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A GENERALIZATION OF LIFTING MODULES

УЗАГАЛЬНЕНННЯ ПІДЙОМНИХ МОДУЛІВ

We introduce the notion of $\mathcal{I}$-lifting modules as a proper generalization of the notion of lifting modules and give some properties of this class of modules. It is shown that if M is an $\mathcal{I}$-lifting direct projective module, then S/∇ is regular and $\nabla = JacS$, where S is the ring of all R-endomorphisms of M and $\nabla = \{\phi \in S \mid \operatorname{Im}\phi \ll M\}$. Moreover, we prove that if M is a projective $\mathcal{I}$-lifting module, then M is a direct sum of cyclic modules. The connections between $\mathcal{I}$-lifting modules and dual Rickart modules are given.

Введено поняття $\mathcal{I}$-підйомних модулів як природне узагальнення підйомних модулів. Наведено деякі властивості цього класу модулів. Показано, що якщо M — прямий проективний модуль $\mathcal{I}$-підйому, то S/∇ є регулярною і $\nabla = JacS$, де S — кільце всіх R-ендоморфізмів M, а $\nabla = \{\phi \in S \mid \operatorname{Im}\phi \ll M\}$. Більш того, доведено, що якщо M — проективний $\mathcal{I}$-підйомний модуль, то M є прямою сумою циклічних модулів. Встановлено зв'язки між $\mathcal{I}$-підйомними модулями та дуальними модулями Рікарта.

1. Introduction. Throughout this paper, R will denote an arbitrary associative ring with identity, M a unitary right R-module and $S = \operatorname{End}_R(M)$ the ring of all R-endomorphisms of M. We will use the notation $N \ll M$ to indicate that N is small in M (i.e., $L+N \neq M\ \forall L \lneq M$). The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. $N \trianglelefteq M$ means that N is a fully invariant submodule of M (i.e., $\phi(N) \subseteq N\ \forall \phi \in \operatorname{End}_R(M)$). We denote $D_S(N) = \{\phi \in S \mid \operatorname{Im}\phi \subseteq N\}$ for $N \subseteq M$.

We recall that L is a *cosmall submodule of* K *in* M (denoted by $L \overset{cs}{\hookrightarrow} K$ in M) if $K/L \ll M/L$. Recall that a submodule L of M is called *coclosed* if L has no proper cosmall submodule. It is clear that every direct summand of M is a coclosed submodule of M. A module M is called *lifting* if for every $A \leq M$, there exists a direct summand B of M such that $B \subseteq A$ and $A/B \ll M/B$ [2].

A number of results concerning lifting modules have appeared in the literature in recent years and many generalizations of the concept of lifting modules have been introduced and studied by several authors (see [7–9, 17]). Motivated by the definition of a lifting module, we say that a module M is $\mathcal{I}$*-lifting* if for every $\phi \in \operatorname{End}_R(M)$, there exists a direct summand N of M such that $N \subseteq \operatorname{Im}\phi$ and $\operatorname{Im}\phi/N \ll M/N$. It is obvious that every lifting module is $\mathcal{I}$-lifting. In this note, we study some properties of $\mathcal{I}$-lifting modules. In Section 2, as we state in the abstract, we show that if M is a direct projective module, then M is $\mathcal{I}$-lifting if and only if S_S is f-lifting and $\nabla = JacS$, where $\nabla = \{\phi \in S \mid \operatorname{Im}\phi \ll M\}$ (Corollary 2.3). Moreover, we prove that if M is a projective $\mathcal{I}$-lifting module, then M is a direct sum of cyclic modules (Theorem 2.6).

