UDC 515.126, 515.127

T. V. Rybalkina, V. V. Sergeichuk* (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv)

TOPOLOGICAL CLASSIFICATION OF ORIENTED CYCLES OF LINEAR MAPPINGS

ТОПОЛОГІЧНА КЛАСИФІКАЦІЯ ОРІЄНТОВАНИХ ЦИКЛІВ ЛІНІЙНИХ ВІДОБРАЖЕНЬ

We consider oriented cycles of linear mappings over the fields of real and complex numbers. The problem of their classification to within the homeomorphisms of spaces is reduced to the problem of classification of linear operators to within the homeomorphisms of spaces studied by N. Kuiper and J. Robbin in 1973.

Розглядаються орієнтовані цикли лінійних відображень над полями дійсних та комплексних чисел. Задача їхньої класифікації з точністю до гомеоморфізмів просторів зводиться до задачі класифікації лінійних операторів з точністю до гомеоморфізмів просторів, яку вивчали Н. Койпер та Дж. Роббін у 1973 році.

1. Introduction. We consider the problem of topological classification of oriented cycles of linear mappings.

Let

$$\mathcal{A}\colon\qquad \begin{array}{c} V_1 \xrightarrow{A_1} V_2 \xrightarrow{A_2} \ \dots\ \xrightarrow{A_{t-2}} V_{t-1} \xrightarrow{A_{t-1}} V_t \\ \xleftarrow[\qquad\qquad\qquad A_t \qquad\qquad\qquad]{} \end{array} \tag{1}$$

and

$$\mathcal{B}\colon\qquad \begin{array}{c} W_1 \xrightarrow{B_1} W_2 \xrightarrow{B_2} \ \dots\ \xrightarrow{B_{t-2}} W_{t-1} \xrightarrow{B_{t-1}} W_t \\ \xleftarrow[\qquad\qquad\qquad B_t \qquad\qquad\qquad]{} \end{array} \tag{2}$$

be two oriented cycles of linear mappings of the same *length* t over a field $\mathbb{F}$. We say that a system $\varphi = \{\varphi_i\colon V_i \to W_i\}_{i=1}^{t}$ of bijections *transforms* $\mathcal{A}$ *to* $\mathcal{B}$ if all squares in the diagram

$$\begin{array}{ccccccccc} V_1 & \xrightarrow{A_1} & V_2 & \xrightarrow{A_2} & \dots & \xrightarrow{A_{t-2}} & V_{t-1} & \xrightarrow{A_{t-1}} & V_t \\ & & & & \xleftarrow[\qquad A_t \qquad]{} & & & & \\ \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & & & \downarrow{\scriptstyle\varphi_{t-1}} & & \downarrow{\scriptstyle\varphi_t} \\ W_1 & \xrightarrow{B_1} & W_2 & \xrightarrow{B_2} & \dots & \xrightarrow{B_{t-2}} & W_{t-1} & \xrightarrow{B_{t-1}} & W_t \\ & & & & \xleftarrow[\qquad B_t \qquad]{} & & & & \end{array} \tag{3}$$

are commutative; that is,

$$\varphi_2 A_1 = B_1\varphi_1, \quad \dots, \quad \varphi_t A_{t-1} = B_{t-1}\varphi_{t-1}, \quad \varphi_1 A_t = B_t\varphi_t. \tag{4}$$

* V. V. Sergeichuk was supported in part by the Foundation for Research Support of the State of São Paulo (FAPESP), grant 2012/18139-2.

Definition 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be cycles of linear mappings of the form* (1) *and* (2) *over a field* $\mathbb{F}$.

