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ON THE BEHAVIOR OF SOLUTIONS OF A THIRD-ORDER NONLINEAR DYNAMIC EQUATION ON TIME SCALES*

ПРО ПОВЕДІНКУ РОЗВ'ЯЗКІВ НЕЛІНІЙНОГО ДИНАМІЧНОГО РІВНЯННЯ ТРЕТЬОГО ПОРЯДКУ НА ЧАСОВИХ ШКАЛАХ

The objective of this paper is to study oscillatory and asymptotic properties of the third-order nonlinear dynamic equation

$$\left[\left(\frac{1}{r_2(t)}\left(\left(\frac{1}{r_1(t)}x^{\Delta}(t)\right)^{\gamma_1}\right)^{\Delta}\right)^{\gamma_2}\right]^{\Delta}+f(t,x^{\sigma}(t))=0,\quad t\in\mathbb{T}.$$

By using the Riccati transformation, we present new criteria for the oscillation or certain asymptotic behavior of solutions of this equation. We suppose that the time scale $\mathbb{T}$ is unbounded above.

Метою цієї статті є вивчення осциляційних та асимптотичних властивостей нелінійного динамічного рівняння третього порядку

$$\left[\left(\frac{1}{r_2(t)}\left(\left(\frac{1}{r_1(t)}x^{\Delta}(t)\right)^{\gamma_1}\right)^{\Delta}\right)^{\gamma_2}\right]^{\Delta}+f(t,x^{\sigma}(t))=0,\quad t\in\mathbb{T}.$$

За допомогою перетворення Ріккаті отримано нові критерії осциляції та певної асимптотичної поведінки розв'язків цього рівняння. Часова шкала $\mathbb{T}$ вважається необмеженою зверху.

1. Preliminaries and notation. Much recent attention has been given to dynamic equations on time scales, or measure chains, and we refer the reader to the landmark paper of S. Hilger [1] for a comprehensive treatment of the subject. Since then, several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal, Bohner, O'Regan and Peterson [2]. A book on the subject of time scales by Bohner and Peterson [3] also summarizes and organizes much of the time scale calculus. The various type oscillation and nonoscillation criteria for solutions of ordinary and dynamic equations have been studied extensively in a large cycle of works (see [4 – 12]).

In [4], the authors have considered third-order nonlinear dynamic equation (1.1) for $\gamma_1=\gamma_2=1$. They have studied asymptotic behavior that equation. Yu and Wang [5] have considered the third-order nonlinear dynamic equation

$$(1/(a_2(t)(((1/(a_1(t))(x^{\Delta}(t))^{\alpha_1})^{\Delta})^{\alpha_2})^{\Delta}+q(t)f(x(t))=0,$$

where α_1 and α_2 are quotient of odd positive integers. They have supposed that a_1, a_2 and q are positive, real-valued, rd-continuous functions defined on time scale $\mathbb{T}$.

$f\in C(\mathbb{R},\mathbb{R})$ is assumed to satisfy $xf(x)>0$ $(x\neq 0)$, and for $k>0$, $\exists M=M_k>0$, $\dfrac{f(x)}{x}\geq M$, $|x|\geq k$. The authors have studied the asymptotic behavior of solution of above equation.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward and the backward jump operators on any time scale $\mathbb{T}$ are defined by $\sigma(t):=\inf\{s\in\mathbb{T}\colon s>t\}$, $\rho(t):=\sup\{s\in\mathbb{T}\colon s<t\}$. A point $t\in\mathbb{T}$, $t>\inf\mathbb{T}$, is said to be left-dense if $\rho(t)=t$, right-dense if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, left-scattered if $\rho(t)<t$ and right-scattered if $\sigma(t)>t$. The graininess

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function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. For a function $f\colon \mathbb{T} \to \mathbb{R}$ the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is not right-scattered, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^+} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists. A function $f\colon [a, b] \to \mathbb{R}$ is said to be right-dense continuous if it is right continuous at each right-dense point and there exists a finite left limit at all left-dense points, and f is said to be differentiable if its derivative exists. A useful formula dealing with the time scale is that

$$f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma},$$

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}.$$

The integration by parts formula is

$$\int_a^b f^{\Delta}(t) g(t) \Delta t = f(b) g(b) - f(a) g(a) - \int_a^b f^{\sigma}(t) g^{\Delta}(t) \Delta(t).$$

The function $f\colon \mathbb{T} \to \mathbb{R}$ is called *rd*-continuous if it is continuous at the right-dense points and if the left-sided limits exist in left-dense points. Not only does the new theory of the so-called "dynamic equations" unify the theories of differential and difference equations, but also extends these classical cases to cases "in between", e.g., to the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}} = \{qt\colon t \in \mathbb{N}_0$ for $q > 1\}$ (which has important application in quantum theory).