The notion of right Rickart rings (or right p.p. rings) initially appeared in Maeda [14, p. 510] and was further studied by a number of authors [1, 3–5]. A ring R is called *right Rickart* if the right annihilator of any single element of R is generated by an idempotent as a right ideal. The notion of Rickart modules was introduced by Rizvi and Roman in [16], and was studied recently (see [10, 12, 13]). A module M is said to be *Rickart* if, for every $\phi \in \operatorname{End}_R(M)$, $\operatorname{Ker}\phi \leq^{\oplus} M$. It is clear that for $M = R_R$, the notion of a Rickart module coincides with that of a right Rickart ring. Lee, Rizvi and Roman investigate the dual notion of Rickart modules in [11]. A module M is called *dual Rickart* if for every $\phi \in \operatorname{End}_R(M)$, $\operatorname{Im}\phi \leq^{\oplus} M$. It is easy to see that every dual Rickart module is $\mathcal{I}$-lifting. In Section 3, we investigate the connection between dual Rickart modules and $\mathcal{I}$-lifting modules. It is shown that M is dual Rickart if and only if M is $\mathcal{I}$-lifting and $\mathcal{T}$-noncosingular (Corollary 3.1). We prove that if R is a right V-ring, then an R-module M is dual Rickart if and only if M is $\mathcal{I}$-lifting (Corollary 3.2).

2. $\mathcal{I}$-lifting modules.

Definition 2.1. *A module M is called $\mathcal{I}$-lifting if for every $\phi \in \mathrm{End}_R(M)$, there exists a direct summand N of M such that $N \subseteq \mathrm{Im}\,\phi$ and $\mathrm{Im}\,\phi/N \ll M/N$.*

It is clear that every lifting module is $\mathcal{I}$-lifting, while the converse in not true (the $\mathbb{Z}$-module $\mathbb{Q}$ is $\mathcal{I}$-lifting but it is not lifting).

Proposition 2.1. *The following conditions are equivalent for a module M:*

(1) *M is a $\mathcal{I}$-lifting module.*

(2) *For every $\phi \in S$ there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq \mathrm{Im}\,\phi$ and $M_2 \cap \mathrm{Im}\,\phi \ll M_2$.*

(3) *For every $\phi \in S$, $\mathrm{Im}\,\phi$ can be written as $\mathrm{Im}\,\phi = N \oplus S$ such that $N \leq^{\oplus} M$ and $S \ll M$.*

Proof. It follows from [2] (22.1).

Definition 2.2. *A module M is called a N-$\mathcal{I}$-lifting module if for every homomorphism $\phi\colon M \to N$, there exists $L \leq^{\oplus} N$ such that $L \subseteq \mathrm{Im}\,\phi$ and $\mathrm{Im}\,\phi/L \ll N/L$.*

In view of the above definition, a right module M is $\mathcal{I}$-lifting if and only if M is M-$\mathcal{I}$-lifting.

Theorem 2.1. *Let M and N be right R-modules. Then M is N-$\mathcal{I}$-lifting if and only if for all direct summands $M' \leq^{\oplus} M$ and coclosed submodule N' of N, M' is N'-$\mathcal{I}$-lifting.*

Proof. Let $M' = eM$ for some $e^2 = e \in \mathrm{End}_R(M)$, and N' be a coclosed submodule of N. Assume that $\psi \in \mathrm{Hom}(M', N')$. Since $\psi eM = \psi M' \subseteq N' \subseteq N$ and M is N-$\mathcal{I}$-lifting, there exists a decomposition $N = N_1 \oplus N_2$ such that $N_1 \subseteq \mathrm{Im}\,\psi e$ and $N_2 \cap \mathrm{Im}\,\psi e \ll N_2$. As $N_1 \subseteq \mathrm{Im}\,\psi e \subseteq N'$, $N' = N_1 \oplus (N_2 \cap N')$. By [2] (3.7(3)), $N_2 \cap N' \cap \mathrm{Im}\,\psi \ll N'$. Again by [2] (3.7(3)), $N' \cap N_2 \cap \mathrm{Im}\,\psi \ll N_2 \cap N'$. Therefore M' is N'-$\mathcal{I}$-lifting. The converse is clear.

