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THE MIXED BOUNDARY-VALUE PROBLEM FOR LINEAR SECOND ORDER NONDIVERGENT PARABOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

МІШАНА КРАЙОВА ЗАДАЧА ДЛЯ ЛІНІЙНИХ БЕЗДИВЕРГЕНТНИХ ПАРАБОЛІЧНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ З РОЗРИВНИМИ КОЕФІЦІЄНТАМИ

The mixed boundary-value problem is considered for linear second order nondivergent parabolic equations with discontinuous coefficients satisfying the Cordes conditions. The one-valued strong (almost everywhere) solvability of this problem is proved in the space $\mathring{W}_p^{2,1}$, where p belongs to the same segment containing point 2.

Розглядається мішана крайова задача для лінійних бездивергентних параболічних рівнянь другого порядку з розривними коефіцієнтами, що задовольняють умови Корде. Однозначну сильну (майже скрізь) розв'язність цієї задачі доведено у просторі $\mathring{W}_p^{2,1}$, де p належить тому ж відрізку, що містить точку 2.

1. Introduction. Let E_n and R_{n+1} be n and $(n+1)$-dimensional Euclidean spaces of points $x=(x_1,x_2,\ldots,x_n)$ and $(t,x)=(t,x_1,x_2,\ldots,x_n)$, respectively, let $\Omega\subset E_n$ be a bounded domain with boundary $\partial\Omega\in C^2$, let $B_R^{x^0}$ be an n-dimensional open sphere of radius R with center at the point $x^0=(x_1^0,x_2^0,\ldots,x_n^0)$, $Q_R^{x^0}\times(0,T)\equiv Q_R^T$, $Q_T=\{(t,x)|0<t<T<\infty,\ x\in\Omega\}$, $S_T=\{(t,x)|0<t<T<\infty,\ x\in\partial\Omega\}$, and let $\mathcal{A}(Q_R^T)$ be the set of all functions $u(t,x)$ from $C^\infty(\bar{Q}_R^T)$ with support in $B_\rho^{x^0}\times[0,T]$, $\rho<R$, for which $u(0,x)=0$.

In the domain Q_T, we consider a mixed boundary-value problem for linear parabolic equations of the form

$$\mathcal{L}u=\sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i(t,x)\frac{\partial u}{\partial x_i}+c(t,x)u-\frac{\partial u}{\partial t}=f(t,x), \tag{1}$$

$$u|_{t=0}=0,\quad \left.\frac{\partial u}{\partial n}\right|_{S_T}=0, \tag{2}$$

under the assumptions that $\|a_{ij}(t,x)\|$ is a real symmetric matrix. Moreover, for all $(t,x)\in Q_T$ and $\xi\in E_n$, the conditions

$$\gamma|\xi|^2\le\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j\le\gamma^{-1}|\xi|^2,\quad \gamma\in(0,1]-\mathrm{const}, \tag{3}$$

are satisfied.

In addition, we suppose that all coefficients of the operator $\mathcal{L}$ are real and measurable functions in Q_T.

The aim of the present paper is to find the conditions on thecoefficients of equations (1) under which the mixed boundary-value problem (1), (2) is identically strongly (almost everywhere) solvable in the space $\hat{W}_p^{2,1}$ for any $f(t,x) \in L_p(Q_T)$, $p \in [p_1, p_2]$, where $p_1 \in (1,2)$, $p_2 \in (2,\infty)$.

In case where the leading coefficients of the linear operator are uniformly continuous in the cylindrical domain and the minor coefficients are the elements of the corresponding Lebesgue spaces, the uniform strong (almost everywhere) solvability of the Dirichlet and mixed problems for the parabolic and elliptic equations in the space Sobolev was proved in [1, 2]. An example indicating the exactness of the Cordes conditions is presented in [3]. In [4, 5] the indicated fact is taken to the class of nonlinear parabolic equations of the second order under the stronger condition than the Cordes condition. Note that the Dirichlet problem for linear and quasilinear second-order parabolic and elliptic equations with nondivergent structure and discontinuous coefficients was studied in [6 – 12].

1. Some auxiliary assertions. First, we present some necessary notation and definitions. We denote by u_t, u_i and u_{ij} the derivatives $\dfrac{\partial u}{\partial t}$, $\dfrac{\partial u}{\partial x_i}$, and $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$, $i,j = 1,\ldots,n$, respectively. Let $W_p^{1,0}(Q_T)$ and $W_p^{2,1}(Q_T)$ be Banach spaces of measurable functions $u(t,x)$ given on Q_T with bounded norms

$$\|u\|_{W_p^{1,0}(Q_T)} = \left(\int\limits_{Q_T} \left(|u|^p + \sum_{i=1}^n |u_i|^p \right) dtdx \right)^{1/p}$$

and

$$\|u\|_{W_p^{2,1}(Q_T)} = \left(\int\limits_{Q_T} \left(|u|^p + \sum_{i=1}^n |u_i|^p + \sum_{i,j=1}^n |u_{ij}|^p + |u_t|^p \right) dtdx \right)^{1/p},$$

respectively. By $\hat{W}_p^{2,1}(Q_T)$, we denote the subspace $W_p^{2,1}(Q_T)$ in which the dense set is the collection of all functions from $C^\infty(\overline{Q}_T)$ vanishing at $t = 0$ and $\left.\dfrac{\partial u}{\partial n}\right|_{S_T} = 0$.

Definition. *A function $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ is called a strong solution of the mixed boundary-value problem* (1), (2) *if it satisfies equation* (1) *almost everywhere in Q_T.*

Further, throughout the paper, the notation $C(\ldots)$ means that the positive constant C depends only on the content of the parentheses.

Lemma 1. *If $u(t,x) \in \mathcal{A}(Q_R^T)$, then*

$$\int\limits_{Q_R^T} \left(\sum_{i,j=1}^n |u_{ij}|^2 + |u_t|^2 \right) dtdx \le \int\limits_{Q_R^T} (\mathcal{M}_0 u)^2 dtdx,$$

where $\mathcal{M}_0 = \triangle - \dfrac{\partial}{\partial t}$.

Proof. We have

$$\int\limits_{Q_R^T} (\mathcal{M}_0 u)^2 dtdx = \int\limits_{Q_R^T} \left((\triangle u)^2 - 2\triangle u\, u_t + u_t^2 \right) dtdx =$$

$$=\int\limits_{Q_R^T}\left(\sum_{i,j=1}^{n}u_{ii}u_{jj}-2\sum_{i=1}^{n}u_{ii}u_t+u_t^2\right)dtdx=$$

$$=-\int\limits_{Q_R^T}\sum_{i,j=1}^{n}u_iu_{jji}dtdx+2\int\limits_{Q_R^T}\sum_{i=1}^{n}u_iu_{ti}dtdx+\int\limits_{Q_R^T}u_t^2dtdx=$$

$$=\int\limits_{Q_R^T}\left(\sum_{i,j=1}^{n}u_{ij}^2+u_t^2\right)dtdx+\int\limits_{Q_R^T}\sum_{i=1}^{n}(u_i^2)_tdtdx=$$

$$=\int\limits_{Q_R^T}\left(\sum_{i,j=1}^{n}u_{ij}^2+u_t^2\right)dtdx+\int\limits_{B_R^{x^0}}\sum_{i=1}^{n}(u_i^2(T,x)-u_i^2(0,x))dx.$$

Since $u_i(0,x)=0$, we get the required inequality.

Lemma 2. *If* $u(t,x)\in\mathcal{A}(Q_R^T)$ *and* $p\in(1,\infty)$, *then*

$$\int\limits_{Q_R^T}\left(\sum_{i,j=1}^{n}|u_{ij}|^p+|u_t|^p\right)dtdx\le C_1(p,n)\int\limits_{Q_R^T}|\mathcal{M}_0u|^pdtdx.$$

