of equalities (18), they contain terms with non-integrable singularity, except terms for which the inequality (12) holds. We say about the integral term

$$I_{i,j}(s,\tau) = \int\limits_{D_j} (Z_j(s,r_i,\tau,r_j) - Z_j(s,y,\tau,r_j))\mu_i(s,dy) \tag{20}$$

on the right side of the expression for $N_{i,j}(s,\tau)$ and about analogous term (let us denote it by $\hat{I}_{i,n+i-1}(s,\tau)$) in the expression for $\hat{N}_{i,n+j-1}(s,\tau)$.

Indeed, if we take into account the condition b) and the representation

$$I_{i,j}(s,\tau) = \int\limits_{D_j^\delta(r_i)} \left(Z_j(s,r_i,\tau,r_j) - Z_j(s,y,\tau,r_j)\right)\mu_i(s,dy) +$$

$$+ \int\limits_{D_j\setminus D_j^\delta(r_i)} \left(Z_j(s,r_i,\tau,r_j) - Z_j(s,y,\tau,r_j)\right)\mu_i(s,dy) = I_{i,j}^{(1)}(s,\tau) + I_{i,j}^{(2)}(s,\tau),$$

then, applying mean value theorem to the difference $Z_j(s,r_i,\tau,r_j) - Z_j(s,y,\tau,r_j)$ and first inequality in (8) with $r=0,\ p=1$ in the expression for $I_{i,j}^{(1)}$, and first inequality in (8) with $r=0,\ p=0$ to each function $Z_j(s,r_i,\tau,r_j)$ and $Z_j(s,y,\tau,r_j)$ in the expression for $I_{i,j}^{(2)}$, we obtain the estimation $(0\le s<\tau\le t\le T)$

$$|I_{i,j}(s,\tau)| \le C(\tau-s)^{-1}, \tag{21}$$

where constant C just as in the estimation (17) for function $\Psi_i(s,t)$ except its dependence on T and constants from conditions 1)-3) depends also on δ.

It is clear that the same inequality holds also for the function $\hat{I}_{i,n+i-1}(s,\tau)$.

We shall see presently that it is nevertheless possible to obtain the solution of (15), (18) by the ordinary method of successive approximations. Before establishing this important fact, we rewrite system (15), (18) in the form

(22)

$$V_i(s,t) = \sum_{j=i}^{i+1} \int_s^t \left[K_{i,j}(s,\tau)V_j(\tau,t)d\tau + \hat{K}_{i,n+j-1}(s,\tau)V_{n+j-1}(\tau,t)\right]d\tau = f_i(s,t),$$

$$V_{n+i}(s,t) = \sum_{j=i}^{i+1} \int_s^t \left[R_{i,j}(s,\tau)V_j(\tau,t)d\tau + \hat{R}_{i,n+j-1}(s,\tau)V_{n+j-1}(\tau,t)\right]d\tau = f_{n+i}(s,t),$$

$(i=\overline{1,n})$ where

$$f_i(s,t) = d_i(s)\big(\Psi_i(s,t) + \frac{q_{i,i+1}(s)}{\sqrt{b_{i+1}(s,r_i)}}\hat{\Phi}_i(s,t)\big),$$

$$K_{i,j}(s,\tau) = -d_i(s)\big(N_{i,j}(s,\tau) + \frac{q_{i,i+1}(s)}{\sqrt{b_{i+1}(s,r_i)}}M_{i,j}(s,\tau)\big)$$

$$\hat{K}_{i,n+j-1}(s,\tau) = -d_i(s)\big(\hat{N}_{i,n+j-1}(s,\tau) - \frac{q_{i,i+1}(s)}{\sqrt{b_{i+1}(s,r_i)}}\hat{M}_{i,n+j-1}(s,\tau)\big)$$

$$d_i(s) = \frac{b_i(s,r_i)\sqrt{b_{i+1}(s,r_i)}}{q_{i,i}(s)\sqrt{b_{i+1}(s,r_i)} + q_{i,i+1}(s)\sqrt{b_i(s,r_i)}},$$