***Corollary* 2.1.** *The following conditions are equivalent for a module M:*

(1) *M is an $\mathcal{I}$-lifting module.*

(2) *For any coclosed submodule N of M, every direct summand L of M is N-$\mathcal{I}$-lifting.*

***Corollary* 2.2.** *Every direct summand of an $\mathcal{I}$-lifting module is $\mathcal{I}$-lifting.*

An R-module M is called *$\mathcal{T}$-noncosingular* if, $\forall\phi \in \mathrm{End}_R(M)$, $\mathrm{Im}\,\phi \ll M$ implies that $\phi = 0$ [18].

Proposition 2.2. *The following conditions are equivalent for a $\mathcal{T}$-noncosingular module M:*

(1) *M is an indecomposable $\mathcal{I}$-lifting module.*

(2) *Every nonzero endomorphism $\phi \in S$ is an epimorphism.*

Proof. Let M be an indecomposable $\mathcal{I}$-lifting module. Assume that $0 \neq \phi \in \mathrm{End}_R(M)$. Then there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq \mathrm{Im}\,\phi$ and $M_2 \cap \mathrm{Im}\,\phi \ll M_2$. Since M is indecomposable, $M_1 = 0$ or $M_1 = M$. If $M_1 = 0$, then $\mathrm{Im}\,\phi \ll M$. By $\mathcal{T}$- noncosingularity, $\phi = 0$, a contradiction. Thus $M_1 = M$ and so ϕ is an epimorphism. The converse follows easily.

Recall that a module M is said to be *Hopfian* if every epimorphism $\phi \in \mathrm{End}_R(M)$ is an isomorphism.

Proposition 2.3. *Let M be a $\mathcal{T}$-noncosingular Noetherian $\mathcal{I}$-lifting module. Then there exists a decomposition $M = M_1 \oplus \ldots \oplus M_n$, where M_i is an indecomposable Noetherian $\mathcal{I}$-lifting module with $\mathrm{End}_R(M_i)$ a division ring.*

Proof. Since M is Noetherian, it has a finite decomposition with indecomposable Noetherian direct summands. By Corollary 2.2, every direct summand is $\mathcal{I}$-lifting. By Proposition 2.2 and since every Noetherian module is Hopfian, each indecomposable direct summand has a division ring.

An R-module M is called *direct projective* if, for every direct summand X of M, every epimorphism $M \to X$ splits. A module M is called *finitely lifting*, or *f-lifting* for short, if for every finitely generated submodule A of M, there exists a direct summand B of M such that $B \subseteq A$ and $A/B \ll M/B$.

Proposition 2.4. *Let M be an $\mathcal{I}$-lifting direct projective module. Then*:

(1) *S/∇ is regular and $\nabla = JacS$, where $\nabla = \{\phi \in S \mid \operatorname{Im}\phi \ll M\}$.*

(2) *S_S is f-lifting.*

Proof. (1) Let f be an arbitrary element of S. As M is $\mathcal{I}$-lifting, there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq \operatorname{Im} f$ and $\operatorname{Im} f \cap M_2 \ll M$. Let π denote the projection $M_1 \oplus M_2 \to M_1$. Since $\pi f \colon M \to M_1$ is an epimorphism and M is direct projective, $\operatorname{Ker}\pi f \leq^{\oplus} M$. So $M = \operatorname{Ker}\pi f \oplus U$ for some $U \subseteq M$. The restriction of πf to U is an isomorphism onto M_1 and the inverse isomorphism of M_1 to U can be extended to an element $g \in S$. Note that $g\pi f = 1_U$. Now $(f - fg\pi f)M = (f - fg\pi f)(\operatorname{Ker}\pi f \oplus U) = f(\operatorname{Ker}\pi f) \leq f(M) \cap M_2$. Hence $(f - fg\pi f)M \ll M$ and so $f - fg\pi f \in \nabla$. Therefore S/∇ is a regular ring. It follows that $JacS \subseteq \nabla$. Now we want to show that $\nabla \subseteq JacS$. Let $f \in \nabla$. Since $M = fM + (1-f)M$ and $\operatorname{Im} f \ll M$, $(1-f)M = M$. As M is direct projective, $1 - f$ is right invertible. But ∇ is an ideal, so $\nabla \subseteq JacS$.