(i) *$\mathcal{A}$ and $\mathcal{B}$ are isomorphic if there exists a system of linear bijections that transforms $\mathcal{A}$ to $\mathcal{B}$.*

(ii) *$\mathcal{A}$ and $\mathcal{B}$ are topologically equivalent if* $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{R}$,

$$V_i = \mathbb{F}^{m_i}, \qquad W_i = \mathbb{F}^{n_i} \qquad \text{for all} \quad i = 1, \ldots, t,$$

and there exists a system of homeomorphisms[1] *that transforms $\mathcal{A}$ to $\mathcal{B}$.*

The *direct sum* of cycles (1) and (2) is the cycle

$$\mathcal{A} \oplus \mathcal{B}\colon \quad V_1 \oplus W_1 \xrightarrow{A_1 \oplus B_1} V_2 \oplus W_2 \xrightarrow{A_2 \oplus B_2} \ \ldots \ \xrightarrow{A_{t-1} \oplus B_{t-1}} V_t \oplus W_t \quad (V_t \oplus W_t \xrightarrow{A_t \oplus B_t} V_1 \oplus W_1).$$

The vector $\dim \mathcal{A} := (\dim V_1, \ldots, \dim V_t)$ is the *dimension* of $\mathcal{A}$. A cycle $\mathcal{A}$ is *indecomposable* if its dimension is nonzero and $\mathcal{A}$ cannot be decomposed into a direct sum of cycles of smaller dimensions.

A cycle $\mathcal{A}$ is *regular* if all $A_1, \ldots, A_t$ are bijections, and *singular* otherwise. Each cycle $\mathcal{A}$ possesses a *regularizing decomposition*

$$\mathcal{A} = \mathcal{A}_{\text{reg}} \oplus \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_r, \tag{5}$$

in which $\mathcal{A}_{\text{reg}}$ is regular and all $\mathcal{A}_1, \ldots, \mathcal{A}_r$ are indecomposable singular. An algorithm that constructs a regularizing decomposition of a nonoriented cycle of linear mappings over $\mathbb{C}$ and uses only unitary transformations was given in [3].

The following theorem reduces the problem of topological classification of oriented cycles of linear mappings to the problem of topological classification of linear operators.

Theorem 1. (a) *Let* $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{R}$, *and let*

$$\mathcal{A}\colon \quad \mathbb{F}^{m_1} \xrightarrow{A_1} \mathbb{F}^{m_2} \xrightarrow{A_2} \ \ldots \ \xrightarrow{A_{t-2}} \mathbb{F}^{m_{t-1}} \xrightarrow{A_{t-1}} \mathbb{F}^{m_t} \quad (\mathbb{F}^{m_t} \xrightarrow{A_t} \mathbb{F}^{m_1}) \tag{6}$$

and

$$\mathcal{B}\colon \quad \mathbb{F}^{n_1} \xrightarrow{B_1} \mathbb{F}^{n_2} \xrightarrow{B_2} \ \ldots \ \xrightarrow{B_{t-2}} \mathbb{F}^{n_{t-1}} \xrightarrow{B_{t-1}} \mathbb{F}^{n_t} \quad (\mathbb{F}^{n_t} \xrightarrow{B_t} \mathbb{F}^{n_1}) \tag{7}$$

be topologically equivalent. Let

$$\mathcal{A} = \mathcal{A}_{\text{reg}} \oplus \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_r, \qquad \mathcal{B} = \mathcal{B}_{\text{reg}} \oplus \mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_s \tag{8}$$

be their regularizing decompositions. Then their regular parts $\mathcal{A}_{\text{reg}}$ and $\mathcal{B}_{\text{reg}}$ are topologically equivalent, $r = s$, and after a suitable renumbering their indecomposable singular summands $\mathcal{A}_i$ and $\mathcal{B}_i$ are isomorphic for all $i = 1, \ldots, r$.

[1] By [1] (Corollary 19.10) or [2] (Section 11) $m_1 = n_1, \ldots, m_t = n_t$.

