UDC 515.14

**H. Hosseinzadeh, A. Jabbari** (Islamic Azad Univ., Ardabil, Iran),

**A. Razani** (Imam Khomeini Int. Univ., Qazvin, Iran)

# FIXED-POINT THEOREMS AND COMMON FIXED-POINT THEOREMS ON SPACES EQUIPPED WITH VECTOR-VALUED METRICS

# ТЕОРЕМИ ПРО НЕРУХОМУ ТОЧКУ ТА СПІЛЬНУ НЕРУХОМУ ТОЧКУ НА ПРОСТОРАХ ІЗ ВЕКТОРНОЗНАЧНОЮ МЕТРИКОЮ

We show the existence of a fixed point and common fixed point for single-valued generalized contractions on spaces equipped with vector-valued metrics.

Показано існування нерухомої точки та спільної нерухомої точки для однозначних узагальнених стискувальних відображень на просторах із векторнозначною метрикою.

**1. Introduction.** The classical Banach contraction principle was extended for contraction mappings on spaces endowed with vector-valued metrics by Perov in 1964 [4]. Filip et al. [2] studied fixed point property of a self mapping on generalized metric space $(X, d)$ and generalized the results of Perov. In this paper, Theorem 2.1 of [2] is generalized, and local fixed point property of a self mapping on generalized metric space $(X, d)$ is considered. Finally, common fixed point property of two single-valued self mappings on generalized metric space $(X, d)$ is studied.

Throughout this paper $\mathcal{C}, \mathcal{R}$ and $\mathcal{N}$ are the sets of all complex, real and natural numbers, respectively.

Let $(\mathcal{V}, \preceq)$ be an ordered Banach space. The cone $\mathcal{V}_+ = \{v \in \mathcal{V} : \theta \preceq v\}$, where $\theta$ is the zero-vector of $\mathcal{V}$, satisfies the usual properties

1)  $\mathcal{V}_+ \cap -\mathcal{V}_+ = \{\theta\}$;

2)  $\mathcal{V}_+ + \mathcal{V}_+ \subset \mathcal{V}_+$;

3)  $\alpha \mathcal{V}_+ \subset \mathcal{V}_+$ for $\alpha \geq 0$.

Let $X$ be a nonempty set. A mapping $d \colon X \times X \longrightarrow \mathcal{V}$ is called a vector-valued metric on $X$, if the following properties are satisfied:

1)  $d(x, y) \geq \theta$ for each $x, y \in X$, if $d(x, y) = \theta$, then $x = y$;

2)  $d(x, y) = d(y, x)$ for each $x, y \in X$;

3)  $d(x, y) \preceq d(x, z) + d(z, y)$ for each $x, y, z \in X$.

The pair $(X, d)$ is called vector-valued metric space. Agarwal and Khamsi [3] (Theorem 2) show that for lower semi-continuous function $F$ from complete vector-valued metric space $(X, d)$ over an order complete and order continuous Banach lattice $\mathcal{V}$, if function $T \colon X \longrightarrow X$ satisfied in the following condition for every $x \in X$

$$d(x, T(x)) \leq F(x) - F(T(x)),$$

then $\operatorname{Fix}(T) \neq \varnothing$. Now, we replace $\mathcal{V}$ by $\mathcal{R}^m$ and have the following definition for vector-valued metric space.

Let $X$ be a nonempty set. A mapping $d \colon X \times X \longrightarrow \mathcal{R}^m$ is called a vector-valued metric on $X$, if the following properties are satisfied:

1) $d(x, y) \geq 0$ for each $x, y \in X$, if $d(x, y) = 0$, then $x = y$;

2) $d(x, y) = d(y, x)$ for each $x, y \in X$;

3) $d(x, y) \leq d(x, z) + d(z, y)$ for each $x, y, z \in X$.

A set $X$ equipped with a vector-valued metric $d$ is called a generalized metric space and denoted by $(X, d)$. Let $x_1$ be an element of generalized metric space $X$ and $r = (r_i)_{i=1}^m \in \mathcal{R}^m$, with $r_i > 0$ for each $1 \leq i \leq n$, then $B(x_1, r) = \{x \in X | d(x_1, x) < r\}$ is the open ball centered in $x_1$ with radius $r$, also $\widetilde{B}(x_1, r) = \{x \in X | d(x_1, x) \leq r\}$ is the closed ball centered in $x_1$ with radius $r$.