(2) Let $f \in S$ and fS be an arbitrary cyclic right ideal of S. Consider the proof of (1). Set $h = fg\pi \in S$. It is clear that $h^2 = h$ and $hS \subseteq fS$. Since $(f - fg\pi f)M \ll M$, $(f - fg\pi f) \in \nabla = JacS$. So $(1-h)fS \ll S$. By [2] (22.7), S_S is f-lifting.

***Corollary* 2.3.** *Let M be a direct projective module. Then M is $\mathcal{I}$-lifting if and only if S_S is f-lifting and $\nabla = JacS$.*

Proof. Let M be an $\mathcal{I}$-lifting direct projective module. Then, by Proposition 2.4, S_S is f-lifting and $\nabla = JacS$. Conversely, let S_S be f-lifting and $\nabla = JacS$. Assume that $f \in S$. Then there exists an idempotent $e \in S$ such that $eS \subseteq fS$ and $(1-e)fS \subseteq JacS = \nabla$. Therefore $M = eM + (1-e)M$, $eM \subseteq fM$ and $(1-e)fM \ll M$.

Let K and N be submodules of M. K is called a *supplement* of N in M if $M = K + N$ and K is minimal with respect to this property, or equivalently, $M = K + N$ and $K \cap N \ll K$. A module M is called *supplemented* if every submodule of M has a supplement in M. We say a module M is *$\mathcal{I}$-supplemented* if for every $f \in S$, $\operatorname{Im} f$ has a supplement in M. Recall that a submodule U of the R-module M has *ample supplements* in M if, for every $V \subseteq M$ with $M = U + V$, there is a supplement V' of U with $V' \subseteq V$. We call M *amply $\mathcal{I}$-supplemented* if, for every $f \in S$, $\operatorname{Im} f$ has ample supplements in M.

Proposition 2.5. *Let M be an amply $\mathcal{I}$-supplemented R-module. Then every direct summand of M is amply $\mathcal{I}$-supplemented.*

Proof. Let V be a direct summand of M. Then $M = V \oplus U$ for some $U \subseteq M$. Assume that $f \in \operatorname{End}_R(V)$ and $V = \operatorname{Im} f + X$. Thus $M = \operatorname{Im} f + X + U$. Note that $\operatorname{Im} f = \operatorname{Im}\iota f\pi$, where ι is the injection map from V to M and π is the projection map from M onto V. Since M is amply $\mathcal{I}$-supplemented, there exists a supplement Y of $U + X$ with $Y \subseteq \operatorname{Im} f$. We get $X \cap Y \subseteq (U + X) \cap Y \ll Y$ and $M = Y + X + U$. Thus $X + Y = V$ and $X \cap Y \ll Y$. Therefore V is amply $\mathcal{I}$-supplemented.

Proposition 2.6. *Let M be an amply $\mathcal{I}$-supplemented module and let for every supplement submodule X of M we have $X \leq^{\oplus} M$, then M is $\mathcal{I}$-lifting.*

Proof. Let $f \in S$ and V be a supplement of $\operatorname{Im} f$ in M and X a supplement of V in M with $X \subseteq \operatorname{Im} f$. By hypothesis, $M = X \oplus X'$ for some $X' \leq M$. Since $\operatorname{Im} f \cap V \ll M$, this X' is a supplement of $X + (\operatorname{Im} f \cap V) = \operatorname{Im} f$ (see [19] (41.1)). Hence $\operatorname{Im} f \cap X' \ll X'$.