(b) *Each regular cycle* $\mathcal{A}$ *of the form* (6) *is isomorphic to the cycle*

$$\mathcal{A}': \quad \mathbb{F}^{m_1} \xrightarrow{\mathbb{1}} \mathbb{F}^{m_2} \xrightarrow{\mathbb{1}} \ldots \xrightarrow{\mathbb{1}} \mathbb{F}^{m_{t-1}} \xrightarrow{\mathbb{1}} \mathbb{F}^{m_t} \quad (\mathbb{F}^{m_t} \xrightarrow{A_t \ldots A_2 A_1} \mathbb{F}^{m_1}). \tag{9}$$

If cycles (6) *and* (7) *are regular, then they are topologically equivalent if and only if the linear operators* $A_t \ldots A_2 A_1$ *and* $B_t \ldots B_2 B_1$ *are topologically equivalent* (*as the cycles* $\mathbb{F}^{m_1} \circlearrowleft A_t \ldots A_2 A_1$ *and* $\mathbb{F}^{n_1} \circlearrowleft B_t \ldots B_2 B_1$ *of length* 1).

Kuiper and Robbin [4, 5] gave a criterion for topological equivalence of linear operators over $\mathbb{R}$ without eigenvalues that are roots of 1. Budnitska [6] (Theorem 2.2) found a canonical form with respect to topological equivalence of linear operators over $\mathbb{R}$ and $\mathbb{C}$ without eigenvalues that are roots of 1. The problem of topological classification of linear operators with an eigenvalue that is a root of 1 was studied by Kuiper and Robbin [4, 5], Cappell and Shaneson [7 – 11], and Hsiang and Pardon [12]. The problem of topological classification of affine operators was studied in [6, 13 – 16]. The topological classifications of pairs of counter mappings $V_1 \rightleftarrows V_2$ (i.e., oriented cycles of length 2) and of chains of linear mappings were given in [17] and [18].

2. Oriented cycles of linear mappings up to isomorphism. This section is not topological; we construct a regularizing decomposition of an oriented cycle of linear mappings over an arbitrary field $\mathbb{F}$.

A classification of cycles of length 1 (i.e., linear operators $V \circlearrowleft$) over any field is given by the Frobenius canonical form of a square matrix under similarity. The oriented cycles of length 2 (i.e., pairs of counter mappings $V_1 \rightleftarrows V_2$) are classified in [19, 20]. The classification of cycles of arbitrary length and with arbitrary orientation of its arrows is well known in the theory of representations of quivers; see [21] (Section 11.1).

For each $c \in \mathbb{Z}$, we denote by $[c]$ the natural number such that

$$1 \leqslant [c] \leqslant t, \qquad [c] \equiv c \pmod{t}.$$

By the Jordan theorem, for each indecomposable singular cycle $V \circlearrowleft A$ there exists a basis $e_1, \ldots, e_n$ of V in which the matrix of A is a singular Jordan block. This means that the basis vectors form a *Jordan chain*

$$e_1 \xrightarrow{A} e_2 \xrightarrow{A} e_3 \xrightarrow{A} \ldots \xrightarrow{A} e_n \xrightarrow{A} 0.$$

In the same manner, each indecomposable singular cycle $\mathcal{A}$ of an arbitrary length t also can be given by a chain

$$e_p \xrightarrow{A_p} e_{p+1} \xrightarrow{A_{[p+1]}} e_{p+2} \xrightarrow{A_{[p+2]}} \ldots \xrightarrow{A_{[q-1]}} e_q \xrightarrow{A_{[q]}} 0$$

in which $1 \leqslant p \leqslant q \leqslant t$ and for each $l = 1, 2, \ldots, t$ the set $\{e_i | i \equiv l \pmod{t}\}$ is a basis of V_l; see [21] (Section 11.1). We say that this chain *ends in* $V_{[q]}$ since $e_q \in V_{[q]}$. The number $q - p$ is called the *length* of the chain.

For example, the chain

$$\begin{array}{ccccccccc} & & & & & & e_4 & \to & e_5 \\ e_6 & \to & e_7 & \to & e_8 & \to & e_9 & \to & e_{10} \\ e_{11} & \to & e_{12} & \to & 0 & & & & \end{array}$$

of length 8 gives an indecomposable singular cycle on the spaces $V_1 = \mathbb{F}e_6 \oplus \mathbb{F}e_{11}$, $V_2 = \mathbb{F}e_7 \oplus \mathbb{F}e_{12}$, $V_3 = \mathbb{F}e_8$, $V_4 = \mathbb{F}e_4 \oplus \mathbb{F}e_9$, $V_5 = \mathbb{F}e_5 \oplus \mathbb{F}e_{10}$.