We will study the asymptotic behavior or oscillation of solutions of third-order nonlinear dynamic equation

$$\left[\left(\frac{1}{r_2(t)}\left(\left(\left(\frac{1}{r_1(t)} x^{\Delta}(t)\right)^{\gamma_1}\right)^{\Delta}\right)^{\gamma_2}\right]^{\Delta} + f(t, x^{\sigma}(t)) = 0 \qquad t \in \mathbb{T}, \tag{1.1}$$

or for short,

$$L_3 x(t) + f(t, x^{\sigma}(t)) = 0, \quad t \in \mathbb{T}, \tag{1.2}$$

where $\mathbb{T}$ is a time scale,

$$L_1 x(t) = \left(\frac{1}{r_1(t)} x^{\Delta}(t)\right)^{\gamma_1},$$

$$L_2x(t) = \left(\frac{1}{r_2(t)}\left(\left(\frac{1}{r_1(t)}x^{\Delta}(t)\right)^{\gamma_1}\right)^{\Delta}\right)^{\gamma_2},$$

$$L_3x(t) = [L_2x(t)]^{\Delta}.$$

In the sequel we will assume:

(H_1) $r_n(t)$ are positive, real-value, rd-continuous functions defined on the time scales such that

$$\int\limits_{T_0}^{\infty} r_n(s)\Delta s = \infty, \quad n = 1, 2;$$

(H_2) γ_n is a quotient of odd positive integers, $n = 1, 2$;

(H_3) $f\colon \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous function, there exists $k > 0$ such that $uf(t,u) > 0$, $u \neq 0$, $\dfrac{f(t,u)}{u} \geq q(t)$, $|u| \geq k$. $q(t)$ is positive, real-valued, rd-continuous function defined on time scales.

A solution $x(t)$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

2. Main results. We need the following lemmas which play an important role in the proof of main results.

Lemma 2.1. *Assume* (H_1)–(H_3) *hold,* $x(t)$ *is an eventually positive solution of* (1.1). *Then there exists a* $T_1 \in [T_0, \infty)$ *such either*:

(i) $x(t) > 0$, $L_1x(t) > 0$, $L_2x(t) > 0$, $t \in [T_1, \infty)$,

or

(ii) $x(t) > 0$, $L_1x(t) < 0$, $L_2x(t) > 0$, $t \in [T_1, \infty)$.

Proof. Let $x(t)$ be a eventually positive solution of (1.1), then there exists $T_1 \in [T_0, \infty)$ such that $x(t) > 0$ for $t \in [T_1, \infty)$. Since $x^{\sigma}(t) > x(t) > 0$, $f(t, x^{\sigma}(t)) > 0$ for $t \in [T_1, \infty)$ and from (1.1) we have

$$L_3x(t) = -f(t, x^{\sigma}(t)) < 0, \quad t \in [T_1, \infty),$$

which implies that $L_2x(t)$ is strictly decreasing on $t \in [T_1, \infty)$. We claim that $L_2x(t) > 0$. Otherwise, there exists a $T_2 \in [T_1, \infty)$ such that

$$L_2x(t) \leq L_2x(T_2) < 0, \quad t \in [T_2, \infty),$$

that is,

$$\left[\frac{1}{r_2(t)}\left(\left(\frac{1}{r_1(t)}x^{\Delta}(t)\right)^{\gamma_1}\right)^{\Delta}\right]^{\gamma_2} \leq L_2x(T_2) < 0, \quad t \in [T_2, \infty).$$

Hence we have

$$(L_1x(t))^{\Delta} \leq r_2(t)(L_2x(T_2))^{\frac{1}{\gamma_2}} \tag{2.1}$$

which implies that $L_1x(t)$ is strictly decreasing on $[T_2, \infty)$. Integrating (2.1) from T_2 to t, we obtain

$$L_1x(t) \leq L_1x(T_2) + (L_2x(T_2))^{\frac{1}{\gamma_2}} \int\limits_{T_2}^{t} r_2(s)\Delta s.$$