Proposition 2.7. *Let M be an $\mathcal{I}$-lifting module and N be a submodule of M invariant under all maps $f \in \operatorname{End}_R(M)$ with $\operatorname{Im} f$ a direct summand of M. Then N is a fully invariant submodule of M.*

Proof. Let $f \in \mathrm{End}_R(M)$. Since M is $\mathcal{I}$-lifting, $M = M_1 \oplus M_2$ where $M_1 \subseteq \mathrm{Im}\, f$ and $M_2 \cap \mathrm{Im}\, f \ll M_2$. We have $\mathrm{Im}\, f = M_1 \oplus (M_2 \cap \mathrm{Im}\, f)$. Consider the projection maps $\pi_{M_1} : M \to M$ and $\pi_{M_2} : M \to M$ of M onto M_1 and M_2, respectively. Note that $\mathrm{Im}(\pi_{M_1} f) = M_1$ is a direct summand of M. By hypothesis, $\pi_{M_1} f(N) \subseteq N$. As $\mathrm{Im}(\pi_{M_2} f) \ll M$, we have $\mathrm{Im}(1_M - \pi_{M_2} f) = M$. By assumption, $(1_M - \pi_{M_2} f)(N) \subseteq N$. So $\pi_{M_2} f(N) \subseteq N$. Therefore $f(N) \subseteq N$.

An element a of the ring R is called (*von Neumann*) *regular* if $axa = a$ for some $x \in R$.

***Corollary* 2.4.** *Suppose that M is a direct projective $\mathcal{I}$-lifting module and $N \subseteq M$. Then the following are equivalent*:

(1) *N is invariant under all (von Neumann) regular elements of* $\mathrm{End}_R(M)$;

(2) *N is invariant under all $f \in \mathrm{End}_R(M)$ with $\mathrm{Im}\, f$ a direct summand of M*;

(3) *N is a fully invariant submodule of M.*

Proof. (1) $\Rightarrow$ (2). Let $f : M \to M$ be any homomorphism with $\mathrm{Im}\, f$ a direct summand of M. Since M is direct projective, $\mathrm{Ker}\, f$ is also direct summand. By [19] (37.7), f is a (von Neumann) regular element of $\mathrm{End}_R(M)$.

(2) $\Rightarrow$ (3). By Proposition 2.7, N is a fully invariant submodule of M.

(3) $\Rightarrow$ (1). It is clear.

A module M is called a *N-$\mathcal{I}$-supplemented* module if for every homomorphism $\phi : M \to N$, there exists $L \leq N$ such that $\mathrm{Im}\, \phi + L = N$ and $\mathrm{Im}\, \phi \cap L \ll L$. It is clear that a right module M is $\mathcal{I}$-supplemented if and only if M is M-$\mathcal{I}$-supplemented.

Theorem 2.2. *Let M_1, M_2 and N be modules. If N is M_i-$\mathcal{I}$-supplemented for $i = 1, 2$, then N is $M_1 \oplus M_2$-$\mathcal{I}$-supplemented. The converse is true if $M_1 \oplus M_2$ is a duo module.*