Let $f \colon X \longrightarrow X$ be a single-valued map. $\mathrm{Fix}(f) = \{x \in X | f(x) = x\}$ is the set of all fixed points of $f$.

$M_{m,m}(\mathcal{R}^+)$ means the set of all $m \times m$ matrices with positive elements, $\Theta$ the zero matrix, and $I$ the identity $m \times m$ matrix. Let $A \in M_{m,m}(\mathcal{R}^+)$, $A$ is said to be convergent to zero, if and only if $A^n \longrightarrow 0$ as $n \longrightarrow \infty$ (see [7], for more details).

Let $\alpha, \beta \in \mathcal{R}^m$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ and $c \in \mathcal{R}$. Note that $\alpha \leq \beta$ (resp. $\alpha < \beta$) means $\alpha_i \leq \beta_i$ (resp. $\alpha_i < \beta_i$) for each $1 \leq i \leq m$, and also $\alpha \leq c$ (resp. $\alpha < c$) means $\alpha_i \leq c$ (resp. $\alpha_i < c$) for $1 \leq i \leq m$, respectively. As well as, we can define addition and multiplication on $\mathcal{R}^m$ as follows:

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_m + \beta_m),$$

and

$$\alpha \cdot \beta = (\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_m \beta_m),$$

for every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathcal{R}^m$.

Now, we need the following equivalent statements:

1) $A$ is convergent towards zero;

2) $A^n \longrightarrow 0$ as $n \longrightarrow \infty$;

3) the eigenvalues of $A$ are in the open unit disc, that is, $|\lambda| < 1$, for each $\lambda \in \mathcal{C}$ with $\det(A - \lambda I) = 0$;

4) the matrix $I - A$ is nonsingular and

$$(I - A)^{-1} = I + A + \ldots + A^n + \ldots;$$

5) $A^n q \longrightarrow 0$ and $q A^n \longrightarrow 0$ as $n \longrightarrow \infty$, for each $q \in \mathcal{R}^m$.

The proof of the above statements are the classical results in matrix analysis (see [1, 5, 6] for more details). For the sake of simplicity, we make an identification between row and column vectors in $\mathcal{R}^m$.

**2. Fixed point property.** Let $(X, d)$ be a vector-valued metric space and $A$ be an operator such as [3] (Theorem 1). In [3], $d_A(x, y)$ identified by $(I - A)d(x, y)$, for every $x, y \in X$. Then $d_A$ is a vector-valued metric on $X$, and the vector-valued metric space $(X, d_A)$ is complete.

**Theorem 2.1.** *Let $(X, d)$ be a complete generalized metric space, and $f \colon X \longrightarrow X$ be a continuous map with the property that, there exists $A, B, C \in M_{m,m}(\mathcal{R}_+)$ such that*

$$d(f(x), f(y)) \leq Ad(x, y) + Bd(x, y)d(y, f(x))[d(x, f(x)) + d(y, f(y))]+$$

$$+ Cd(x, y)[d(x, f(y))d(y, f(x))] \tag{2.1}$$

*for every $x, y \in X$. Suppose $A$ is a matrix converging to zero. Then $\mathrm{Fix}(f) \neq \varnothing$.*

**Proof.** It is sufficient to provide conditions [3] (Theorem 2). By condition (2.1), we have

$$d(f(x), f^2(x)) \leq Ad(x, f(x)), \tag{2.2}$$

for every $x \in X$. Therefore

$$d_A(x, f(x)) = (I - A)d(x, f(x)) \leq d(x, f(x)) - d(f(x), f^2(x)).$$

Similar to [3] (Corollary 1), we define $F(x) := d(x, f(x))$, for every $x \in X$. Then $F$ is a continuous map, and by [3] (Theorem 2]), $\text{Fix}(f) \neq \varnothing$.

We recall $\widetilde{B}(x_1, r) = \{x \in X | d(x_1, x) \leq r\}$, is the closed ball centered in $x_1$ with radius $r$.