Lemma 1. *Let*

$$\mathcal{A}\colon \qquad V_1 \xrightarrow{A_1} V_2 \xrightarrow{A_2} \ \dots \ \xrightarrow{A_{t-2}} V_{t-1} \xrightarrow{A_{t-1}} V_t \qquad (A_t\colon V_t \to V_1)$$

be an oriented cycle of linear mappings, and let (5) *be its regularizing decomposition.*

(a) *Write*

$$\hat{A}_i := A_{[i+t-1]} \dots A_{[i+1]} A_i \colon V_i \to V_i \tag{10}$$

and fix a natural number z such that

$$\tilde{V}_i := \hat{A}_i^z V_i = \hat{A}_i^{z+1} V_i \qquad \textit{for all} \quad i = 1, \dots, t.$$

Let

$$\tilde{\mathcal{A}}\colon \qquad \tilde{V}_1 \xrightarrow{\tilde{A}_1} \tilde{V}_2 \xrightarrow{\tilde{A}_2} \ \dots \ \xrightarrow{\tilde{A}_{t-2}} \tilde{V}_{t-1} \xrightarrow{\tilde{A}_{t-1}} \tilde{V}_t \qquad (\tilde{A}_t\colon \tilde{V}_t \to \tilde{V}_1)$$

be the cycle formed by the restrictions $\tilde{A}_i \colon \tilde{V}_i \to \tilde{V}_{[i+1]}$ of $A_i \colon V_i \to V_{[i+1]}$. Then $\mathcal{A}_{\text{reg}} = \tilde{\mathcal{A}}$ (and so the regular part is uniquely determined by $\mathcal{A}$).

(b) *The numbers*

$$k_{ij} := \dim \operatorname{Ker}(A_{[i+j]} \dots A_{[i+1]} A_i), \quad i = 1, \dots, t \quad \textit{and} \quad j \geqslant 0,$$

determine the singular summands $\mathcal{A}_1, \dots, \mathcal{A}_r$ of regularizing decomposition (5) *up to isomorphism since the number n_{lj} ($l = 1, \dots, t$ and $j \geqslant 0$) of singular summands given by chains of length j that end in V_l can be calculated by the formula*

$$n_{lj} = k_{[l-j],j} - k_{[l-j],j-1} - k_{[l-j-1],j+1} + k_{[l-j-1],j} \tag{11}$$

in which $k_{i,-1} := 0$.

Proof. (a) Let (5) be a regularizing decomposition of $\mathcal{A}$. Let

$$V_i = V_{i,\text{reg}} \oplus V_{i1} \oplus \dots \oplus V_{ir}, \qquad i = 1, \dots, t,$$

be the corresponding decompositions of its vector spaces. Then $\hat{A}_i^z V_{i,\text{reg}} = V_{i,\text{reg}}$ (since all linear mappings in $\mathcal{A}_{\text{reg}}$ are bijections) and $\hat{A}_i^z V_{i1} = \dots = \hat{A}_i^z V_{ir} = 0$. Hence $V_{i,\text{reg}} = \tilde{V}_i$, and so $\mathcal{A}_{\text{reg}} = \tilde{\mathcal{A}}$.