Letting $t \to \infty$, we have $L_1x(t) \to -\infty$. Thus, there exists $T_3 \in [T_2, \infty)$ such that

$$L_1x(t) \le L_1x(T_3) < 0, \quad t \in [T_3, \infty),$$

that is,

$$\left(\frac{1}{r_1(t)}x^{\Delta}(t)\right)^{\gamma_1} \le L_1x(T_3) < 0, \quad t \in [T_3, \infty).$$

It follows that

$$x^{\Delta}(t) \le (L_1x(T_3))^{\frac{1}{\gamma_1}} r_1(t), \quad t \in [T_3, \infty).$$

Integrating from T_3 to t, we have

$$x(t) \le x(T_3) + (L_1x(T_3))^{\frac{1}{\gamma_1}} \int\limits_{T_3}^{t} r_1(s)\Delta s.$$

Letting $t \to \infty$, we have $x(t) \to -\infty$, which is a contradiction with the fact that $x(t) > 0$. Hence $L_2x(t) > 0, \ \ t \in [T_1, \infty)$. This implies that $L_1x(t)$ is strictly increasing on $[T_1, \infty)$. It follows that either $L_1x(t) > 0$ or $L_1x(t) < 0$.

Lemma 2.1 is proved.

Lemma 2.2. *Assume* $(H_1)-(H_3)$ *and* $\int_{T_0}^{\infty} q(s)\Delta s = \infty$ *hold. If* $x(t)$ *is a solution of* (1.1) *that satisfies Case* (ii) *in Lemma* 2.1, *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ be solution of (1.1) satisfying case (ii) in Lemma 2.1. Then from $L_1x(t) < 0$, we get

$$\left(\frac{1}{r_1(t)}x^{\Delta}(t)\right)^{\gamma_1} < 0, \quad t \ge T_1.$$

So, $x^{\Delta}(t) < 0$ for $t \ge T_1$ and $\lim_{t\to\infty} x(t) = b \ge 0$. We claim that $b = 0$. Assume not, that is, let be $x(t) \ge b > 0$, $t \ge T_1$. With $k = b$, from (H_3) and $x^{\sigma}(t) > x(t)$,

$$L_3x(t) = -f(t, x^{\sigma}(t)) \le -q(t)x^{\sigma}(t) \le -q(t)x(t) < -bq(t), \quad t \ge T_1.$$

Letting $y(t) := L_2x(t) > 0$, $t \ge T_1$, then

$$y^{\Delta}(t) = L_3x(t) < -q(t)b, \quad t \ge T_1.$$

Integrating the last inequality from T_1 to t, we have

$$y(t) \le y(T_1) - b\int\limits_{T_1}^{t} q(s)\Delta s.$$

Letting $t \to \infty$, we have $y(t) \to -\infty$, which is a contradiction. Therefore, $b = 0$, that is, $\lim_{t\to\infty} x(t) = 0$.

Lemma 2.2 is proved.

Lemma 2.3. *Suppose that* $(H_1)-(H_3)$ *hold. If* $x(t)$ *is a solution of* (1.1) *satisfying Case* (i) *of Lemma* 2.1, *then there exists* $T_1 \in [T_0, \infty)$ *such that*

$$L_1x(t) \ge R(t, T_1)(L_2x(t))^{\frac{1}{\gamma_2}}$$

or

$$x^{\Delta}(t) \geq r_1(t)(R(t,T_1))^{\frac{1}{\gamma_1}}(L_2x(t))^{\frac{1}{\gamma_1\gamma_2}},$$

and $\dfrac{L_1x(t)}{R(t,T_1)}$ *is decreasing on* (T_1,∞), *where* $R(t,T_1)=\int_{T_1}^{t} r_2(s)\Delta s.$

Proof. Let $x(t)$ be a solution of (1.1) satisfying case (i) of Lemma 2.1. Then from (1.1) we have $L_3x(t)<0$ for $t\in[T_1,\infty)$, so $L_2x(t)$ strictly decreasing on $[T_1,\infty)$. From $L_2x(t) = \left(\dfrac{1}{r_2(t)}\left(\left(\dfrac{1}{r_1(t)}x^{\Delta}(t)\right)^{\gamma_1}\right)^{\Delta}\right)^{\gamma_2}$, we obtain

$$(L_1x(t))^{\Delta} = r_2(t)(L_2x(t))^{\frac{1}{\gamma_2}}.$$

Then for $t\geq T_1$, we have

$$\int\limits_{T_1}^{t}(L_1x(s))^{\Delta}\Delta s = L_1x(t)-L_1x(T_1) =$$

$$=\int\limits_{T_1}^{t} r_2(s)(L_2x(s))^{\frac{1}{\gamma_2}}\Delta s \geq (L_2x(t))^{\frac{1}{\gamma_2}}\int\limits_{T_1}^{t} r_2(s)\Delta s.$$