**Theorem 2.2.** *Let $(X, d)$ be a complete generalized metric space, $r := (r_i)_{i=1}^m \in \mathcal{R}_+^m$ with $0 < r_i \leq 1$ for each $i \in \{1, 2, \ldots, m\}$ and $f \colon \widetilde{B}(x_1, r) \longrightarrow X$ having the property that, there exists $A, B, C \in M_{m,m}(\mathcal{R}_+)$ such that*

$$d(f(x), f(y)) \leq A(d(x, y))d(x, y) + Bd(x, y)d(y, f(x))[d(x, f(x)) + d(y, f(y))]+$$

$$+Cd(x, y)[d(x, f(y))d(y, f(x))]$$

*for every $x, y \in X$. Suppose*

(i) *$A$ is a matrix converging to zero*;
(ii) *if $u \in \mathcal{R}_+^m$ is such that $u(I - A)^{-1} \leq (I - A)^{-1}r$, then $u \leq r$;*
(iii) *$d(x_1, f(x_1))(I - A)^{-1} \leq r$.*

*Then $\text{Fix}(f) \neq \varnothing$.*

**Proof.** Construct the sequence $(x_n)_{n \in \mathcal{N}}$ as follows: for each $n \in \mathcal{N}$, set $x_{n+1} = f(x_n)$. By (iii), $d(x_1, x_2)(I - A)^{-1} = d(x_1, f(x_1))(I - A)^{-1} \leq r \leq (I - A)^{-1}r$. Therefore, by using (ii), $d(x_1, x_2) \leq r$ and

$$d(x_1, x_2)(d(x_1, x_2)(I - A)^{-1}) = d(x_1, x_2)(d(x_1, f(x_1))(I - A)^{-1} \leq$$

$$\leq d(x_1, x_2)r \leq d(x_1, x_2)(I - A)^{-1}r \leq r^2.$$

Therefore, $A(d(x_1, x_2))d(x_1, x_2)(I - A)^{-1} \leq Ar^2$. Similarly,

$$d(x_2, x_3)(I - A)^{-1} = d(f(x_1), f(x_2))(I - A)^{-1} \leq$$

$$\leq A(d(x_1, x_2))d(x_1, x_2)(I - A)^{-1}+$$

$$+Bd(x_1, x_2)d(x_2, f(x_1))[d(x_1, f(x_1)) + d(x_2, f(x_2))](I - A)^{-1}+$$

$$+ Cd(x_1, x_2)[d(x_1, f(x_2)d(x_2, f(x_1))](I - A)^{-1} \leq$$

$$\leq Ar^2 + \Theta = Ar^2.$$

Above inequality hold by this fact since $x_2 = f(x_1)$, thus $d(x_2, f(x_1)) = \Theta$,

$$Bd(x_1, x_2)d(x_2, f(x_1))[d(x_1, f(x_1)) + d(x_2, f(x_2))](I - A)^{-1} = \Theta,$$

and $Cd(x_1, x_2)[d(x_1, f(x_2)d(x_2, f(x_1))](I - A)^{-1} = \Theta$.

As well as, since $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$, $0 < r < 1$ and $r^2 < r$, therefore,

$$d(x_1, x_3)(I - A)^{-1} \leq d(x_1, x_2)(I - A)^{-1} + d(x_2, x_3)(I - A)^{-1} \leq$$

$$\leq Ir + Ar^2 \leq Ir + Ar \leq (I + A + A^2 + \ldots + A^n + \ldots)r =$$

$$= (I - A)^{-1}r.$$

By induction, for all $n \in \mathcal{N}$, the sequence $(x_n)$ is in $\widetilde{B}(x_1, r)$ and satisfying
(1) $x_{n+1} = f(x_n)$;
(2) $d(x_0, x_n)(I - A)^{-1} \leq (I - A)^{-1}r$;
(3) $d(x_n, x_{n+1})(I - A)^{-1} \leq A^{n+1}r$.
We show, $(x_n)$ is a Cauchy sequence. For every $n, m \in \mathcal{N}$, $n \leq m$,

$$d(x_n, x_m)(I - A)^{-1} \leq d(x_n, x_{n+1})(I - A)^{-1} + d(x_{n+1}, x_{n+2})(I - A)^{-1} + \ldots$$