Proof. Suppose N is M_i-$\mathcal{I}$-supplemented for $i = 1, 2$. We will prove that N is $M_1 \oplus M_2$-$\mathcal{I}$-supplemented. Let $\phi = (\pi_1 \phi, \pi_2 \phi)$ be any homomorphism from N to $M_1 \oplus M_2$, where π_i is the projection map from $M_1 \oplus M_2$ to M_i for $i = 1, 2$. Since N is M_i-$\mathcal{I}$-supplemented, there exists a submodule K_i of M_i such that $\pi_i \phi N + K_i = M_i$ and $\pi_i \phi N \cap K_i \ll K_i$, for $i = 1, 2$. Let $K = K_1 \oplus K_2$. Then $M_1 \oplus M_2 = \pi_1 \phi N + \pi_2 \phi N + K_1 + K_2 = \phi N + K$. Since $\phi N \cap (K_1 + K_2) \leq$ $\leq (\phi N + K_1) \cap K_2 + (\phi N + K_2) \cap K_1$, we have $\phi N \cap (K_1 + K_2) \leq (\phi N + M_1) \cap K_2 + (\phi N + M_2) \cap K_1$. As $\phi N + M_1 = \pi_2 \phi N \oplus M_1$ and $\phi N + M_2 = \pi_1 \phi N \oplus M_2$, thus $\phi N \cap K \subseteq (\pi_2 \phi N \cap K_2) + (\pi_1 \phi N \cap$ $\cap K_1)$. Since $\pi_i \phi N \cap K_i \ll K_i$ for $i = 1, 2$, $\phi N \cap K \ll K_1 + K_2 = K$. Hence N is $M_1 \oplus M_2$-$\mathcal{I}$-supplemented. Conversely, let N be $M_1 \oplus M_2$-$\mathcal{I}$-supplemented. Let ϕ be a homomorphism from N to M_1. Then $\mathrm{Im}\, \iota\phi = \mathrm{Im}\, \phi$, where ι is the canonical inclusion from M_1 to $M_1 \oplus M_2$. Since N is $M_1 \oplus M_2$-$\mathcal{I}$-supplemented, there exists $K \subseteq M_1 \oplus M_2$ such that $M_1 \oplus M_2 = \mathrm{Im}\, \phi + K$ and $\mathrm{Im}\, \phi \cap K \ll K$. Thus $M_1 = \mathrm{Im}\, \phi + (K \cap M_1)$ and $\mathrm{Im}\, \phi \cap K \cap M_1 = \mathrm{Im}\, \phi \cap K \ll K$. As $M_1 \oplus M_2$ is a duo module, $K \trianglelefteq M_1 \oplus M_2$ and so $K \cap M_1$ is a direct summand of K. Hence $\mathrm{Im}\, \phi \cap K \cap M_1 \ll (K \cap M_1)$. Therefore N is M_1-$\mathcal{I}$-supplemented.

***Corollary* 2.5.** *Suppose $M = M_1 \oplus M_2$ and M is M_i-$\mathcal{I}$-supplemented module for $i = 1, 2$. Then*:

(1) *M is $\mathcal{I}$-supplemented and for every $f \in S$, $\mathrm{Im}\, f$ has a supplement of the form $K_1 + K_2$ with $K_1 \subseteq M_1$ and $K_2 \subseteq M_2$.*

(2) *Let $f \in S$ and $\mathrm{Im}\, f$ be a supplement submodule of M. Then K_1 and $\mathrm{Im}\, f + K_2$ are mutual supplements in M and the same is true for K_2 and $\mathrm{Im}\, f + K_1$.*

Proof. (1) By using the proof of Theorem 2.2.

(2) Let $f \in S$ and $\mathrm{Im}\, f$ be a supplement submodule of M. Consider the proof of Theorem 2.2. Then, by [2] (20.2(c)), since $\mathrm{Im}\, f \cap (K_1 + K_2) \ll M$ we have $\mathrm{Im}\, f \cap (K_1 + K_2) \ll \mathrm{Im}\, f$.

Hence $K_1 \cap (\operatorname{Im} f + K_2) \subseteq [\operatorname{Im} f \cap (K_1 + K_2)] + [K_2 \cap (\operatorname{Im} f + K_1)] \ll \operatorname{Im} f + K_2$. Similarly, $K_2 \cap (\operatorname{Im} f + K_1) \ll \operatorname{Im} f + K_1$.

A module K is said to be *generalized M-projective* if, for any epimorphism $g\colon M \to X$ and morphism $f: K \to X$, there exist decompositions $K = K_1 \oplus K_2$, $M = M_1 \oplus M_2$, a morphism $h_1 : K_1 \to M_1$ and an epimorphism $h_2 \colon M_2 \to K_2$, such that $h_1 g = f|_{K_1}$ and $h_2 f = g|_{M_2}$.