and functions $f_{n+i}(s,t),\ R_{i,j}(s,\tau)$ and $\hat{R}_{i,n+j-1}(s,\tau)$ are defined by formulas analogous to formulas for $f_i(s,t),\ K_{i,j}(s,\tau)$ and $\hat{K}_{i,n+j-1}(s,\tau)$ respectively, with expressions $d_i(s)$ and $\frac{q_{i,i+1}(s)}{\sqrt{b_{i+1}(s,r_i)}}$ replaced by $l_i(s) = \frac{b_{i+1}(s,r_i)\sqrt{b_i(s,r_i)}}{q_{i,i}(s)\sqrt{b_{i+1}(s,r_i)}+q_{i,i+1}(s)\sqrt{b_i(s,r_i)}}$ and $-\frac{q_{i,i}(s)}{\sqrt{b_i(s,r_i)}}$ respectively.

By the above properties of functions $\Psi_i(s,t),\ \hat{\Phi}_i(s,t),\ i=\overline{1,n}$, and the conditions $1), a), b)$, we have that the functions $f_i(s,t), i=\overline{1,2n}$, are continuous in $s\in[0,t)$ and for them the estimation

$$|f_i(s,t)| \le M(\delta)\|\varphi\|(t-s)^{-\frac{1}{2}} \tag{23}$$

($M(\delta)$ positive constant) holds in domain $0 \le s < t \le T$ for any $\delta > 0$.

Let us investigate kernels of integral equations in (22). All of them are estimated by the same scheme. Consider first the function $K_{i,j}(s,\tau),\ i=\overline{1,n},\ j=i,i+1$. We shall use for it the following representation:

$$K_{i,j}(s,\tau) = K^{(1)}_{i,j}(s,\tau) + K^{(2)}_{i,j}(s,\tau), \tag{24}$$

where

$$K^{(1)}_{i,j}(s,\tau) = -d_i(s)\Bigg[\frac{q_{i,i+1}(s)}{\sqrt{b_{i+1}(s,r_i)}}M_{i,j}(s,\tau) + (-1)^{i+j}q_{i,j}(s)D^1_xG_j(s,r_i,\tau,r_j)+$$
$$+\int\limits_{D^\delta_j(r_i)}(G_j(s,r_i,\tau,r_j)-G_j(s,y,\tau,r_j))\mu_i(s,dy)+$$
$$+\int\limits_{D_j\setminus D^\delta_j(r_i)}(Z'_j(s,r_i,\tau,r_j)-Z'_j(s,y,\tau,r_j))\mu_i(s,dy)\Bigg],$$
$$K^{(2)}_{i,j}(s,\tau) = -d_i(s)\int\limits_{D_j\setminus D^\delta_j(r_i)}(Z_j(s,r_i,\tau,r_j)-Z_j(s,y,\tau,r_j))\mu_i(s,dy).$$

Then, using estimations (8), (12), condition b) and, in addition, the mean value theorem for differences in integrals over the domain $D_j\setminus D^\delta_j(r_i)$, we obtain inequalities

$$|K^{(1)}_{i,j}(s,\tau)| \le N(\delta)(\tau-s)^{-1+\frac{\alpha}{2}}, \quad i=\overline{1,n},\ j=i,i+1, \tag{25}$$

$$|K^{(2)}_{i,j}(s,\tau)| \le C(\delta)(\tau-s)^{-1}, \quad i=\overline{1,n},\ j=i,i+1, \tag{26}$$

which are true for $0\le s<\tau\le T$ and some constants $N(\delta)$ and $C(\delta)$.

Similar representations can be written also for other kernels of system of integral equations (22). Furthermore, in equalities

$$\hat K_{i,n+j-1}(s,\tau) = \hat K^{(1)}_{i,n+j-1}(s,\tau) + \hat K^{(2)}_{i,n+j-1}(s,\tau),$$
$$R_{i,j}(s,\tau) = R^{(1)}_{i,j}(s,\tau) + R^{(2)}_{i,j}(s,\tau),$$
$$\hat R_{i,n+j-1}(s,\tau) = \hat R^{(1)}_{i,n+j-1}(s,\tau) + \hat R^{(2)}_{i,n+j-1}(s,\tau)$$

functions

$$\hat K^{(1)}_{i,n+j-1}(s,\tau), R^{(1)}_{i,j}(s,\tau), \hat R^{(1)}_{i,n+j-1}(s,\tau),\ \text{and}\ \hat K^{(2)}_{i,n+j-1}(s,\tau), R^{(2)}_{i,j}(s,\tau), \hat R^{(2)}_{i,n+j-1}(s,\tau)$$

satisfy inequalities (25) and (26) respectively.