Proof. Let

$$F(t,x)=\triangle u(t,x)-u_t(t,x),$$

$$G(t,x)=\begin{cases}a_0t^{-n/2}\exp\left(-\dfrac{|x|^2}{4t}\right), & \text{at}\quad t>0,\\ 0 & \text{at}\quad t\le0,\ \text{(except for } t=|x|=0),\end{cases}$$

where $a_0=2^{-n}\pi^{-n/2}$. Then

$$u(t,x)=\int\limits_{Q_R^T}G(t-\tau,x-y)F(\tau,y)d\tau dy.$$

For $i=1,\ldots,n$ we have

$$u_i(t,x)=\int\limits_{Q_R^T}G_i(t-\tau,x-y)F(\tau,y)d\tau dy=\int\limits_{Q_R^T}G_i(t-\tau,y-x)F(\tau,y)d\tau dy=$$

$$=\int\limits_{\mathbb{R}_{n+1}}G_i(t-\tau,v)F(\tau,v+x)d\tau dv.$$

Further, acting as in the differentiation of integrals with weak singularity [12], we obtain

$$u_{ij}(t,x)=\int\limits_{\mathbb{R}_{n+1}}G_i(t-\tau,v)F(\tau,v+x)d\tau dv=$$

$$= \lim_{\rho\to 0}\left\{-\int\limits_{B_{0,1/\rho}^{(t,x)}} G_i(t-\tau,x-y)F(\tau,y)d\tau dy +\right.$$

$$\left.+\int\limits_{\partial B_{0,1/\rho}^{(t,x)}} G_i(t-\tau,x-y)F(\tau,y)\cos(\overline{n},y_j)ds_{\tau,y}\right\} =$$

$$= \lim_{\rho\to 0}\left\{-\int\limits_{\partial B_{0,1/\rho}^{(t,x)}} G_i(t-\tau,x-y)F(\tau,y)d\tau dy\right\}+$$

$$+\lim_{\rho\to 0}\left\{F(t,x)\int\limits_{\partial B_{0,1/\rho}^{(t,x)}} G_i(t-\tau,x-y)\cos(\overline{n},y_j)ds_{\tau,y} +\right.$$

$$\left.+\int\limits_{\partial B_{0,1/\rho}^{(t,x)}} [F(\tau,y)-F(t,x)]G_i(t-\tau,x-y)\cos(\overline{n},y_j)ds_{\tau,y}\right\} =$$

$$= G_{ij}*F + F(t,x)\lim_{\rho\to 0}\int\limits_{\partial B_{0,1/\rho}^{(t,x)}} G_i(t-\tau,x-y)\cos(\overline{n},y_j)ds_{\tau,y}+$$

$$+\lim_{\rho\to 0}\int\limits_{\partial B_{0,1/\rho}^{(t,x)}} K_{ij}(\rho)G_i(t-\tau,x-y)*\cos(\overline{n},y_j)ds_{\tau,y},$$

where

$$G_{ij}*F = \lim_{\rho\to 0}\int\limits_{B_{0,1/\rho}^{(t,x)}} G_i(t-\tau,x-y)F(\tau,x)d\tau dy,$$

$$K_{ij}(\rho) = F(\tau,y)-F(t,x),$$

$$B_{0,1/\rho}^{(t,x)} = \left\{(\tau,y): 0 < \frac{G(t-\tau,x-y)}{t-\tau} < \frac{1}{\rho}\right\},$$

and $\partial B_{0,1/\rho}^{(t,x)}$ is its boundary.

We now find

$$J_{ij}(\rho)=\int\limits_{\partial B^{(x,t)}_{0,1/\rho}} G_i(t-\tau,x-y)\cos(\overline{n},y_j)ds_{\tau,y}=$$

$$=\int\limits_{\partial B^{(0,0)}_{0,1/\rho}} G_i(t-\tau,x-y)\cos(\overline{n},y_j)ds_{\tau,y}=\frac{1}{\rho}\int\limits_{\partial B^{(0,0)}_{0,1/\rho}}\frac{y_i}{2}\cos(\overline{n},y_j)ds_{\tau,y}.$$

If $i\neq j$, then $J_{ij}=0$. Now let $i=j$. Consider, e.g., the case $i=j=n$, because, in all remaining cases, the proof is similar. Denote by S_ρ the part of $\partial B^{(x,t)}_{0,1/\rho}$ in which $y_n>0$. By Π_ρ we denote the projection of S_ρ onto the hyperline $y_n=0$. Then

$$J_{nn}(\rho)=\frac{2}{\rho}\int\limits_{S_\rho}\frac{y_n}{2}\cos(\overline{n},y_n)ds_{\tau,y}=$$

$$=\frac{2}{\rho}\int\limits_{\Pi_\rho}\frac{y_n}{2}\cos(\overline{n},y_n)\frac{1}{\cos(\overline{n},y_n)}d\tau dy_1\ldots dy_{n-1}=$$

$$=\frac{2}{\rho}\int\limits_{\Pi_\rho}\frac{y_n}{2}d\tau dy_1\ldots dy_{n-1}=\frac{2}{\rho}\int\limits_{\Pi_\rho}\sum_{i=1}^{n}\frac{y_i^2}{4}d\tau dy_1\ldots dy_{n-1}=$$

$$=\frac{2}{\rho}\int\limits_{\Pi_\rho}\sqrt{\frac{n+2}{2}(-\tau)\ln\frac{(a_0\rho)^{\frac{2}{n+2}}}{-\tau}-\sum_{i=1}^{n-1}\frac{y_i^2}{4}}\,d\tau dy_1\ldots dy_{n-1}.$$

We now perform the change of variables $u=-\tau(a_0\rho)^{-\frac{2}{n+2}}$, $v_i=y_i(a_0\rho)^{-\frac{1}{n+2}}$, $i=1,2,\ldots,n-1$. Let Π^+ be the image of transformation Π_ρ. We get

$$J_{nn}(\rho)=2a_0\int\limits_{\Pi^+}\sqrt{\frac{n+2}{2}u\ln\frac{1}{u}-\sum_{i=1}^{n-1}\frac{v_i^2}{4}}\,dudv_1\ldots dv_{n-1}=$$

$$=\frac{2^{n+1}}{n+2}\int\limits_0^1\sqrt{\ln\frac{1}{r}}dr\int\limits_{\mathbb{E}_n}\exp\left[-\sum_{i=1}^{n-1}\xi_i^2\right]d\xi_1\ldots d\xi_{n-1},$$

where $r=\exp\left[\sum_{i=1}^{n-1}\frac{v_i^2}{4u}-\frac{n+2}{2}u\ln\frac{1}{u}\right]$, $\xi_i=\frac{v_i}{2\sqrt{u}}$, $i=1,2,\ldots,n-1$.

It is easy to see that the last integral is equal to $\frac{1}{n+2}$.

$$K_{ij}(\rho)=\int\limits_{\partial B^{(t,x)}_{0,1/\rho}}[F(\tau,y)-F(t,x)]G_i(t-\tau,x-y)\cos(\overline{n},y_j)ds_{\tau,y}=$$

$$=\int\limits_{\partial B^{(0,0)}_{0,1/\rho}}[F(\tau,y)-F(0,0)]G_i(-\tau,-y)\cos(\overline{n},y_j)ds_{\tau,y}=$$

$$= \frac{1}{\rho} \int\limits_{\partial B^{(0,0)}_{0,1/\rho}} [F(\tau, y) - F(0,0)] \frac{y_i}{2} \cos(\overline{n}, y_j) ds_{\tau,y} =$$

$$= \frac{2}{\rho} \int\limits_{S_\rho} [F(\tau, y) - F(0,0)] \frac{y_i}{2} \cos(\overline{n}, y_j) ds_{\tau,y} =$$

$$= \frac{2}{\rho} \int\limits_{\Pi_\rho} [F(\tau, y) - F(0,0)] \frac{y_i}{2} d\tau dy_1 \ldots dy_i dy_{i+1} \ldots dy_n.$$

Since

$$|F(\tau, y) - F(0,0)| = \begin{cases} C(n, F) \left(|x|^{1/2} + |y|\right), & \text{for} \quad (\tau, y) \in B^{(0,0)}_{1,\infty}, \\ 2 \max |F|, & \text{for} \quad (\tau, y) \in B^{(0,0)}_{0,\infty}, \end{cases}$$

then, for $i \neq j$, we get

$$K_{ij}(\rho) \leq C(n, F) \rho^{\frac{1}{n+2}} \frac{2}{\rho} \int\limits_{\Pi_\rho} \frac{y_i}{2} d\tau dy_1 \ldots dy_i dy_{i+1} \ldots dy_n = C(n, F) \rho^{\frac{1}{n+2}} J_{ij}(\rho) = 0$$

and, for $i = j$, we find $K_{ii}(\rho) \leq C(n, F) \rho^{\frac{1}{n+2}}$, $\rho^{\frac{1}{n+2}} \to 0$ as $\rho \to 0$, where $C_1(n, F) = \dfrac{C(n, F)}{n+2}$.