$$\ldots + d(x_{m-1}, x_m)(I - A)^{-1} \leq$$

$$\leq A^{n+1}r + A^{n+2}r + \ldots + A^m r \leq$$

$$\leq A^{n+1}(I + A + \ldots + A^{m-1} + \ldots)r =$$

$$= A^{n+1}(I - A)^{-1}r \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Therefore $(x_n)$ is a Cauchy sequence. Since the space $\left(\widetilde{B}(x_1, r), d\right)$ is a complete, there exists an element $x^* \in \widetilde{B}(x_1, r)$ such that $x_n \longrightarrow x^*$. Now, $x^*$ is a fixed point of $f$, because

$$d(x^*, f(x^*)) \leq d(x^*, x_n) + d(x_n, f(x^*)) =$$

$$= d(x^*, x_n) + d(f(x_{n-1}), f(x^*)) \leq$$

$$\leq d(x^*, x_n) + Ad(x_{n-1}, x^*)d(x_{n-1}, x^*) +$$

$$+ Bd(x^*, f(x_{n-1}))[d(x_{n-1}, f(x_{n-1})) + d(x^*, f(x^*))] +$$

$$+ Cd(x^*, f(x_{n-1})d(x_{n-1}, x^*) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Then $\text{Fix}(f) \neq \varnothing$.

An application for above stated theorem is the following operator system in Banach space $X$ with norm $\| \cdot \|$:

$$f_1(u_1, u_2) = u_1,$$

$$f_2(u_1, u_2) = u_2,$$

(2.3)

where $f_i \colon X^2 \to X$, $i = 1, 2$, are given nonlinear operator.

It is obvious that the system (2.3) can be viewed as a fixed point problem as follows:

$$f(u) = u$$

in the space $X^2$, where $u = (u_1, u_2)$ and $f = (f_1, f_2)$. Now, we have the following theorem for an application to Theorem 2.2.

**Theorem 2.3.** *Assume that for $i \in 1, 2$, there exist nonnegative number $a_i$, $a_i'$, $b_i$, $b_i'$, $c_i$, $c_i'$ such that*

$$\|f_i(u_1, u_2) - f_i(v_1, v_2)\| \leq a_i \|u_1 - v_1\|^2 + a_i' \|u_2 - v_2\|^2 +$$

$$+ b_i \|u_1 - v_1\| \big[ (\|v_1 - f_i(u_1, u_2)\| + \|v_2 - f_i(u_1, u_2)\|) \times$$

$$\times (\|u_1 + v_1 - f_i(u_1, u_2) - f_i(v_1, v_2)\| + \|u_2 + v_2 - f_i(u_1, u_2) - f_i(v_1, v_2)\|) \big] +$$

$$+ b_i' \|u_2 - v_2\| \big[ (\|v_1 - f_i(u_1, u_2)\| + \|v_2 - f_i(u_1, u_2)\|) \times$$

$$\times (\|u_1 + v_1 - f_i(u_1, u_2) - f_i(v_1, v_2)\| + \|u_2 + v_2 - f_i(u_1, u_2) - f_i(v_1, v_2)\|) \big] +$$

$$+ c_i \|u_1 - v_1\| \big[ (\|u_1 - f_i(v_1, v_2)\| + \|u_2 - f_i(v_1, v_2)\|)(\|v_1 - f_i(u_1, u_2)\| + \|v_2 - f_i(u_1, u_2)\|) \big] +$$

$$+ c_i' \|u_2 - v_2\| \big[ (\|u_1 - f_i(v_1, v_2)\| + \|u_2 - f_i(v_1, v_2)\|) \times$$

$$\times (\|v_1 - f_i(u_1, u_2)\| + \|v_2 - f_i(u_1, u_2)\|) \big] \tag{2.4}$$

*for all $u_1, u_2, v_1, v_2 \in X$. In addition assume that $A = \begin{pmatrix} a_1 & a_1' \\ a_2 & a_2' \end{pmatrix}$ is a convergence to zero matrix, and $B = \begin{pmatrix} b_1 & b_1' \\ b_2 & b_2' \end{pmatrix}$ and $\begin{pmatrix} c_1 & c_1' \\ c_2 & c_2' \end{pmatrix}$ belong to $M_{2 \times 2}(\mathcal{R}^+)$. Then (2.3) has a unique solution $u = (u_1, u_2)$ in $X^2$.*