It follows that

$$\left(\frac{1}{r_1(t)}x^{\Delta}(t)\right)^{\gamma_1} = L_1x(t)\geq L_1x(T_1)+R(t,T_1)(L_2x(t))^{\frac{1}{\gamma_2}}\geq$$

$$\geq R(t,T_1)(L_2x(t))^{\frac{1}{\gamma_2}}, \quad t\geq T_1, \tag{2.2}$$

so, we get

$$x^{\Delta}(t)\geq r_1(t)(R(t,T_1))^{\frac{1}{\gamma_1}}((L_2x(t))^{\frac{1}{\gamma_1\gamma_2}}, \quad t\geq T_1.$$

We claim that $\dfrac{L_1x(t)}{R(t,T_1)}$ is decreasing on (T_1,∞). For $t>T_1$, from (2.2) we get

$$\left(\frac{L_1x(t)}{R(t,T_1)}\right)^{\Delta} = \frac{(L_1x(t))^{\Delta}R(t,T_1)-L_1x(t)(R(t,T_1))^{\Delta}}{R(t,T_1)R(\sigma(t),T_1)} =$$

$$=\frac{r_2(t)(L_2x(t))^{\frac{1}{\gamma_2}}R(t,T_1)-L_1x(t)r_2(t)}{R(t,T_1)R(\sigma(t),T_1)}\leq\frac{r_2(t)(L_1x(t)-L_1x(t))}{R(t,T_1)R(\sigma(t),T_1)}=0.$$

Hence, $\dfrac{L_1x(t)}{R(t,T_1)}$ is decreasing on (T_1,∞).

Lemma 2.3 is proved.

Theorem 2.1. *Suppose that* (H_1)–(H_3) *and* $\gamma_1\gamma_2=1$ *hold, and assume that there exists a positive function* $z(t)$ *such that* z^{Δ} *is* rd*-continuous on* $[T_0,\infty)$, *we have*

$$\limsup_{t\to\infty} \int\limits_{T_0}^{t} \left[z(s)q(s) - \frac{(z^{\Delta}(s))^2}{4B(s,T_1)} \right] \Delta s = \infty, \quad t > T_1 \geq T_0, \tag{2.3}$$

where $B(t,T_1) = z(t)r_1(t)(R(t,T_1))^{\frac{1}{\gamma_1}}$, $R(t,T_1) = \int_{T_1}^{t} r_2(s)\Delta s$. *Then every solution* $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Assume that $x(t)$ is eventually positive (the case when $x(t) < 0$ is similar). By Lemma 2.1 we see that $x(t)$ satisfies either case (i) or (ii). We claim that case (i) of Lemma 2.1 is not true. Assume not, there exists $T_1 \in [T_0,\infty)$, such that $x(t) > 0,\ L_1x(t) > 0,\ L_2x(t) > 0$ for $t > T_1$. Consider the Riccati substitution

$$w(t) = z(t)\frac{L_2x(t)}{x(t)} > 0, \quad t \in [T_1,\infty). \tag{2.4}$$

From (1.1) we obtain

$$w^{\Delta}(t) = \left[\frac{z(t)}{x(t)}\right]^{\Delta} (L_2x(t))^{\sigma} + \frac{z(t)}{x(t)}(L_2x(t))^{\Delta} =$$

$$= \frac{z^{\Delta}(t)x(t) - z(t)x^{\Delta}(t)}{x(t)x^{\sigma}(t)}(L_2x(t))^{\sigma} - \frac{z(t)}{x(t)}f(t,x^{\sigma}(t)).$$