As a result of these calculations for u_{ij}, we have

$$u_{ij}(t, x) = -G_{ij} * F + \frac{\delta_{ij}}{n+2} F(t, x), \quad i, j = 1, \ldots, n, \tag{4}$$

where δ_{ij} is the Kronecker symbol and $G_{ij} * F$ is a parabolic singular integral with the kernel in G_{ij}. By the Jones theorem [13], for $p \in (1, \infty)$ and $i, j = 1, \ldots, n$, we conclude that

$$\|G_{ij} * F\|_{L_p(Q^T_R)} \leq C_{ij}(p, n) \|F\|_{L_p(Q^T_R)}.$$

By using this inequality in (4), we obtain

$$\sum_{i,j=1}^{n} \|u_{ij}\|_{L_p(Q^T_R)} \leq C_1(p, n) \|F\|_{L_p(Q^T_R)}. \tag{5}$$

We now show that $\|u_t\|_{L_p(Q^T_R)} \leq C_2(p, n) \|F\|_{L_p(Q^T_R)}$. Indeed, from the relations $u_t = \triangle u - F$ and (5), we get

$$\|u_t\|_{L_p(Q^T_R)} \leq \|\triangle u\|_{L_p(Q^T_R)} + \|F\|_{L_p(Q^T_R)} \leq \sum_{i=1}^{n} \|u_{ii}\|_{L_p(Q^T_R)} +$$

$$+ \|F\|_{L_p(Q^T_R)} \leq C_2(p, n) \|F\|_{L_p(Q^T_R)}.$$

Then

$$\left(\int\limits_{Q^T_R} \left(\sum_{i,j=1}^{n} |u_{ij}|^p + |u_t|^p \right) dt dx \right)^{1/p} \leq \left(\int\limits_{Q^T_R} \sum_{i,j=1}^{n} |u_{ij}|^p dt dx \right)^{1/p} +$$

$$+\left(\int\limits_{Q_R^T}|u_t|^p dtdx\right)^{1/p} \le \sum_{i,j=1}^{n}\|u_{ij}\|_{L_p(Q_R^T)}+\|u_t\|_{L_p(Q_R^T)} \le$$

$$\le C_3(p,n)\left(\int\limits_{Q_R^T}\|\mathcal{M}_0 u|^p dtdx\right)^{1/p}.$$

Lemma 2 is proved.

By $\overset{\circ}{W}{}_p^{2,1}(Q_R^T)$ and $\overset{\circ}{V}{}_p^{2,1}(Q_R^T)$, we denote the of closures $\mathcal{A}(Q_R^T)$ in the norms

$$\|u_t\|_{\overset{\circ}{W}{}_p^{2,1}(Q_R^T)}=\left(\int\limits_{Q_R^T}\left(\sum_{i,j=1}^{n}|u_{ij}|^p+|u_t|^p\right)dtdx\right)^{1/p}$$

and

$$\|u_t\|_{\overset{\circ}{V}{}_p^{2,1}(Q_R^T)}=\left(\int\limits_{Q_R^T}\|\mathcal{M}_0 u|^p dtdx\right)^{1/p},$$

respectively, $p\in(1,\infty)$. According to the Friedrichs-type inequality and Lemma 2, the functionals determined above are indeed norms. Denote by $T(p)$ the operator associating each function $u(t,x)\in$ $\in \overset{\circ}{V}{}_p^{2,1}(Q_R^T)$ with it-self as an element of the space $\overset{\circ}{W}{}_p^{2,1}(Q_R^T)$. By Lemma 2, the operator $T(p)$ is bounded. Denote by $K(\rho)$ its norm. By Lemma 1, $K(2)\le 1$. Let p_0 be an arbitrary number from the interval $(1,2)$. According to the Riesz–Thorin theorem on convexity [14], for any $p\in[p_0,2]$,

$$K(p)\le (K(p_0))^{1-\theta}(K(2))^{\theta}\le (K(p_0))^{1-\theta},$$

where $\theta=\dfrac{2(p-p_0)}{p(2-p_0)}$.

Thus,

$$K(p)\le K(p_0)^{\frac{2p_0(p-p_0)}{p(2-p_0)}}.$$

We now fix $p_0=\dfrac{5}{3}$ and denote $a=\max\left\{\left(\dfrac{5}{3}\right)^3,\left(\left(\dfrac{5}{3}\right)\right)^3\right\}$. Since, for $p\in\left[\dfrac{5}{3},2\right]$, $\dfrac{p_0(p-p_0)}{p(2-p_0)}\le\dfrac{2-p}{2-p_0}=3(2-p)$, we finally obtain

$$K(p)\le a^{2-p}.$$

Thus, we have proved the following assertions:

Lemma 3. *If $u(t,x)\in \hat{W}_p^{2,1}(Q_R^T)$, then, for any $p\in\left[\dfrac{5}{3},2\right]$,*

$$\|u_t\|_{\overset{\circ}{W}{}_p^{2,1}(Q_R^T)}\le a^{2-p}\|u_t\|_{\overset{\circ}{V}{}_p^{2,1}(Q_R^T)}.$$

Note that, in this case, the constant $a > 1$ depends only on n. For $p \in \left[\frac{5}{3}, 2\right]$, we denote

$$\sup_{Q_R^T} \left(\sum_{i,j=1}^{n} |a_{ij}(t,x) - \delta_{ij}|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}$$

by δ_p (for the sake of brevity, we write sup instead of ess sup). Also let $\delta_2 = \delta$, $h = \max\left\{\frac{1-\gamma^2}{\gamma}, 1\right\}$.

Lemma 4. *For $p \in \left[\frac{5}{3}, 2\right]$, the following inequality is true:*

$$\delta_p \le h^{\frac{2-p}{p}} \delta^{\frac{2(p-1)}{p}}.$$

Proof. It follows from condition (3) that, for $i = 1, \ldots, n$,

$$\gamma - 1 \le a_{ij}(t,x) - 1 \le \gamma^{-1} - 1,$$

and, since $\gamma - 1 \ge 1 - \gamma^{-1}$, that

$$|a_{ij}(t,x) - 1| \le \frac{1-\gamma}{\gamma}. \tag{6}$$

If $i \ne j$, then

$$2\gamma \le a_{ii}(t,x) + a_{jj}(t,x) + 2a_{ij}(t,x) \le 2\gamma^{-1}.$$

Therefore,

$$|a_{ij}(t,x)| \le \frac{1-\gamma^2}{\gamma}. \tag{7}$$

From (6) and (7), we conclude that, for $i, j = 1, \ldots, n$,

$$|a_{ij}(t,x) - \delta_{ij}| \le h. \tag{8}$$

On the other hand, in view of (8), we obtain

$$\delta_p = \sup_{Q_R^T} \left(\sum_{i,j=1}^{n} (a_{ij}(t,x) - \delta_{ij})^2 |a_{ij}(t,x) - \delta_{ij}|^{\frac{2-p}{p-1}} \right)^{\frac{p-1}{p}} \le h^{\frac{2-p}{p}} \delta^{\frac{2(p-1)}{p}}.$$

Lemma 4 is proved.

Lemma 5. *Let $\delta < 1$. Then there exists $p_1(\gamma, \delta, n) \in \left[\frac{5}{3}, 2\right]$, such that for all $p \in [p_1, 2]$*

$$a^{2-p} \delta_p \le \delta^{1/3}.$$

Proof. According to the previous lemma,

$$a^{2-p} \delta_p \le a^{2-p} h^{\frac{2-p}{p}} \delta^{\frac{2(p-1)}{p}}.$$

But $h^{1/p} \le h^{\frac{3}{5}} = h_1$, $\frac{p-1}{p} \ge \frac{1}{3}$. Therefore,

$$a^{2-p} \delta_p \le (ah_1)^{2-p} \delta^{\frac{2}{3}}. \tag{9}$$

Now let $p_1 = \max\left\{\frac{5}{3}, 2 - \frac{\ln(1/\delta)}{3\ln(ah_1)}\right\}$. Then, for $p \in [p_1, 2]$, we have $(ah_1)^{2-p} \le \delta^{-1/3}$ and the assertions of the lemma follow from (9).

2. Internal priory estimation. Consider an operator

$$\mathcal{L}_0 = \sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t},$$

together with the operator $\mathcal{L}$.

Lemma 6. *If condition* (3) *and inequality* $\delta < 1$ *are satisfied for the coefficients of the operator* $\mathcal{L}_0$*, then, for all* $p \in [p_1, 2]$ *and any function* $u(t,x) \in \overset{\circ}{W}{}_p^{2,1}(Q_R^T)$ *the estimation*

$$\|u\|_{\overset{\circ}{W}{}_p^{2,1}(Q_R^T)} \le C_4(\gamma,\delta,n)\|\mathcal{L}_0 u\|_{L_p(Q_R^T)}$$

is true.