In the sequel we shall need the exact expressions only for second terms in these representations. In particular, the function $\hat K^{(2)}_{i,n+j-1}(s,\tau)$ is expressed by formula

$$\hat K^{(2)}_{i,n+j-1}(s,\tau) = -d_i(s)\int\limits_{D_j\setminus D^\delta_j(r_i)}(Z_j(s,r_i,\tau,r_{j-1})-Z_j(s,y,\tau,r_{j-1}))\mu_i(s,dy). \tag{27}$$

Similarly, functions $R^{(2)}_{i,j}(s,\tau)$ and $\hat R^{(2)}_{i,n+j-1}(s,\tau)$ are defined by replacing in formulas for $K^{(2)}_{i,j}(s,\tau)$ and $\hat K^{(2)}_{i,n+j-1}(s,\tau)$ the factor $d_i(s)$ by $l_i(s)$.

Let us prove that the system of equations (22) has the solution $V_i(s,t),\ i=\overline{1,2n}$, of the form

$$V_i(s,t) = \sum_{k=0}^{\infty} V^{(k)}_i(s,t), \quad V_{n+i}(s,t) = \sum_{k=0}^{\infty} V^{(k)}_{n+i}(s,t), \quad i=\overline{1,n}, \tag{28}$$

where

$$V^{(0)}_i(s,t) = f_i(s,t), \quad V^{(0)}_{n+i}(s,t) = f_{n+i}(s,t),\ i=\overline{1,n},$$

and $(i=\overline{1,n},\ k\in\mathbb{N})$

$$\begin{aligned}
V_i^{(k)}(s,t) &= \sum_{j=i}^{i+1}\int_s^t [K_{i,j}(s,\tau)V_j^{(k-1)}(\tau,t)+\hat{K}_{i,n+j-1}(s,\tau)V_{n+j-1}^{(k-1)}(\tau,t)]d\tau,\\
V_{n+i}^{(k)}(s,t) &= \sum_{j=i}^{i+1}\int_s^t [R_{i,j}(s,\tau)V_j^{(k-1)}(\tau,t)+\hat{R}_{i,n+j-1}(s,\tau)V_{n+j-1}^{(k-1)}(\tau,t)]d\tau.
\end{aligned}\tag{29}$$

Let us estimate $V_i^{(1)}(s,t)$ and $V_{n+i}^{(1)}(s,t)$, $i=\overline{1,n}$. To this end, taking into account (24) as well as the analogous representations for $\hat{K}_{i,n+j-1}(s,\tau)$, $R_{i,j}(s,\tau)$ and $\hat{R}_{i,n+j-1}(s,\tau)$, $j=i,i+1$, we get the relation

$$\begin{aligned}
V_i^{(1)}(s,t) &= \sum_{j=i}^{i+1}\int_s^t [K_{i,j}^{(1)}(s,\tau)f_j(\tau,t)+\hat{K}_{i,n+j-1}^{(1)}(s,\tau)f_{n+j-1}(\tau,t)]d\tau+\\
&\quad+\sum_{j=i}^{i+1}\int_s^t [K_{i,j}^{(2)}(s,\tau)f_j(\tau,t)+\hat{K}_{i,n+j-1}^{(2)}(s,\tau)f_{n+j-1}(\tau,t)]d\tau,\\
V_{n+i}^{(1)}(s,t) &= \sum_{j=i}^{i+1}\int_s^t [R_{i,j}^{(1)}(s,\tau)f_j(\tau,t)+\hat{R}_{i,n+j-1}^{(1)}(s,\tau)f_{n+j-1}(\tau,t)]d\tau+\\
&\quad+\sum_{j=i}^{i+1}\int_s^t [R_{i,j}^{(2)}(s,\tau)f_j(\tau,t)+\hat{R}_{i,n+j-1}^{(2)}(s,\tau)f_{n+j-1}(\tau,t)]d\tau.
\end{aligned}\tag{30}$$