**Proof.** Condition (2.4) can be rewritten as

$$\|f(u) - f(v)\| \leq A\|u - v\|^2 + B[\|u - v\| \|v - f(v)\| (\|u - f(u)\| + \|v - f(v)\|)] +$$

$$+ C[\|u - v\| (\|u - f(v)\| \|v - f(u)\|)].$$

Thus Theorem 2.2 implies our desire. Here $X^2 = R^2$ and $d(u, v) = \|u - v\|$.

**3. Common fixed-point theorem.** Let $f$ and $g$ be two self mappings on complete generalized metric space $(X, d)$. In this section, we study the existence of a common fixed point for these mappings. Due to this, we need the following lemma.

**Lemma 3.1.** *Let $(X, d)$ be a complete generalized metric space and $(y_n)_{n \in \mathcal{N} \cup \{0\}}$ be a sequence in $X$. If $A \in M_{m,m}(\mathcal{R}^+)$ is a matrix converging to zero, and for every $n \in \mathcal{N}$*

$$d(y_n, y_{n+1}) \leq A d(y_n, y_{n-1}). \tag{3.1}$$

*Then $(y_n)_{n \in \mathcal{N}}$ is converging in $X$.*

**Proof.** By (3.1) we have

$$d(y_n, y_{n-1}) \leq A d(y_{n-1}, y_{n-2}), d(y_{n-1}, y_{n-2}) \leq A d(y_{n-2}, y_{n-3}), \ldots, d(y_2, y_1) \leq A d(y_1, y_0).$$

Thus

$$d(y_{n+1}, y_n) \leq A d(y_n, y_{n-1}) \leq A^2 d(y_{n-1}, y_{n-2}) \leq \ldots \leq A^n d(y_1, y_0) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Therefore $(y_n)$ is a Cauchy sequence. Since $(X, d)$ is a complete, there exists an element $x \in X$ such that $y_n \longrightarrow x$.

**Theorem 3.1.** *Let $(X, d)$ be a complete generalized metric space, $f \colon X \longrightarrow X$ be a continuous function and $A \in M_{m,m}(\mathcal{R}^+)$ be a nonzero matrix converging to zero. Suppose $g$ is a selfmap function such that $g \circ f = f \circ g$, $g(X) \subset f(X)$, and*

$$d(g(x), g(y)) \leq A d(f(x), f(y)), \quad x, y \in X. \tag{3.2}$$

*Then $f$ and $g$ have a unique common fixed point.*

**Proof.** Choose the elements $x_0, x_1 \in X$ such that $f(x_1) = g(x_0)$. Construct the sequence $(x_n)_{n \in \mathcal{N} \cup \{0\}}$, as follows

$$f(x_n) = g(x_{n-1}).$$

By the assumption $g(X) \subset f(X)$, and (3.2), we have

$$d(g(x_n), g(x_{n-1})) \leq A d(f(x_n), f(x_{n-1})),$$

$$d(f(x_n), f(x_{n-1})) = d(g(x_{n-1}), g(x_{n-2})) \leq A d(f(x_{n-1}), f(x_{n-2})),$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$d(f(x_2), f(x_1)) = A d(g(x_1), g(x_0)) \leq A d(f(x_1), f(x_0)).$$

Thus

$$d(f(x_{n+1}), f(x_n)) = d(g(x_n), g(x_{n-1})) \leq$$

$$\leq A d(f(x_n), f(x_{n-1})) = A d(g(x_{n-1}), g(x_{n-2})) \leq$$

$$\leq A^2 d(f(x_{n-1}), f(x_{n-2})) \leq \ldots$$

$$\ldots \leq A^n d(f(x_1), f(x_0)) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Therefore $(f(x_n))$ is a Cauchy sequence. Thus by Lemma 3.1, there exists a $t \in X$ such that $f(x_n) \longrightarrow t$. Also, by the definition of $g$, $g(x_n) \longrightarrow t$. Continuity of $f$ implies that $g$ is a continuous map. Thus $g(f(x_n)) \longrightarrow g(t)$ and since $g \circ f = f \circ g$, so $f(g(x_n)) \longrightarrow f(t)$. By these results, $f(t) = g(t)$ and $f(f(t)) = f(g(t)) = g(g(t))$. But,