Proof. According to Lemma 3,

$$\|u\|_{\overset{\circ}{W}{}_p^{2,1}(Q_R^T)} \le a^{2-p}\|\mathcal{M}_0 u\|_{L_p(Q_R^T)} \le$$

$$\le a^{2-p}\|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + a^{2-p}\left\|\sum_{i,j=1}^{n}(a_{ij}(t,x)-\delta_{ij})u_{ij}\right\|_{L_p(Q_R^T)} \le$$

$$\le a^{2/5}\|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + a^{2-p}\left\|\sum_{i,j=1}^{n}(a_{ij}(t,x)-\delta_{ij})u_{ij}\right\|_{L_p(Q_R^T)}. \qquad (10)$$

But, on the other hand,

$$\left\|\sum_{i,j=1}^{n}(a_{ij}(t,x)-\delta_{ij})u_{ij}\right\|_{L_p(Q_R^T)} \le$$

$$\le \left(\int\limits_{Q_R^T}\left(\sum_{i,j=1}^{n}|u_{ij}|^p\right)\left(\sum_{i,j=1}^{n}(a_{ij}(t,x)-\delta_{ij})^{\frac{p}{p-1}}\right)^{p-1} dtdx\right)^{1/p} \le \delta_p\|u\|_{\overset{\circ}{W}{}_p^{2,1}(Q_R^T)}.$$

Therefore, from (10) and Lemma 5, we conclude that

$$\|u\|_{\overset{\circ}{W}{}_p^{2,1}(Q_R^T)} \le a^{2/5}\|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + a^{2-p}\delta_p\|u\|_{\overset{\circ}{W}{}_p^{2,1}(Q_R^T)} \le$$

$$\le a^{2/5}\|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \delta^{1/3}\|u\|_{\overset{\circ}{W}{}_p^{2,1}(Q_R^T)}$$

and the assertion of the lemma is proved.

In what follows, we everywhere assume that the radius R of the sphere $B_R^{x^0}$ ($B_R^{x^0}$ is the foundation of the cylinder Q_R^T) does not exceed 1.

Lemma 7. *If the condition of previous lemma is true, then, for all* $p \in [p_1, 2]$ *and any function* $u(t, x) \in \mathcal{A}(Q_R^T)$, *the inequality*

$$\|u\|_{W_p^{2,1}(Q_R^T)} \leq C_5(\gamma, \delta, n)\|\mathcal{L}_0 u\|_{L_p(Q_R^T)}$$

is true.

To prove this, it suffices to apply the Friedrichs inequality and Lemma 6.

We now assume that the following Cordes condition for the leading coefficients of the operator $\mathcal{L}$ is true:

$$\sigma = \frac{\sup_{Q_T} \sum_{i,j=1}^n a_{ij}^2(t,x)}{\left[\inf_{Q_T} \sum_{i=1}^n a_{ii}(t,x)\right]^2} \leq \frac{1}{n-1}. \tag{11}$$

In this case, we suppose that condition (11) is satisfied to within a nonsingular linear transformation, i.e., we can cover the domain Q_T with finite number of subdomains $Q_1, \ldots, Q_m$ and, hence, in every Q_i, there exists a nonsingular linear transformation under which the image of the operator $\mathcal{L}$ satisfies condition (11) in the image of subdomain Q_i, $i = 1, \ldots, m$.

Lemma 8. *To within a nonsingular linear transformation, the condition* $\delta < 1$ *coincides with condition* (11).

Proof. We now perform the transformation $\tau = k^2 t$, $y_i = kx_i$, $i = 1, \ldots, n$, where

$$k = \left(\frac{\sup_{Q_T} \sum_{i,j=1}^n a_{ij}^2(t,x)}{\inf_{Q_T} \sum_{i=1}^n a_{ii}(t,x)}\right)^{-1/2}.$$

Thus, if $\|\mathcal{A}_{ij}(\tau, y)\|$ is the matrix of leading part of the image of the operator $\mathcal{L}$ then $\mathcal{A}_{ij}(\tau, y) = k^2 a_{ij}(t, x)$, $i, j = 1, \ldots, n$. In the new variables, the condition $\delta < 1$ takes the form

$$\sup_{\tilde{Q}_T} \sum_{i,j=1}^n \mathcal{A}_{ij}^2(\tau, y) - 2 \inf_{\tilde{Q}_T} \sum_{i=1}^n \mathcal{A}_{ii}(\tau, y) + n < 1, \tag{12}$$

where $\tilde{Q}_T$ is the image of the domain Q_T. It is clear, that it coincides with the conditions

$$\frac{\sup_{Q_T} \sum_{i,j=1}^n a_{ij}^2(t,x)}{\left[\inf_{Q_T} \sum_{i=1}^n a_{ii}(t,x)\right]^2} \leq \frac{1}{n-1}.$$

Lemma 9. *Let conditions* (3) *and* (11) *be satisfied for the coefficients of the operator* $\mathcal{L}_0$. *Then there exists a constant* $C_6(\gamma, \sigma, n)$ *such that, for any function* $u(t, x) \in C^\infty(\bar{Q}_R^T)$, $u|_{t=0} = 0$ *for every* $p \in [p_1, 2]$, *and* $R_1 \in (0, R)$, *the estimate*

$$\|u\|_{W_p^{2,1}(Q_{R_1}^T)} \leq C_5\|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \frac{C_6}{(R-R_1)^2}\|u\|_{L_p(Q_R^T)} + \frac{C_6}{R-R_1}\|u\|_{W_p^{1,0}(Q_R^T)}$$

is true.

Proof. Let the functions $\eta(x) \in C_0^\infty(B_R^{x^0})$ be such that $\eta(x) = 1$ in $B_{R_1}^{x^0}$, $0 \le \eta(x) \le 1$. Moreover

$$|\eta_1| \le \frac{C_7}{R - R_1}, \qquad |\eta_{ij}| \le \frac{C_7}{(R - R_1)^2}, \quad i, j = 1, \ldots, n, \tag{13}$$

where $C_7 = C_7(n)$. Applying Lemma 7 to the functions $u\eta$, we obtain

$$\|u\|_{W_p^{2,1}(Q_{R_1}^T)} \le C_5 \|\mathcal{L}_0(u\eta)\|_{L_p(Q_R^T)}. \tag{14}$$

But, on the other hand,

$$|\mathcal{L}_0(u\eta)| \le |\mathcal{L}_0 u| + |u| \left| \sum_{i=1}^n a_{ij}(t,x)\eta_{ij} \right| + 2 \left| \sum_{i,j=1}^n a_{ij}(t,x) u_i \eta_i \right|. \tag{15}$$

Further, in view of (13), we get

$$\left| \sum_{i=1}^n a_{ij}(t,x)\eta_{ij} \right| \le \frac{C_8(\gamma, n)}{(R - R_1)^2},$$

$$2 \left| \sum_{i,j=1}^n a_{ij}(t,x) u_i \eta_i \right| \le 2 \left(\sum_{i,j=1}^n a_{ij}(t,x) u_i u_j \right)^{1/2} \left(\sum_{i,j=1}^n a_{ij}(t,x) \eta_i \eta_j \right)^{1/2} \le$$

$$\le 2\gamma^{-1} \left(\sum_{i=1}^n u_i^2 \right)^{1/2} \left(\sum_{i=1}^n \eta_i^2 \right)^{1/2} \le 2\gamma^{-1} \sum_{i=1}^n |u_i| \sum_{i=1}^n |\eta_i| \le \frac{2n\gamma^{-1} C_7}{R - R_1} \sum_{i=1}^n |u_i|.$$

Thus, from (15), we conclude

$$\|\mathcal{L}_0(u\eta)\|_{L_p(Q_R^T)} \le \|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \frac{C_8}{(R - R_1)^2} \|u\|_{L_p(Q_R^T)} +$$

$$+ \frac{C_9(\gamma, n)}{R - R_1} \sum_{i=1}^n \|u_i\|_{L_p(Q_R^T)} \le \|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \frac{C_8}{(R - R_1)^2} \|u\|_{L_p(Q_R^T)} +$$

$$+ \frac{C_9(\gamma, n)}{R - R_1} \|u\|_{W_p^{1,0}(Q_R^T)}. \tag{16}$$

In view of (16) and (14), we denote $\max\{C_5 C_8, C_5 C_9\}$ by C_{10} and arrive at the required estimate (13).

Lemma 10. *Let the conditions of the previous lemma be satisfied for the coefficients of the operator* $\mathcal{L}_0$. *Then there exists a constant* $C_{11}(\gamma, \sigma, n)$ *such that, for any function* $u(t, x) \in C^\infty(\bar{Q}_R^T)$, $u|_{t=0} = 0$ *for any* $\varepsilon > 0$, *and* $p \in [p_1, 2]$, *the estimate*

$$\|u\|_{W_p^{2,1}(Q_{\frac{R}{2}}^T)} \le C_5 \|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \varepsilon \|u\|_{W_p^{2,1}(Q_R^T)} + \frac{C_{11}}{\varepsilon R^2} \|u\|_{L_p(Q_R^T)}$$

is true.