Take the function $V_i^{(1)}(s,t)$ from (30). Denote by $J_i^{(1)}$ and $J_i^{(2)}$ the integral terms on the left side of its expression. Using (23) and (25), we find that

$$\begin{aligned}
|J_i^{(1)}| &\le M(\delta)\|\varphi\|\cdot 2N(\delta)\sum_{j=i}^{i+1}\int_s^t (t-\tau)^{-\frac{1}{2}}(\tau-s)^{-1+\frac{\alpha}{2}}d\tau=\\
&= M(\delta)\|\varphi\|(t-s)^{-\frac{1}{2}}\cdot 4N(\delta)\frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1+\alpha}{2})}(t-s)^{\frac{\alpha}{2}},\quad i=\overline{1,n}.
\end{aligned}\tag{31}$$

To estimate $J_i^{(2)}$, notice that

$$\begin{aligned}
&|Z_j(s,r_i,\tau,r_j)-Z_j(s,y,\tau,r_j)|=\\
&\quad=\left|[2\pi b_j(\tau,r_j)(\tau-s)]^{-\frac{1}{2}}\int_0^1 \frac{d}{d\theta}\exp\left\{-\frac{[(r_j-y)+\theta(y-r_i)]^2}{2b_j(\tau,r_j)(\tau-s)}\right\}d\theta\right|\le\\
&\quad\le \frac{|y-r_i|}{\sqrt{2\pi b}\cdot b(\tau-s)^{\frac{3}{2}}}\int_0^1 |(r_j-y)+\theta(y-r_i)|\exp\left\{-\frac{[(r_j-y)+\theta(y-r_i)]^2}{2B(\tau-s)}\right\}d\theta
\end{aligned}$$

and

$$d_i(s)\le\frac{B\sqrt{B}}{\sqrt{b}\cdot q_0},\quad i=\overline{1,n},\quad s\in[0,T].$$

Here b and B are the constants from condition 1) and

$$q_0=\min_i\min_s(q_{i,i}(s)+q_{i,i+1}(s)).$$

Next, in view of the estimation (23) and the relation

$$\int_s^t (t-\tau)^{-\frac{1}{2}}(\tau-s)^{-\frac{3}{2}} \exp\left\{-\frac{[(r_j-y)+\theta(y-r_i)]^2}{2B(\tau-s)}\right\}d\tau =$$
$$= \frac{\sqrt{2\pi B}}{|(r_j-y)+\theta(y-r_i)|\sqrt{t-s}} \exp\left\{-\frac{[(r_j-y)+\theta(y-r_i)]^2}{2B(t-s)}\right\},$$

one can show that

$$|J_i^{(2)}| \le M(\delta)\|\varphi\|(t-s)^{-\frac{1}{2}} d\cdot\lambda_i(\delta,s), \quad i=\overline{1,n}, \tag{32}$$

where

$$d=\frac{2}{q_0}\left(\frac{B}{b}\right)^2, \quad \lambda_i(\delta,s)=\sum_{j=i}^{i+1}\int\limits_{D_j\setminus D_j^\delta(r_i)} |y-r_i|\mu_i(s,dy).$$

Consider the expression $d\cdot\lambda_i(\delta,t)$. From condition b) it follows that there exists a sufficiently small $\delta=\delta_0>0$ such that for all $i=\overline{1,n}$ and $s\in[0,T]$

$$d\cdot\lambda_i(\delta_0,s)\le d_0<1. \tag{33}$$

In the sequel we shall assume that $\delta=\delta_0$ is substituted into all the above constants depending on δ. Combining (33), (32) and (31), we arrive at the inequality $(0\le s<t\le T)$

$$|V_i^{(1)}(s,t)| \le M(\delta_0)\|\varphi\|(t-s)^{-\frac{1}{2}}\left[\frac{4N(\delta_0)\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1+\alpha}{2})}(t-s)^{\frac{\alpha}{2}}+d_0\right]. \tag{34}$$

It is easy to verify that the same inequality holds also for the function $V_{n+i}^{(1)}(s,t)$.