(b) Denote by

$$\sigma_{ij} := n_{ij} + n_{i,j+1} + n_{i,j+2} + \dots$$

the number of chains of length $\geqslant j$ that end in V_i. Clearly, $k_{i0} = \sigma_{i0}$, $k_{i1} = \sigma_{i0} + \sigma_{[i+1],1}$, ..., and

$$k_{ij} = \sigma_{i0} + \sigma_{[i+1],1} + \dots + \sigma_{[i+j],j}$$

for each $1 \leqslant i \leqslant t$ and $j \geqslant 0$. Therefore,

$$\sigma_{lj} = k_{ij} - k_{i,j-1}, \qquad l := [i+j]$$

(recall that $k_{i,-1} = 0$). This means that $l \equiv i + j \pmod{t}$, $i \equiv l - j \pmod{t}$, $i = [l-j]$, and so

$$\sigma_{lj} = k_{[l-j],j} - k_{[l-j],j-1}.$$

We get

$$n_{lj} = \sigma_{lj} - \sigma_{l,j+1} = k_{[l-j],j} - k_{[l-j],j-1} - k_{[l-j-1],j+1} + k_{[l-j-1],j}.$$

Lemma 1 is proved.

3. Proof of Theorem 1. In this section, $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

(a) Let $\mathcal{A}$ and $\mathcal{B}$ be cycles (6) and (7). Let them be topologically equivalent; that is, $\mathcal{A}$ is transformed to B by a system $\{\varphi_i : \mathbb{F}^{m_i} \to \mathbb{F}^{n_i}\}_{i=1}^{t}$ of homeomorphisms. Let (8) be regularizing decompositions of $\mathcal{A}$ and $\mathcal{B}$.

First we prove that their regular parts $\mathcal{A}_{\text{reg}}$ and $\mathcal{B}_{\text{reg}}$ are topologically equivalent. In notation (10),

$$\hat{A}_i = A_{[i+t-1]} \ldots A_{[i+1]} A_i, \qquad \hat{B}_i = B_{[i+t-1]} \ldots B_{[i+1]} B_i.$$

Let z be a natural number that satisfies both $\hat{A}_i^z \mathbb{F}^{m_i} = \hat{A}_i^{z+1} \mathbb{F}^{m_i}$ and $\hat{B}_i^z \mathbb{F}^{n_i} = \hat{B}_i^{z+1} \mathbb{F}^{n_i}$ for all $i = 1, \ldots, t$. By (3), the diagram

$$\begin{CD} \mathbb{F}^{m_i} @>{\hat{A}^z}>> \mathbb{F}^{m_i} \\ @V{\varphi_i}VV @VV{\varphi_i}V \\ \mathbb{F}^{n_i} @>{\hat{B}^z}>> \mathbb{F}^{n_i} \end{CD}$$

is commutative. Then $\varphi_i \operatorname{Im} \hat{A}_i^z = \operatorname{Im} \hat{B}_i^z$ for all i. Therefore, the restriction $\hat{\varphi}_i : \operatorname{Im} \hat{A}_i^z \to \operatorname{Im} \hat{B}_i^z$ is a homeomorphism. The system of homeomorphisms $\hat{\varphi}_1, \ldots, \hat{\varphi}_t$ transforms $\tilde{\mathcal{A}}$ to $\tilde{\mathcal{B}}$, which are the regular parts of $\mathcal{A}$ and $\mathcal{B}$ by Lemma 1(a).

Let us prove that $r = s$, and, after a suitable renumbering, $\mathcal{A}_i$ and $\mathcal{B}_i$ are isomorphic for all $i = 1, \ldots, r$. Since all summands $\mathcal{A}_i$ and $\mathcal{B}_i$ with $i \geqslant 1$ can be given by chains of basic vectors, it suffices to prove that $n_{ij} = n'_{ij}$ for all i and j, where n'_{ij} is the number of singular summands $\mathcal{B}_1, \ldots, \mathcal{B}_s$ in (8) given by chains of length j that end in the i th space $\mathbb{F}^{n_i}$.

Due to (11), it suffices to prove that the numbers k_{ij} are invariant with respect to topological equivalence.