Proof. We use the following interpolation inequality [1]: let $p \in (0, \infty)$. Then, for any functions $u(t,x) \in W_p^{2,1}(Q_R^T)$, any $\varepsilon > 0$, and $p \in [p_1, 2]$ the following estimate is true:

$$\|u\|_{W_p^{1,0}(Q_R^T)} \leq \varepsilon \|u\|_{W_p^{2,1}(Q_R^T)} + \frac{C_{12}}{\varepsilon} \|u\|_{L_p(Q_R^T)}. \tag{17}$$

We now fix an arbitrary number $\varepsilon > 0$ and let $\varepsilon_1 > 0$ be a number which will be chosen later. According to Lemma 9 and inequality (17), we have

$$\|u\|_{W_p^{2,1}(Q_{R/2}^T)} \leq C_5 \|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \frac{4C_6}{R^2} \|u\|_{L_p(Q_R^T)} + \frac{2C_6}{R^2} \|u\|_{W_p^{1,0}(Q_R^T)} \leq$$

$$\leq C_5 \|\mathcal{L}_0 u\|_{L_p(Q_R^T)} + \frac{4C_6}{R^2} \|u\|_{L_p(Q_R^T)} + \frac{2C_6 \varepsilon_1}{R} \|u\|_{W_p^{2,1}(Q_R^T)} + \frac{2C_6 C_{13}}{R \varepsilon_1} \|u\|_{L_p(Q_R^T)},$$

where $C_{13} = \sup_{p \in [p_1, 2]} C_{12}(p, n)$. It is now sufficient to choose $\varepsilon_1 = \dfrac{\varepsilon R}{2C_6}$.

Lemma 10 is proved.

***Remark* 1.** If the minor coefficients of the operator $\mathcal{L}$ are bounded, then there exists $R_0(\gamma, \sigma_0, n, \mathbb{B}, c)$ such that, for $R \leq R_0$, the assertion of Lemma 10 is also true for the operator $\mathcal{L}$. Here, $\mathbb{B} = (b_1(t,x), \ldots, b_n(t,x))$. For $\rho > 0$, the set $\{x : x \in \Omega,\ \mathrm{dist}\,(x, \partial\Omega) > \rho\}$ is denoted by Ω_ρ.

Lemma 11. *Let conditions* (3) *and* (11) *be satisfied for the coefficients of the operator* $\mathcal{L}$. *Then, for any function* $u(t,x) \in C^\infty(\bar{Q}_R^T)$, $u|_{t=0} = 0$ *for any* $\varepsilon > 0$, $\rho > 0$, *and* $p \in [p_1, 2]$, *the estimate*

$$\|u\|_{W_p^{2,1}(Q_\rho \times (0,T))} \leq C_{14}(\gamma, \sigma, n, \rho, \Omega) \|\mathcal{L}_0 u\|_{L_p(Q_R^T)} +$$

$$+ \varepsilon \|u\|_{W_p^{2,1}(Q_R^T)} + \frac{C_{15}(\gamma, \sigma, n, \rho, \Omega)}{\varepsilon} \|u\|_{L_p(Q_R^T)}$$

is true.

Proof. We now fix arbitrary $\varepsilon > 0$ and $\rho > 0$. Let $\varepsilon_2 > 0$ be a number chosen in what follows. Consider a covering $\bar{\Omega}_\rho$ by a system of spheres $\{B_{\rho/2}^{x^i}\}$ and choose a finite subcovering $B^1, \ldots, B^N$ from this covering. It is evident that the number N depends only on ρ, n, and $\mathrm{diam}\,\Omega$. Applying, for every $i = 1, \ldots, N$, Lemma 10, we obtain

$$\|u\|_{W_p^{2,1}(B^i \times (0,T))} \leq 3^{p-1} \left(C_5^p \|\mathcal{L}_0 u\|_{L_p(Q_T)}^p + \varepsilon_2^p \|u\|_{W_p^{2,1}(Q_T)}^p + \frac{C_{11}^p}{\varepsilon_2^p \rho^{2p}} \|u\|_{L_p(Q_T)}^p \right).$$

Finding the sum of these inequalities over i from 1 to N, we conclude that

$$\|u\|_{W_p^{2,1}(\Omega_\rho \times (0,T))} \leq 3^{p-1} N \left(C_5^p \|\mathcal{L}_0 u\|_{L_p(Q_T)}^p + \varepsilon_2^p \|u\|_{W_p^{2,1}(Q_T)}^p + \frac{C_{11}^p}{\varepsilon_2^p \rho^{2p}} \|u\|_{L_p(Q_T)}^p \right).$$

It is now sufficient to choose $\varepsilon_2 = \dfrac{\varepsilon}{3N}$ and the lemma is proved.

3. Basic coercive estimation. The assertion of Lemma 11 is true without any requirements imposed on the domain $\partial\Omega$. All subsequent assertions of the present paper hold under the condition $\partial\Omega \in C^2$, and we always assume that this condition is satisfied.

Lemma 12. *Let conditions* (3) *and* (11) *be satisfied for the coefficients of the operator* $\mathcal{L}_0$. *Then there exists positive constants* p_1, C_{16}, *and* C_{17} *depending on* γ, σ_0, *and* n *and a domain* Ω *such that, for any function* $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$, *any* $\varepsilon > 0$, *and* $p \in [p_1, 2]$ *the estimate*

$$\|u\|_{W_p^{2,1}((\Omega\setminus\Omega_{\rho_1}\times(0,T))} \leq C_{16}\|\mathcal{L}_0 u\|_{L_p(Q_T)} + \varepsilon\|u\|_{W_p^{2,1}(Q_T)} + \frac{C_{17}}{\varepsilon}\|u\|_{L_p(Q_T)}$$

is true.

Proof. It is sufficient to prove the lemma for the functions $u(t,x) \in C^\infty(\bar{Q}_T)$, $u|_{\Gamma(Q_T)} = 0$. Moreover, without loss of generality, we can suppose that the coefficients of the operator $\mathcal{L}_0$ are infinitely differentiable $\bar{Q}_T$. We now fix an arbitrary number $\varepsilon > 0$ and a point $x^0 \in \partial\Omega$. We perform an orthogonal transformation of the coordinate $x \to y$ such that the tangent hyperline to $\partial\tilde{\Omega}$ at the point y^0 is perpendicular to the axis Oy_n. Here, $\tilde{\Omega}$ and y^0 are images of the domain Ω and the point x^0, under this transformation, respectively. Denote by $\tilde{u}(t,x)$ the image of the function $u(t,x)$. For simplicity, we suppose that the domain $\partial\tilde{\Omega}$ at the intersection of $\partial\tilde{\Omega}$ with some neighborhood O_h of the point y^0 is given by the equation $y_n = \varphi(y_1,\ldots,y_{n-1})$ with twice continuously differentiable function φ and the part $\tilde{\Omega}$ adjacent to $\partial\tilde{\Omega} \cap O_h$ belongs to the set $\{y\colon y_n > \varphi(y_1,\ldots,y_{n-1})$. Let $\mathcal{A}(t,x) = \|a_{ij}(t,x)\|$ be a matrix of coefficients of the operator $\mathcal{L}_0$, $\tilde{\mathcal{A}}(t,x) = \|\tilde{a}_{ij}(t,y)\|$, where $\tilde{a}_{ij}(t,y)$ are leading coefficients of the image $\tilde{\mathcal{L}}_0$ of the operator $\mathcal{L}_0$ under our transformation; $i,j = 1,\ldots,n$. We now show that the eigenvalues of the matrices $\mathcal{A}$ and $\tilde{\mathcal{A}}$ coincide. Indeed, we fix an arbitrary point $(t,x) \in Q_T$; λ is an arbitrary eigenvalue of the matrix $\mathcal{A}$ and x^λ corresponds to its eigenvector. By virtue of the orthogonality of our transformation, there exists a nondegenerate matrix T such that $\tilde{\mathcal{A}} = T^{-1}\mathcal{A}T$. Denote $T^{-1}x^\lambda$. We get

$$\tilde{\mathcal{A}}y^\lambda = T^{-1}\mathcal{A}x^\lambda = \lambda y^\lambda.$$

On the other hand, we can write condition (11) in the following form:

$$\sigma = \sup_{Q_T} \frac{\sum_{i=1}^n \lambda_i^2(t,x)}{\left[\sum_{i=1}^n \lambda_i(t,x)\right]^2} \leq \frac{1}{n-1},$$

where $\lambda_i(t,x)$ are eigenvalues of the matrix $\mathcal{A}(t,x)$, $i = 1,\ldots,n$. Thus, condition (11) is also satisfied for the operator $\tilde{\mathcal{L}}_0$ and, moreover, with the same constant σ. Analogously, it can be shown that conditions (3) are satisfied for the operator $\tilde{\mathcal{L}}_0$ (with the same constant γ). We now perform one more transformation: $z_i = y_i$, $i = 1,\ldots,n-1$, $z_n = y_n - \varphi(y_1,\ldots,y_{n-1})$. Let $\mathcal{L}_0'$, Ω', and z^0 be the images of the operator $\tilde{\mathcal{L}}_0$, domain $\tilde{\Omega}$, and point y^0, respectively, under our transformation, and let $a'_{ij}(t,z)$ be the leading coefficients of the operator $\mathcal{L}_0'$; $i,j = 1,\ldots,n$. It is easy to see that

$$a'_{ij}(t,z) = \sum_{k,l=1}^n \tilde{a}_{kl}(t,y)\frac{\partial z_l}{\partial y_k}\frac{\partial z_j}{\partial y_l}, \quad i,j = 1,\ldots,n.$$