Since, $x^{\Delta}(t) > 0$, $x^{\sigma}(t) > x(t)$, and $L_2x(t) \geq (L_2x(t))^{\sigma}$, $t \geq T_1$ and Lemma 2.3, we get

$$w^{\Delta}(t) \leq z^{\Delta}(t)\frac{(L_2x(t))^{\sigma}}{x^{\sigma}(t)} - z(t)\frac{x^{\Delta}(t)}{(x^{\sigma}(t))^2}(L_2x(t))^{\sigma} - z(t)\frac{f(t,x^{\sigma}(t))}{x^{\sigma}(t)} \leq$$

$$\leq -z(t)q(t) + z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - z(t)\frac{r_1(t)(R(t,T_1))^{\frac{1}{\gamma_1}}L_2x(t)(L_2x(t))^{\sigma}}{(x^{\sigma}(t))^2} =$$

$$= -z(t)q(t) + z^{\Delta}(t)\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - B(t,T_1)\frac{(w^{\sigma}(t))^2}{(z^{\sigma}(t))^2} =$$

$$= -z(t)q(t) + \frac{(z^{\Delta}(t))^2}{4B(t,T_1)} - \left[\sqrt{B(t,T_1)}\frac{w^{\sigma}(t)}{z^{\sigma}(t)} - \frac{z^{\Delta}(t)}{2\sqrt{B(t,T_1)}}\right]^2 \leq$$

$$\leq -z(t)q(t) + \frac{(z^{\Delta}(t))^2}{4B(t,T_1)},$$

hence we have

$$w^{\Delta}(t) \leq -\left[z(t)q(t) - \frac{(z^{\Delta}(t))^2}{4B(t,T_1)}\right]. \tag{2.5}$$

Integrating (2.5) from T_2 to t, we find that

$$w(t) - w(T_2) \leq -\int\limits_{T_2}^{t} \left[z(s)q(s) - \frac{(z^{\Delta}(s))^2}{4B(s,.)}\right] \Delta s \geq -w(T_2),$$

that is

$$\int_{T_2}^{t} \left[z(s)q(s) - \frac{(z^{\Delta}(s))^2}{4B(s,.)} \right] \Delta s \le w(T_2),$$

which is contradiction with (2.3). Hence, case (i) of Lemma 2.1 is not true. If case (ii) of Lemma 2.1 holds, then clearly $\lim_{t\to\infty} x(t)$ exists.

Theorem 2.1 is proved.

***Corollary* 2.1.** *Suppose that* (H_1)–(H_3) *and* $\gamma_1\gamma_2 = 1$ *hold. If*

$$\int_{T_0}^{\infty} q(s)\Delta s = \infty, \tag{2.6}$$

then every solution $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof. If we take $z(t) = 1$ in Theorem 2.1, by the proof of Theorem 2.1 we have that every solution $x(t)$ of (1.1) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists. For the last case, by Lemma 2.2 we obtain $\lim_{t\to\infty} x(t) = 0$.

***Corollary* 2.2.** *Suppose that* (H_1)–(H_3) *and* $\gamma_1\gamma_2 = 1$ *hold. If*

$$\limsup_{t\to\infty} \int_{T_0}^{t} \left[sq(s) - \frac{1}{4sr_1(s)(R(s,T_1))^{\frac{1}{\gamma_1}}} \right] \Delta s = \infty, \tag{2.7}$$

then every solution $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Proof. If we take $z(t) = t$ in Theorem 2.1, by the proof of Theorem 2.1 we have that every solution $x(t)$ of (1.1) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists.

Corollary 2.2 is proved.

***Example* 2.1.** Consider the equation

$$\left[\left(\left(\left(\frac{1}{t} x^{\Delta}(t) \right)^{\frac{1}{3}} \right)^{\Delta} \right)^{3} \right]^{\Delta} + \frac{1}{t}|x^{\sigma}(t)| = 0, \tag{2.8}$$

where $t \in \mathbb{T} = q_0^{\mathbb{N}}$, $q_0 > 1$, $r_2(t) = 1$, $r_1(t) = 1/t$ and $f(t, x^{\sigma}(t)) = 1/t|x^{\sigma}(t)|$, $q(t) \ge \dfrac{1}{t}$, $\gamma_1 = 1/3$, $\gamma_2 = 3$. For sufficient large T_1,

$$R(t,T_1) = \int_{T_1}^{t} \Delta s = t - T_1$$

and for $T_2 > T_1$,

$$\limsup_{t\to\infty} \int_{T_2}^{t} \left[sq(s) - \frac{1}{4sr_1(s)(R(s,T_1))^{\frac{1}{\gamma_1}}} \right] \Delta s = \limsup_{t\to\infty} \int_{T_2}^{t} \left[s\frac{1}{s} - \frac{1}{4s\frac{1}{s}(s-T_1)^3} \right] \Delta s = \infty.$$

We get that all conditions of Corollary 2.2 are satisfied and then every solution $x(t)$ of (2.8) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists.