$$d(g(t), g(g(t))) \leq A d(f(t), f(g(t))) = A d(g(t), g(g(t))),$$

and so

$$d(g(t), g(g(t)))(I - A) \leq 0.$$

But $I \neq A$, hence $d(g(t), g(g(t))) = 0$. This means

$$g(g(t)) = g(t) = f(g(t)) = g(f(t)).$$

Therefore $g(t) \in \mathrm{Fix}(f) \cap \mathrm{Fix}(g)$. Finally, suppose there are $x, y \in X$ such that $x, y \in \mathrm{Fix}(f) \cap \cap \mathrm{Fix}(g)$. Then by (2.2),

$$d(x,y) = d(g(x), g(y)) \leq Ad(f(x), f(y)) = Ad(x,y),$$

and so

$$d(x,y)(I - A) \leq 0.$$

Therefore $x = y$ and the common fixed point of $f$ and $g$ is unique.

**Corollary 3.1.** *Let $f$ and $g$ be two commuting and self mappings on a complete generalized metric space $(X, d)$. Suppose $f$ is continuous and $g(X) \subset f(X)$. If, there exists a matrix $A \in \in M_{m,m}(\mathcal{R}^+)$ such that $A$ converges to zero, and for each $x, y \in X$, following condition holds*

$$d(g^k(x), g^k(y)) \leq Ad(f(x), f(y)), \quad k \in \mathcal{N}.$$

*Then $f$ and $g$ have a common fixed point.*

**Proof.** For every $k > 1$,

$$g^k \circ f = g^{k-1} \circ g \circ f = g^{k-1} \circ f \circ g = \ldots = f \circ g^k,$$

also $g^k(X) \subseteq g(X) \subset f(X)$. Therefore, by Theorem 3.1, $g^k$ and $f$ have a unique common fixed point. Let $a \in X$ be the unique common fixed point of $g^k$ and $f$. Thus $a = f(a) = g^k(a)$. Since $f$ and $g$ are commuting mappings, then

$$g(a) = g(f(a)) = g(g^k(a)) = g^k(g(a)) = f(g(a)).$$

Thus $g(a)$ is a common fixed point of $g^k$ and $f$. Since, the common fixed point of $g^k$ and $f$ was unique. Hence, we should have $a = g(a) = f(a)$.

**Corollary 3.2.** *Let $n \in \mathcal{N}$ and $A$ invertible matrix which $A > I$. Suppose $g$ is a continuous self mapping on a complete generalized metric space $(X, d)$ satisfying*:

$$d(g^n(x), g^n(y)) \geq Bd(x,y), \quad x, y \in X,$$

*where $B$ is the matrix $A^{-1}$. Then $g$ has a unique fixed point.*

**Proof.** Take $f = g^{n+1}$, then proof is clear.

1. *Allaire G., Kaber S. M.* Numerical linear algebra // Texts Appl. Math. – 2008. – **55**.

2. *Filip A. D., Petruşel.* Fixed point theorems on spaces endowed with vector-valued metrics // Fixed Point Theory and Appl. – 2010. – Article ID 281381. – 15 p.

3. *Agarwal R. P., Khamsi M. A.* Extension of Caristis fixed point theorem to vector valued metric spaces // Nonlinear Anal. / DOI: 10.1016/j.na.2010.8.025.

4. *Perov A. I.* On the Cauchy problemma for a system of ordinary differential equations // Priblizhen. Met. Reshen. Differents. Uravn. – 1964. – **2**. – P. 115 – 134.

5. *Precup R.* The role of matrices that are convergent to zero in the study of semilinear operator systems // Math. and Comput. Modelling. – 2009. – **49**. – P. 703 – 708.

6. *Rus I. A.* Principles and applications of the fixed point theory. – Romania: Dacia, Cluj-Napoca, 1979.

7. *Varga R. S.* Matrix iterative analysis // Springer Ser. Comput. Math. – 2000. – **27**.