Therefore,

$$a'_{ij}(t,z) = \tilde{a}_{ij}(t,y) \quad \text{if} \quad 1 \leq i,j \leq n-1,$$

$$a'_{nj}(t,z) = -\sum_{k=1}^{n-1} \tilde{a}_{kj}(t,y)\frac{\partial\varphi}{\partial y_k} + \tilde{a}_{nj}(t,y) \quad \text{if} \quad 1 \leq i,j \leq n-1,$$

$$a'_{nn}(t,z) = \sum_{k,l=1}^{n} \tilde{a}_{kl}(t,y)\frac{\partial\varphi}{\partial y_k}\frac{\partial\varphi}{\partial y_l} - 2\sum_{k=1}^{n-1} \tilde{a}_{nk}(t,y)\frac{\partial\varphi}{\partial y_k} + \tilde{a}_{nn}(t,y).$$

Since $\dfrac{\partial\varphi}{\partial y_i}(y^0) = 0$ for $i = 1,\ldots,n-1$, there exists $h_1(y^0,\varphi)$ such that the condition (11) $\left(\text{with the same constant } \sigma' = \dfrac{\sigma + 1/(n-1)}{2}\right)$ is satisfied for $h \le h_1$ in the intersection $\Omega' \cap$ $\cap \left(B_h^{z^0} \times (0,T)\right)$. Moreover, conditions (3) are satisfied $\left(\text{with the constant } \dfrac{\gamma}{2}\right)$ for the operator $\mathcal{L}'_0$ in the indicated intersection. Assume that $r = r(z^0) = h_1(y_0,\varphi)$. Let $u'(t,z)$ be the image of the function $\tilde{u}(t,y)$ under our transformation. It is clear that, in the variables z, the intersection $\Omega' \cap B_r^{z^0}$ represents a hemisphere $B_r^+ = \{z\colon |z - z^0| < r, z_n > 0\}$. We continue the function $u'(t,z)$ and the coefficients of the operator $\mathcal{L}'_0$ by evenness relative to the hyperplane $z_n = 0$ in $B_r^{z^0}\backslash B_r^+$ and denote by $u'(t,z)$ and $\mathcal{L}'_0$, respectively, the function and operator obtained in this case. Since $u'(t,z) \in W_p^{2,1}\left(B_h^{z^0} \times (0,T)\right)$, according to Lemma 10, we find

$$\|u'\|_{W_p^{2,1}\left(B_{\frac{r}{2}}^{z^0}\times(0,T)\right)} \le C_5\|\mathcal{L}'_0 u'\|_{L_p\left(B_r^{z^0}\times(0,T)\right)} + \varepsilon_3\|u'\|_{W_p^{2,1}\left(B_r^{z^0}\times(0,T)\right)} +$$

$$+\frac{C_{11}}{\varepsilon_3 r^2}\|u'\|_{L_p\left(B_r^{z^0}\times(0,T)\right)}, \tag{18}$$

where $\varepsilon_3 > 0$ is chosen in what follows. However, on the other hand, each norm on the right-hand side of (18) represents the corresponding norm taken for a semicylinder $Q_r^+ = B_r^+ \times (0,T)$ and multiplied by $2^1/p$. Therefore, from (18), we conclude

$$\|u'\|_{W_p^{2,1}\left(Q_{\frac{r}{2}}^+\right)} \le C_5\|\mathcal{L}'_0 u'\|_{L_p(Q_r^+)} + \varepsilon_3\|u'\|_{W_p^{2,1}(Q_r^+)} + \frac{C_{11}}{\varepsilon_3 r^2}\|u'\|_{L_p(Q_r^+)}. \tag{19}$$

We cover $\partial\Omega'$ by a system of spheres $\{B_{\frac{r}{2}}^{z^i}\}$ and choose from this covering a finite subcovering $B^1,\ldots,B^M$. In this case, the number M is determined only by the quantities γ, σ_0, and h and the domain Ω. We write an inequality of the form (19) for every semicylinder $B_r^+(z^i) \times (0,T)$, $i = 1,\ldots,M$, raise both sides of the obtained inequalities to the power p, and find the sum of these inequalities over i from 1 to M. This yields

$$\|u'\|^p_{W_p^{2,1}(\mathcal{B}\times(0,T))} \le 3^{p-1}M\left(C_5\|\mathcal{L}'_0 u'\|^p_{L_p(\Omega'\times(0,T))} + \varepsilon_3^p\|u'\|_{W_p^{2,1}(\Omega'\times(0,T))} + \right.$$

$$\left. + \frac{C_{11}^p}{\varepsilon_3^p r_0^{2p}}\|u'\|^p_{L_p(\Omega'\times(0,T))}\right),$$

where $\mathcal{B} = \bigcup_{i=1}^M B_{\frac{r}{2}}^+(z^i)$, and $r_0 = \min\{r(z_1),\ldots,r(z_M)\}$. We return to the variables x and note that the preimage $\mathcal{B}$ contains the set $\Omega\backslash\Omega_{\rho_1}$ with some $\rho_1(\gamma,\sigma,n,\Omega)$. This enables us to conclude that

$$\|u\|_{W_p^{2,1}((\Omega\backslash\Omega_{\rho_1}\times(0,T))} \le C_{18}\|\mathcal{L}_0 u\|_{L_p(Q_T)} + C_{19}\varepsilon_3\|u\|_{W_p^{2,1}(Q_T)} + \frac{C_{20}}{\varepsilon_3}\|u\|_{L_p(Q_T)},$$

where the constants C_{18}, C_{19}, and C_{20} depend only on γ, σ, and n and the domain Ω. It is now sufficient to choose $\varepsilon_3 = \dfrac{\varepsilon}{C_{19}}$, and the lemma is proved.

Lemmas 11 and 12 now imply the following assertion:

Lemma 13. *Let conditions* (3) *and* (11) *be satisfied for coefficients of the operator* $\mathcal{L}_0$. *Then, for any function* $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ *and any* $p \in [p_1, 2]$, *the estimate*

$$\|u\|_{W_p^{2,1}(Q_T)} \le C_{21}(\gamma,\sigma,n,\Omega)\left(\|\mathcal{L}_0 u\|_{L_p(Q_T)} + \|u\|_{L_p(Q_T)}\right)$$

is true.

We now impose the following conditions on the minor coefficient of the operator $\mathcal{L}$. For $p \in$ $\in [p_1, 2]$,

$$b_i(t,x) \in L_{n+2}(Q_T), \quad i = 1,\ldots,n, \tag{20}$$

$$c(t,x) \in L_l(Q_T), \quad l = \begin{cases} \max\left(p, \dfrac{n+2}{2}\right), & \text{for} \quad p \ne \dfrac{n+2}{2}, \\ 2+\nu, & \text{for} \quad n = p = 2, \end{cases} \tag{21}$$

where ν is a positive constant. Let $\psi(t,x) \in L_p(Q_T)$, $1 < p < \infty$. The quantity

$$\omega_{\psi;p}(\delta) = \sup_{e \in Q_T,\, \operatorname{mes} e \le \delta} \left(\int_e |\psi|^p dt dx\right)^{1/p}$$

is called the $\mathcal{AC}$ modulus of the function $\psi(t,x)$. Denote $\max_{1\le i\le n}\{\omega_{b_i;p}(\delta)\}$ by $\omega_{B;p}(\delta)$.

Let $K = \sum_{i=1}^n \|b_i\|_{L_{n+2}(Q_T)} + \|c\|_{L_m(Q_T)}$.

Everywhere in what follows, the symbol $C(\mathcal{L})$ means that the positive constant C depends only on γ, σ, K, and ν.

Lemma 14. *Let conditions* (3), (11), *and* (20) *be satisfied for the coefficients of the operator* $\mathcal{L}$. *Then there exist positive constants* $C_{22}(\mathcal{L}, n, \Omega)$ *and* $T_0(\mathcal{L}, n)$ *such that if* $T \le T_0$, *then, for any function* $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$, *and any* $p \in [p_1, 2]$ *the estimate*

$$\|u\|_{W_p^{2,1}(Q_T)} \le C_{22}\|\mathcal{L}u\|_{L_p(Q_T)},$$

is true.