In the same manner as k_{ij} is constructed by $\mathcal{A}$, we construct k'_{ij} by $\mathcal{B}$. Let us fix i and j and prove that $k_{ij} = k'_{ij}$. Write

$$A := A_{[i+j]} \ldots A_{[i+1]} A_i, \qquad B := B_{[i+j]} \ldots B_{[i+1]} B_i, \quad q := [i + j + 1]$$

and consider the commutative diagram

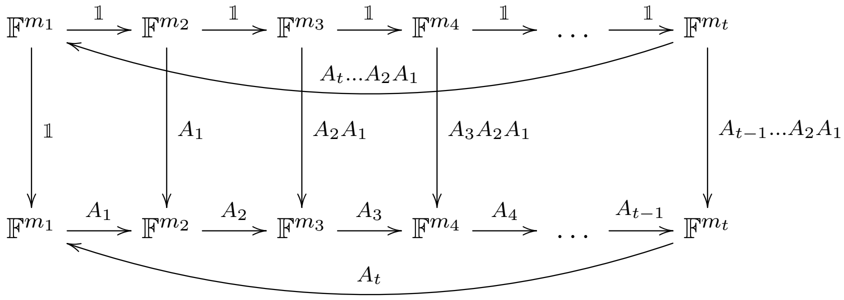
$$\begin{CD} \mathbb{F}^{m_i} @>{A}>> \mathbb{F}^{m_q} \\ @V{\varphi_i}VV @VV{\varphi_q}V \\ \mathbb{F}^{n_i} @>{B}>> \mathbb{F}^{n_q} \end{CD} \tag{12}$$

which is a fragment of (3). We have

$$k_{ij} = \dim \operatorname{Ker} A = m_i - \dim \operatorname{Im} A, \qquad k'_{ij} = n_i - \dim \operatorname{Im} B.$$

Because $\varphi_i : \mathbb{F}^{m_i} \to \mathbb{F}^{n_i}$ is a homeomorphism, $m_i = n_i$ (see [1], Corollary 19.10, or [2], Section 11). Since the diagram (12) is commutative, $\varphi_q(\operatorname{Im} A) = \operatorname{Im} B$. Hence, the vector spaces $\operatorname{Im} A$ and $\operatorname{Im} B$ are homeomorphic, and so $\dim \operatorname{Im} A = \dim \operatorname{Im} B$, which proves $k_{ij} = k'_{ij}$.

(b) Each regular cycle $\mathcal{A}$ of the form (6) is isomorphic to the cycle $\mathcal{A}'$ of the form (9) since the diagram



(13)

is commutative.

Let $\mathcal{A}$ and $\mathcal{B}$ be regular cycles of the form (6) and (7). Let them be topologically equivalent; that is, $\mathcal{A}$ is transformed to B by a system $\varphi = (\varphi_1, \ldots, \varphi_t)$ of homeomorphisms; see (3). By (4),

$$\varphi_1 A_t A_{t-1} \ldots A_1 = B_t \varphi_t A_{t-1} \ldots A_1 = B_t B_{t-1} \varphi_{t-1} A_{t-2} \ldots A_1 = \ldots = B_t B_{t-1} \ldots B_1 \varphi_1,$$

and so the cycles $\mathbb{F}^{m_1} \circlearrowleft A_t \ldots A_2 A_1$ and $\mathbb{F}^{m_1} \circlearrowleft B_t \ldots B_2 B_1$ are topologically equivalent via φ_1.

Conversely, let $\mathbb{F}^{m_1} \circlearrowleft A_t \ldots A_2 A_1$ and $\mathbb{F}^{m_1} \circlearrowleft B_t \ldots B_2 B_1$ be topologically equivalent via some homeomorphism φ_1, and let $\mathcal{A}'$ and $\mathcal{B}'$ be constructed by $\mathcal{A}$ and $\mathcal{B}$ as in (9). Then $\mathcal{A}'$ and $\mathcal{B}'$ are topologically equivalent via the system of homeomorphisms $\varphi = (\varphi_1, \varphi_1, \ldots, \varphi_1)$. Let ε and δ be systems of linear bijections that transform $\mathcal{A}'$ to $\mathcal{A}$ and $\mathcal{B}'$ to $\mathcal{B}$; see (13). Then $\mathcal{A}$ and $\mathcal{B}$ are topologically equivalent via the system of homeomorphisms $\delta\varphi\varepsilon^{-1}$.