Letting on the right side of (34)

$$M=M(\delta_0), \quad N=N(\delta_0), \quad h_{s,t}=\frac{4N\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1+\alpha}{2})}(t-s)^{\frac{\alpha}{2}}$$

and proceeding by induction, we have

$$|V_i^{(k)}(s,t)| \le M\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{m=0}^{k}\binom{k}{m} h_{s,t}^{(k-m)} d_0^m, \quad k\in\{0\}\cup\mathbb{N}, \quad i=\overline{1,2n},$$

where

$$h_{s,t}^{(m)}=\frac{\big(4N\Gamma(\frac{\alpha}{2})\big)^m\Gamma(\frac{1}{2})}{\Gamma(\frac{1+m\alpha}{2})}(t-s)^{m\cdot\frac{\alpha}{2}}, \quad m=\overline{0,k}.$$

Hence for $0\le s<t\le T$, $i=\overline{1,2n}$

$$\sum_{k=0}^{\infty}|V_i^{(k)}(s,t)| \le M\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m} h_{s,t}^{(k-m)} d_0^m = \tag{35}$$
$$= M\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}h_{s,t}^{(k)}\sum_{m=0}^{\infty}\binom{k+m}{m} d_0^m =$$
$$= M\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{h_{s,t}^{(k)}}{(1-d_0)^{k+1}} =$$
$$= M\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{\big(4N\Gamma(\frac{\alpha}{2})\big)^k\Gamma(\frac{1}{2})}{\Gamma(\frac{1+k\alpha}{2})(1-d_0)^{k+1}}.$$

The inequality (35) ensures the convergence of series (28) in $s \in [0,t)$ and gives for $V_i(s,t)$ the estimation

$$|V_i(s,t)| \leq C\|\varphi\|(t-s)^{-\frac{1}{2}}, \quad i=\overline{1,2n}, \tag{36}$$

where $0 \leq s < t \leq T$ and C is some constant.

From estimations (8) and (36) it follows that simple layer potentials in (9) exist and satisfy initial conditions $\lim\limits_{s\uparrow t} u^{(i)}(s,x,t) = 0$ and inequalities (10) with $r = p = 0$. This means that the function $u(s,x,t)$ defined by relations (9), (28) is the required classical solution of the problem (3)-(6). The proof of its uniqueness is based on maximum principle for parabolic equations and is a repetition of the proof of the analogous assertion in [12] with obvious changes.

Thus, we have proved the following theorem.

Theorem 3.1. *Let the coefficients of operators $A_s^{(i)}$, $s \in [0,T]$, $i = \overline{1,n+1}$, in (1) satisfy conditions $1) - 3)$ and coefficients $q_{i,i}(s)$, $q_{i,i+1}(s)$ and measures μ_i, $i = \overline{1,n}$, in (2) have properties $a), b)$. Then for any function $\varphi \in C_b(\mathbb{R})$ there exist the unique solution $u(s,x,t)$ of the conjugation problem (3)-(6) which belongs to the class $C^{1,2}([0,t)\times \cup_{i=1}^{n+1} D_i) \cap C([0,t]\times\mathbb{R})$. Furthermore, this solution is of the form (9), (28) and for it the estimation*

$$|u(s,x,t)| \leq C\|\varphi\|, \quad 0 \leq s < t \leq T, \quad x \in \mathbb{R},$$

holds.

4. Construction of Feller semigroup

We introduce the two-parameter family of linear operators $T_{s,t},\ 0 \leq s < t \leq T$ acting on the space $C_b(\mathbb{R})$ by the rule:

$$T_{s,t}\varphi(x) = u(s,x,t,\varphi), \tag{37}$$

where $u(s,x,t,\varphi) \equiv u(s,x,t)$ is the solution of problem (3)-(6) defined by formulas (9), (28).

Let us show that the family of operators $T_{s,t}$ is the required semigroup. To this end, notice first that operators $T_{s,t}$ have the following property: if the sequence $\varphi_n \in C_b(\mathbb{R})$ is such that $\lim\limits_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in \mathbb{R}$ and, in addition, $\sup\limits_n \|\varphi_n\| < \infty$, then $\lim\limits_{n\to\infty} T_{s,t}\varphi_n(x) = T_{s,t}\varphi(x)$ for all $0 \leq s < t \leq T,\ x \in \mathbb{R}$. The proof of this property is based on well known assertions of calculus on passage of the limit under the summation and integral signs (here this concerns series (28) and integrals on the right side of equality (9)). This property allows us to prove the next properties of the operator family $T_{s,t}$, without loss of generality, under the assumption that the function φ has a compact support.