Proof. We use the following embedding theorems [1]: For any function $u(t,x) \in \hat{W}_q^{2,1}(Q_T)$, the following inequalities are true:

$$\|u_i\|_{L_{\frac{q(n+2)}{n+2-q}}(Q_T)} \le C_{23}(q,n)\|u\|_{W_q^{2,1}(Q_T)} \qquad \text{for} \quad 1 \le q < n+2, \tag{22}$$

$$\|u\|_{L_{\frac{q(n+2)}{n+2-2q}}(Q_T)} \le C_{24}(q,n)\|u\|_{W_q^{2,1}(Q_T)} \qquad \text{for} \quad 1 \le q < \frac{n+2}{2}. \tag{23}$$

According to Lemma 13,

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C_{21}\|\mathcal{L}u\|_{L_p(Q_T)} + C_{21}\|(\mathcal{L}-\mathcal{L}_0)u\|_{L_p(Q_T)} + C_{21}\|u\|_{L_p(Q_T)} \leq$$

$$\leq C_{21}\|\mathcal{L}u\|_{L_p(Q_T)} + C_{21}\sum_{i=1}^{n}\|b_iu_i\|_{L_p(Q_T)} + C_{21}\|cu\|_{L_p(Q_T)} + C_{21}\|u\|_{L_p(Q_T)}. \qquad (24)$$

We now fix an arbitrary i, $1 \leq i \leq n$ and assume that $q = p$ in (21). We find

$$\|b_iu_i\|_{L_p(Q_T)} \leq \|b_i\|_{L_{n+2}(Q_T)}\|u_i\|_{L_{\frac{p(n+2)}{n+2-p}}(Q_T)} \leq C_{23}\|b_i\|_{L_{n+2}(Q_T)}\|u\|_{W_p^{2,1}(Q_T)}.$$

Thus,

$$\sum_{i=1}^{n}\|b_iu_i\|_{L_p(Q_T)} \leq C_{23}\sum_{i=1}^{n}\|b_i\|_{L_{n+2}(Q_T)}\|u\|_{W_p^{2,1}(Q_T)} \leq$$

$$\leq C_{25}(n)\omega_{\mathbb{B};n+2}(\delta)\|u\|_{W_p^{2,1}(Q_T)}, \qquad (25)$$

where $\delta = T\operatorname{mes}\Omega$ and $C_{25} = \sup_{p\in[p_1,2]} C_{23}(p,n)$.

Similarly, by virtue of (23), for $n \leq 3$, we get

$$\|cu\|_{L_p(Q_T)} \leq \|c\|_{L_{\frac{n+2}{2}}(Q_T)}\|u\|_{L_{\frac{p(n+2)}{n+2-2p}}(Q_T)} \leq C_{24}\|c\|_{L_{\frac{n+2}{2}}(Q_T)}\|u\|_{W_p^{2,1}(Q_T)} \leq$$

$$\leq C_{26}(n)\omega_{c;\frac{n+2}{2}}(\delta)\|u\|_{W_p^{2,1}(Q_T)},$$

where $C_{26} = \sup_{p\in[p_1,2]} C_{24}(p,n)$.

It is easy to see that an analogous estimate holds for $n = 2$ and $p \neq 2$. Now let $n = p = 2$. Thus, according to embedding theorem [1], for any function $u(t,x) \in W_p^{2,1}(Q_T)$ and every $q \in [1,\infty]$, the following estimate is true:

$$\|u\|_{L_q(Q_T)} \leq C_{27}(q,n)\|u\|_{W_p^{2,1}(Q_T)}.$$

Therefore, if $c(t,x) \in L_{2+\nu_1}(Q_T)$, then

$$\|cu\|_{L_2(Q_T)} \leq \|c\|_{L_{2+\nu_1}(Q_T)}\|u\|_{L_{\frac{2(2+\nu_1)}{\nu_1}}(Q_T)} \leq C_{28}(\nu)\omega_{c;2+\nu_1}(\delta)\|u\|_{W_2^{2,1}(Q_T)}.$$

Finally, let $n = 1$. Then, according to the embedding theorem [1], for any function $u(t,x) \in W_p^{2,1}(Q_T)$, the estimate

$$\sup_{Q_T}|u| \leq C_{28}\|u\|_{W_p^{2,1}(Q_T)}$$

is true. Therefore,

$$\|cu\|_{L_p(Q_T)} \leq \sup_{Q_T}|u|\|c\|_{L_p(Q_T)} \leq C_{28}\omega_{c;p}(\delta)\|u\|_{W_p^{2,1}(Q_T)}.$$

Thus, in any case,we get the inequality

$$\|cu\|_{L_p(Q_T)} \leq C_{29}(n)\omega_{c;l}(\delta)\|u\|_{W_p^{2,1}(Q_T)}. \qquad (26)$$

Now let $t \in (0,T)$. We use the following inequality:

$$\|u\|_{L_p(Q_T)} \le T\|u_t\|_{L_p(Q_T)}. \tag{27}$$

In view of (25), (26), (27), and (24) we arrive at the inequality

$$\|u\|_{W_p^{2,1}(Q_T)} \le C_{21}\|\mathcal{L}u\|_{L_p(Q_T)} + C_{21}(C_{25}\omega_{\mathbb{B};n+2}(\delta) + C_{29}\omega_{c;l}(\delta) + T)\|u\|_{W_p^{2,1}(Q_T)}.$$

Then there exists a constant $T_0(\mathcal{L}, n)$ such that, for $T \le T_0$,

$$C_{25}\omega_{\mathbb{B};n+2}(\delta) + C_{29}\omega_{c;\frac{n+2}{n}}(\delta) + T < \frac{1}{2C_{21}}.$$

Lemma 14 is proved.

4. Case $p > 2$. Let $p \in \left[2, \frac{7}{3}\right]$ and let $K(p)$ have the same meaning as in Lemma 3. By the Riesz – Theorin theorem, for any $p \in \left[2, \frac{7}{3}\right]$,

$$K(p) \le (K(2))^{1-\theta}\left(K\left(\frac{7}{3}\right)\right)^{\theta} \le \left(K\left(\frac{7}{3}\right)\right)^{\theta},$$

where $\theta = \dfrac{2(p-2)}{p\left(\dfrac{7}{3}-2\right)}$. Denote $\max\left\{\left(\frac{7}{3}\right)^3, K\left(\left(\frac{7}{3}\right)\right)^3\right\}$ by $a_1(n)$. We obtain

$$K(p) \le a_1^{p-2}.$$

Thus, the following analog of Lemma 3 is true:

Lemma 15. *If $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$, then, for any $p \in \left[2, \frac{7}{3}\right]$, the inequality*

$$\|u\|_{\overset{\circ}{W}_p^{2,1}(Q_T)} \le a_1^{p-2}\|u\|_{\overset{\circ}{V}_p^{2,1}(Q_R^T)}$$

is true.

The analogs of Lemmas 4 and 5 are proved in an absolutely similar way:

Lemma 16. *For $p \in \left[2, \frac{7}{3}\right]$, the following inequality is true:*

$$\delta_p \le h^{\frac{p-2}{p}}\delta.$$

Lemma 17. *Let $\delta < 1$. Then there exists $p_2(\gamma, \delta, n) \in \left(2, \frac{7}{3}\right]$ such that, for all $p \in [2, p_2]$,*

$$a_1^{p-2}\delta_p \le \delta^{1/3}.$$

We now impose the following restrictions on the minor coefficients of the operator $\mathcal{L}$ for $p \in$ $\in (2, p_2]$:

$$b_i(t,x) \in L_{n+2}(Q_T), \quad i = 1, \ldots, n, \tag{28}$$

$$c(t,x) \in L_{l'}(Q_T), \qquad l' = \max\left(p, \frac{n+2}{2}\right). \tag{29}$$

By using the scheme realized in Lemmas 6 – 12 and applied to Lemmas 14 – 16, we conclude that Lemma 1 is true for $p \in (2, p_2]$ and $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ if conditions (3), (11), (18), and (29) are satisfied only for the coefficients of the operator $\mathcal{L}$. We combine conditions (21) and (29) by assuming that $p \in [p_1, p_2]$, i.e., we suppose that

$$c(t,x) \in L_m(Q_T), \qquad \text{where} \quad m = \begin{cases} l, & \text{for} \quad p \in [p_1, 2], \\ l', & \text{for} \quad p \in (2, p_2]. \end{cases} \tag{30}$$

Theorem 1. *Let conditions* (3), (11), (18), *and* (29) *be satisfied for the coefficients of the operator* $\mathcal{L}$. *Then there exists positive constants* $T_0(\mathcal{L}, n)$ *and* $C_{30}(\gamma, \sigma, K, n, \Omega)$ *such that, for any functions* $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ *with* $T \le T_0$ *and any* $p \in [p_1, 2]$, *the estimate*

$$\|u\|_{W_p^{2,1}(Q_T)} \le C_{30} \|\mathcal{L}u\|_{L_p(Q_T)},$$

is true.