1. *Bredon G. E.* Topology and geometry. – New York: Springer-Verlag, 1993. – 587 p.
2. *McCleary J.* A first course in topology: continuity and dimension. – Providence: Amer. Math. Soc., 2006. – 141 p.
3. *Sergeichuk V. V.* Computation of canonical matrices for chains and cycles of linear mappings // Linear Algebra and Appl. – 2004. – **376**. – P. 235 – 263.
4. *Robbin J. W.* Topological conjugacy and structural stability for discrete dynamical systems // Bull. Amer. Math. Soc. – 1972. – **78**. – P. 923 – 952.
5. *Kuiper N. H., Robbin J. W.* Topological classification of linear endomorphisms // Invent. Math.– 1973. – **19**, № 2. – P. 83 – 106.
6. *Budnitska T.* Topological classification of affine operators on unitary and Euclidean spaces // Linear Algebra and Appl. – 2011. – **434**. – P. 582 – 592.
7. *Cappell S. E., Shaneson J. L.* Linear algebra and topology // Bull. Amer. Math. Soc. (N. S.). – 1979. – **1**. – P. 685 – 687.
8. *Cappell S. E., Shaneson J. L.* Nonlinear similarity of matrices // Bull. Amer. Math. Soc. (N. S.). – 1979. – **1**. – P. 899 – 902.
9. *Cappell S. E., Shaneson J. L.* Non-linear similarity // Ann. Math. – 1981. – **113**, № 2. – P. 315 – 355.
10. *Cappell S. E., Shaneson J. L.* Non-linear similarity and linear similarity are equivariant below dimension 6 // Contemp. Math. – 1999. – **231**. – P. 59 – 66.
11. *Cappell S. E., Shaneson J. L., Steinberger M., West J. E.* Nonlinear similarity begins in dimension six // Amer. J. Math. – 1989. – **111**. – P. 717 – 752.
12. *Hsiang W. C., Pardon W.* When are topologically equivalent orthogonal transformations linearly equivalent? // Invent. Math. – 1982. – **68**, № 2. – P. 275 – 316.
13. *Ephrämowitsch W.* Topologische Klassifikation affiner Abbildungen der Ebene // Mat. Sb. – 1935. – **42**, № 1. – P. 23 – 36.
14. *Blanc J.* Conjugacy classes of affine automorphisms of $\mathbb{K}^n$ and linear automorphisms of $\mathbb{P}^n$ in the Cremona groups // Manuscr. Math. – 2006. – **119**, № 2. – P. 225 – 241.
15. *Budnitska T. V.* Classification of topological conjugate affine mappings // Ukr. Math. J. – 2009. – **61**, № 2. – P. 164 – 170.
16. *Budnitska T., Budnitska N.* Classification of affine operators up to biregular conjugacy // Linear Algebra and Appl. – 2011. – **434**. – P. 1195 – 1199.

17. *Rybalkina T.* Topological classification of pairs of counter linear maps // Mat. Stud. – 2013. – **39**, № 1. – P. 21 – 28 (in Ukrainian).
18. *Rybalkina T., Sergeichuk V. V.* Topological classification of chains of linear mappings // Linear Algebra and Appl. – 2012. – **437**. – P. 860 – 869.
19. *Dobrovol'skaya N. M., Ponomarev V. A.* A pair of counter-operators // Uspekhi Mat. Nauk. – 1965. – **20**, № 6. – P. 80 – 86 (in Russian).
20. *Horn R. A., Merino D. I.* Contragredient equivalence: a canonical form and some applications // Linear Algebra and Appl. – 1995. – **214**. – P. 43 – 92.
21. *Gabriel P., Roiter A. V.* Representations of finite-dimensional algebras // Encycl. Math. Sci. – Berlin: Springer-Verlag, 1992. – **73**. – 177 p.

Received 16.07.13