***Corollary* 2.3.** *Suppose that* $(H_1)-(H_3)$ *and* $\gamma_1\gamma_2 = 1$ *hold. If there is* $\alpha > 1$ *such that*

$$\limsup_{t\to\infty} \int_{T_0}^{t} \left[s^\alpha q(s) - \frac{((s^\alpha)^\Delta)^2}{4s^\alpha r_1(s)(R(s,T_1))^{\frac{1}{\gamma_1}}} \right] \Delta s = \infty, \tag{2.9}$$

then every solution $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Proof. We take $z(t) = t^\alpha,\ \alpha > 1$ in Theorem 2.1, by the proof of Theorem 2.1 we have that every solution $x(t)$ of (1.1) is either oscillatory or $\lim_{t\to\infty} x(t)$ exists.

Corollary 2.3 is proved.

Theorem 2.2. *Assume* $(H_1)-(H_3)$ *and* $\gamma_1\gamma_2 = 1$ *hold. If there exist* $m \geq 1$ *and a positive function* $z(t)$ *such that* z^Δ *is* rd*-continuous on* $[T_0,\infty)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{T_0}^{t} (t-s)^m \left[z(s)q(s) - \frac{(z^\Delta(s))^2}{4B(t,T_1)} \right] \Delta s = \infty, \tag{2.10}$$

where $B(t,T_1) = z(t)r_1(t)(R(t,T_1))^{\frac{1}{\gamma_1}}$, $R(t,T_1) = \displaystyle\int_{T_1}^{t} r_2(s)\Delta s$. *Then every solution* $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Proof. Proceeding as in Theorem 2.1, we suppose that (1.1) has a nonoscillatory solution. Let be $x(t) > 0,\ t \geq T_1$. Multiplying (2.5) by $(t-s)^m$ (with *t* replaced by *s*) and then integrating from T_2 to *t* ($t \geq T_2 > T_1$), we have

$$\int_{T_2}^{t} (t-s)^m w^\Delta(s)\Delta s \leq -\int_{T_2}^{t} (t-s)^m \left[z(s)q(s) - \frac{(z^\Delta(s))^2}{4B(t,T_1)} \right] \Delta s.$$

An integrating by parts of left-hand side leads to

$$\int_{T_2}^{t} (t-s)^m w^\Delta(s)\Delta s = (t-s)^m w(s)\Big|_{T_2}^{t} - \int_{T_2}^{t} ((t-s)^m)^{\Delta_s} w(\sigma(s))\Delta s.$$

Let be $h(t,s) := ((t-s)^m)^{\Delta_s}$. Since

$$h(t,s) = \begin{cases} -m(t-s)^{m-1}, & \mu(s) = 0, \\ \dfrac{(t-\sigma(s))^m - (t-s)^m}{\mu(s)}, & \mu(s) > 0, \end{cases}$$

and when $m \geq 1$ for $t \geq \sigma(s)$, it follows that

$$\int_{T_2}^{t} (t-s)^m w^\Delta(s) \geq -(t-T_2)^m w(T_2)$$

or

$$\frac{1}{t^m}\int\limits_{T_2}^{t}(t-s)^m\left[z(s)q(s)-\frac{(z^{\Delta}(s))^2}{4B(t,T_1)}\right]\Delta s\le\left(\frac{t-T_2}{t}\right)^m w(T_2)\le w(T_2),$$

a contradiction with (2.10). Thus, case (i) in Lemma 2.1 is not true. If case (ii) in Lemma 2.1 holds, then as before, $\lim_{t\to\infty}x(t)$ exists.

Theorem 2.2 is proved.

***Corollary* 2.4.** *Suppose that* (H_1)–(H_2) *and* $\gamma_1\gamma_2=1$ *hold. If there exist* $m\ge 1$,

$$\limsup_{t\to\infty}\frac{1}{t^m}\int\limits_{T_0}^{t}(t-s)^m q(s)\Delta s=\infty,$$

then every solution $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty}x(t)=0$.

Proof. If we take $z(t)=1$ in Theorem 2.2, by the proof of Theorem 2.2 we have that every solution $x(t)$ of (1.1) is either oscillatory or $\lim_{t\to\infty}x(t)$ exists. For the last case, by Lemma 2.2 we obtain $\lim_{t\to\infty}x(t)=0$.

Corollary 2.4 is proved.

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