5. Solvability of the mixed boundary-value problem. We now consider the mixed boundary-value problem (1), (2).

Theorem 2. *Let conditions* (3), (11), (28), *and* (30) *be satisfied for the coefficients of the operator* $\mathcal{L}$ *given in the domain* Q_T. *If* $T \le T_0$ *and* $\partial\Omega \in C^2$, *then the mixed boundary-value problem is identically strongly solvable in the space* $\hat{W}_p^{2,1}(Q_T)$ *for every* $f(t,x) \in L_p(Q_T)$, $p \in [p_1, 2]$. *In this case, for the solution* $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ *the estimate*

$$\|u\|_{W_p^{2,1}(Q_T)} \le C_{30} \|f\|_{L_p(Q_T)}, \tag{31}$$

is true.

Proof. We now prove the theorem by the method of continuation in the parameter. We introduce, for $s \in [0,1]$, the family of operators $\mathcal{L}_s = s\mathcal{L} + (1-s)\mathcal{M}_0$.

It is easy to see that conditions (3) and (11) are satisfied for the operator $\mathcal{L}_s$ with constants γ and σ, respectively. We show this on the example of condition (11). According to Lemma 8, the indicated condition coincides, to within a nonsingular linear transformation, with the condition $\delta < 1$. Let $a_{ij}^s(t,x)$ be the leading coefficients of the operator $\mathcal{L}_s$, $i,j = 1, \ldots, n$, and let

$$\delta^s = \sup_{Q_T} \left(\sum_{i,j=1}^n (a_{ij}^s(t,x) - \delta_{ij})^2 \right)^{1/2}.$$

We have

$$\delta^s = \sup_{Q_T} \left(\sum_{i,j=1}^n (s a_{ij}^s(t,x) + (1-s)\delta_{ij} - \delta_{ij})^2 \right)^{1/2} = s \sup_{Q_T} \left(\sum_{i,j=1}^n (a_{ij}^s(t,x) - \delta_{ij})^2 \right)^{1/2} = s\delta \le \delta.$$

In addition, if $b_i^s(t,x)$, $i,j = 1, \ldots, n$, and $c_s(t,x)$ are minor coefficients of the operator $\mathcal{L}_s$, then the quantity $\sum_{i=1}^n \|b_i^s(t,x)\|_{L_{n+2}(Q_T)} + \|c_s(t,x)\|_{L_m(Q_T)}$ is majorized by a constant depending only on $\sum_{i=1}^{n} \|b_i\|_{L_{n+2}(Q_T)} + \|c_s(t,x)\|_{L_m(Q_T)}$. Hence, the assertion of Theorem 1 is true for the operator $\mathcal{L}_s$ with constant C'_{30} independent of s. Denote by $E = \{s\colon s \in [0,1]\}$ the set in which the problem has a solution. Note that, by virtue of Theorem 2, this solution is unique. We now show that the set E is nonempty and open and closed simultaneously relative to $[0,1]$. Then

$$\mathcal{L}_{s^0} u = f(t,x), \quad (t,x) \in Q_T, \qquad u(t,x) \in \hat{W}_p^{2,1}(Q_T), \tag{32}$$

coincides with the segment $[0,1]$ and, in particular, problem (32) is identically solvable for $s = 1$ when $\mathcal{L}_1 = \mathcal{L}$. In this case, estimate (31) follows from the fact that problem (32) is solvable for $s = 0$ (see [1]).

We now show that the set E is open relative to $[0,1]$. Let $s^0 \in E$, $s \in [0,1]$ be such that $|s - s^0| < \alpha$, where $\alpha > 0$ will be chosen later. We represent problem (32) as

$$\mathcal{L}_{s^0} u = f(t,x) + (\mathcal{L}_{s^0} - \mathcal{L}_s)u, \quad (t,x) \in Q_T, \qquad u(t,x) \in \hat{W}_p^{2,1}(Q_T). \tag{33}$$

It is easy to see that $\mathcal{L}_{s^0} - \mathcal{L}_s = (s^0 - s)(\mathcal{L} - \mathcal{M}_0)$. Consider an auxiliary problem

$$\mathcal{L}_{s^0} u = f(t,x) + (s^0 - s)(\mathcal{L} - \mathcal{M}_0)v, \quad (t,x) \in Q_T, \qquad u(t,x) \in \hat{W}_p^{2,1}(Q_T), \tag{34}$$

where $v(t,x) \in \hat{W}_p^{2,1}(Q_T)$. Acting as in Theorem 1, we can show that

$$\|(\mathcal{L} - \mathcal{M}_0)v\|_{L_p(Q_T)} \leq C_{31}(\mathcal{L}, n)\|v\|_{W_p^{2,1}(Q_T)}.$$

Thus, the operator $\mathcal{M}$ associating every function $v(t,x) \in \hat{W}_p^{2,1}(Q_T)$ with a solution $u(t,x)$ of the problem (34) is determined, i.e., $u = \mathcal{M}v$. We now show that if α is chosen in a certain way, then the operator $\mathcal{M}$ becomes contractive. Let $u^1 = \mathcal{M}v^1$ and $u^2 = \mathcal{M}v^2$. We have

$$\mathcal{L}_{s^0}(u^1 - u^2) = (s^0 - s)(\mathcal{L} - \mathcal{M}_0)(v^1 - v^2), \qquad u^1 - u^2 \in \hat{W}_p^{2,1}(Q_T).$$

Thus, according to Theorem 1

$$\|u^1 - u^2\|_{W_p^{2,1}(Q_T)} \leq C_{30}\alpha C_{31}\|v^1 - v^2\|_{W_p^{2,1}(Q_T)},$$

and it is sufficient to choose $\alpha = \dfrac{1}{2C_{30}C_{31}}$. Then the operator $\mathcal{M}$ has a fixed point $u = \mathcal{M}v$. However, for $v = u$, problem (34) coincides with problem (33), i.e., with (32). The openness of the set E is proved. We now prove that it is closed. Let $s^m \in E$, $m = 1, 2, \ldots$, $s^0 = \lim_{m\to\infty} s^m$. We show that $s^0 \in E$. Denote by $u^m(t,x)$ the solution of the boundary-value problem

$$\mathcal{L}_{s^m} u^m = f(t,x), \qquad (t,x) \in Q_T, \quad u^m \in \hat{W}_p^{2,1}(Q_T).$$

According to Theorem 1,

$$\|u^m\|_{W_p^{2,1}(Q_T)} \leq C_{30}\|f\|_{L_p(Q_T)}.$$

Thus, the sequence $\{u^m(t,x)\}$ is bounded in the norm $W_p^{2,1}(Q_T)$. Hence, it is weakly compact, i.e., there exists a subsequence $m_k \to \infty$ as $k \to \infty$ and a function $u(t,x) \in \hat{W}_p^{2,1}(Q_T)$ such that $u(t,x)$ is the weak limit of the subsequence $\{u^{m_k}(t,x)\}$ as $k \to \infty$ in $\hat{W}_p^{2,1}(Q_T)$. Therefore, in particular, we conclude that, for any function $\hat{W}_p^{2,1}(Q_T)$,

$$\langle \mathcal{L}_{s^0} u^{m_k}, \varphi\rangle \to \langle \mathcal{L}_{s^0}, \varphi\rangle, \quad k\to\infty,$$

where $\langle u, v\rangle = \int_{Q_T} uvdtdx$. But

$$\langle \mathcal{L}_{s^0} u^{m_k}, \varphi\rangle = \langle (\mathcal{L}_{s^0} - \mathcal{L}_{s^{m_k}})u^{m_k}, \varphi\rangle + \langle \mathcal{L}_{s^{m_k}} u^{m_k}, \varphi\rangle = i_1 + i_2.$$

We have

$$|i_1| \le |s^0 - s^{m_k}|\,|\langle (\mathcal{L} - \mathcal{M}_0)u^{m_k}, \varphi\rangle| \le |s^0 - s^{m_k}|\, C_{32}(\varphi, p) C_{31} \|u^{m_k}\|_{W_p^{2,1}(Q_T)} \le$$

$$\le C_{30}C_{32}C_{31}|s^0 - s^{m_k}|\|f\|_{L_p(Q_T)}.$$

Thus, $i_1 \to 0$ as $k\to\infty$. On the other hand, $i_2 = \langle f, \varphi\rangle$. Hence, for any function $\varphi(t,x) \in \in \overset{\circ}{W}{}_p^{2,1}(\bar{Q}_T)$,

$$\langle \mathcal{L}_{s^0} u, \varphi\rangle = \langle f, \varphi\rangle.$$

This means that $\mathcal{L}_{s^0} u = f(t,x)$ almost everywhere in Q_T, i.e., $s^0 \in E$.

The theorem is proved.

***Remark* 2.** For $p = 2$ and the operator $\mathcal{L}$, Theorem 2 is correct without the assumption $T \le T_0$ (see [11]).

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