Now we prove that the operators $T_{s,t},\ 0 \leq s < t \leq T$, remain a cone of nonnegative functions invariant.

Lemma 4.1. *If $\varphi \in C_b(\mathbb{R})$ and $\varphi(x) \geq 0$ for all $x \in \mathbb{R}$, then $T_{s,t}\varphi(x) \geq 0$ for all $0 \leq s < t \leq T,\ x \in \mathbb{R}$.*

Proof. Let φ be any nonnegative function in $C_b(\mathbb{R})$ with a compact support. Denote by γ the minimum of $T_{s,t}\varphi(x)$ in the domain $(s,x) \in [0,t]\times\mathbb{R}$ and assume that $\gamma < 0$. From the minimum principle it follows that there exist $s_0 \in (0,t),\ i_0 \in \{1,\dots,\ n\}$ such that $T_{s_0,t}\varphi(r_{i_0}) = \gamma$. But then the inequalities

$$\frac{\partial T_{s_0,t}\varphi(r_{i_0}-)}{\partial x} \leq 0, \quad \frac{\partial T_{s_0,t}\varphi(r_{i_0}+)}{\partial x} \geq 0,$$

$$\int\limits_{D_{i_0}\cup D_{i_0+1}} (T_{s_0,t}\varphi(r_{i_0}) - T_{s_0,t}\varphi(y))\,\mu_i(s_0,dy) \le 0$$

must hold. Furthermore, Theorem 14 in [16, p. 69] assures us that

$$\frac{\partial T_{s_0,t}\varphi(r_{i_0}-)}{\partial x} < 0, \quad \frac{\partial T_{s_0,t}\varphi(r_{i_0}+)}{\partial x} > 0.$$

Next, since $q_{i_0,i_0}(s_0) + q_{i_0,i_0+1}(s_0) > 0$, it becomes clear that in the case of $s = s_0$ the fulfillment of the conjugation condition (6) is impossible. The contradiction we arrived at indicates that $\gamma \ge 0$. This completes the proof of the lemma. $\square$

Notice also that $T_{s,t}\varphi_0(x) = 1$ for all $0 \le s < t \le T,\ x \in \mathbb{R}$ provided $\varphi_0(x) \equiv 1$. This property together with the assertion of lemma 4.1 allow us to assert that operators $T_{s,t}$ for all $0 \le s < t \le T$ are contractive, that is,

$$\|T_{s,t}\varphi\| \le \|\varphi\|.$$

Finally, we show that the operator family T_{st} has the semigroup property

$$T_{s,t} = T_{s,\tau}T_{\tau,t}, \quad 0 \le s < \tau < t \le T.$$

This property is a consequence of the assertion of uniqueness of the solution of the problem (3)-(6). Indeed, to find $u(s,x,t) = T_{s,t}\varphi(x)$, when it is given that $\lim\limits_{s\uparrow t} u(s,x,t) = \varphi(x)$, one can solve the problem first in time interval $[\tau,t]$ and then solve it in the time interval $[s,\tau]$ with that initial function $u(\tau,x,t) = T_{\tau,t}\varphi(x)$ which was obtained; in other words, $T_{s,t}\varphi(x) = T_{s,\tau}(T_{\tau,t}\varphi)(x),\ \varphi \in C_b(\mathbb{R})$, or $T_{s,t} = T_{s,\tau}T_{\tau,t}$.

The above properties of operators T_{st} imply the following assertion (see. [18, Ch.II, §1]).

Theorem 4.1. *Let the conditions of Theorem 3.1 hold. Then the two-parameter family of operators $T_{s,t},\ 0 \le s \le t \le T$, defined by formula (37), describes the inhomogeneous Feller process on $\mathbb{R}$ which coincides on each domain $D_i,\ i = \overline{1,n+1}$ with the diffusion process generated by the operator $A_s^{(i)}$ and its behavior at points $r_1,\ r_2,\ldots,\ r_n$ is determined by Feller-Wentzell conjugation condition (2).*

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