Lemma 2.1. *Assume that M is a module, $N \leq^{\oplus} M$ and $N = K \oplus L$. Let M be a L-$\mathcal{I}$-lifting module. If K is generalized L-projective, then for every homomorphism $f\colon M \to N$ such that $N = \operatorname{Im} f + L$ and $\pi(\operatorname{Im} f) = \operatorname{Im} f \cap \operatorname{Im} \pi$ where π is an arbitrary projection map of N, there exist $X \overset{cs}{\hookrightarrow} \operatorname{Im} f$ in N, $K' \subseteq K$ and $L' \subseteq L$ such that $N = X \oplus K' \oplus L'$.*

Proof. Let $f: M \to N$ be a homomorphism such that $N = \operatorname{Im} f + L$. Consider the homomorphism $\pi_L f : M \to L$, where π_L is the projection map from N onto L. Note that $\operatorname{Im} \pi_L f = \operatorname{Im} f \cap L$ by hypothesis. Since M is L-$\mathcal{I}$-lifting, there exists a decomposition $L = L_1 \oplus L_2$ such that $L_2 \subseteq L \cap \operatorname{Im} f$ and $(L \cap \operatorname{Im} f) \cap L_1 = L_1 \cap \operatorname{Im} f \ll L_1$. Thus we get $N = \operatorname{Im} f + L = \operatorname{Im} f + L_1$ and $L_1 \cap \operatorname{Im} f \ll L_1$. As $L_2 \subseteq \operatorname{Im} f$, $\operatorname{Im} f = L_2 \oplus ((L_1 \oplus K) \cap \operatorname{Im} f)$. Set $U = L_1 \oplus K$. Since $N = \operatorname{Im} f + L_1$, $U = (U \cap \operatorname{Im} f) + L_1$. By [2] (4.43 and 4.42), there exists a decomposition $U = T \oplus K' \oplus L_1' = T + L_1$ with $T \subseteq U \cap \operatorname{Im} f$, $K' \subseteq K$ and $L_1' \subseteq L_1$. As $T \subseteq U \cap \operatorname{Im} f$ and $\operatorname{Im} f = L_2 + (U \cap \operatorname{Im} f)$, we have $L_2 \oplus T \subseteq \operatorname{Im} f$. Since $N = (L_2 + T) + L_1$ and $\operatorname{Im} f \cap L_1 \ll N$, we have, by [2] (3.2(6)), that $(L_2 \oplus T) \overset{cs}{\hookrightarrow} \operatorname{Im} f$ in N. As $N = L_2 \oplus U$.

Lemma 2.1 is proved.

Theorem 2.3. *Suppose $M = M_1 \oplus M_2$ and M is M_i-$\mathcal{I}$-lifting for $i = 1, 2$. Let M_1 and M_2 be relatively generalized projective modules. Then for every $f \in S$, $\operatorname{Im} f$ is a direct summand of M if $\operatorname{Im} f$ is a coclosed submodule of M and $\pi(\operatorname{Im} f) = \operatorname{Im} f \cap \operatorname{Im} \pi$, where π is any projection map of M. Moreover, $M = M_1 \oplus M_2$ is an exchange decomposition.*

Proof. Let $f \in S$ such that $\operatorname{Im} f$ is a coclosed submodule of M. Since M is M_i-$\mathcal{I}$-supplemented, M is a $\mathcal{I}$-supplemented module and $\operatorname{Im} f$ has a supplement $M_1' \oplus M_2'$, where $M_1' \subseteq M_1$ and $M_2' \subseteq M_2$ (see Corollary 2.5). As M is $\mathcal{I}$-supplemented, the coclosed images of M are precisely the supplement images and $\operatorname{Im} f + M_1'$ and $\operatorname{Im} f + M_2'$ are supplement submodules of M (see Corollary 2.5 again). Since M is M_2-$\mathcal{I}$-lifting, M_1 is generalized M_2-projective and $\operatorname{Im} f + M_1'$ is a supplement, it follows that there exists a decomposition $M = (\operatorname{Im} f + M_1') \oplus M_1'' \oplus M_2''$, with $M_1'' \subseteq M_1$ and $M_2'' \subseteq M_2$ (see Lemma 2.1). Set $U = M_1 + \operatorname{Im} f$ and $N = M_1'' \oplus M_2''$. Then $M = U \oplus N$ and $M/\operatorname{Im} f = U/\operatorname{Im} f \oplus (N + \operatorname{Im} f)/\operatorname{Im} f$. Hence $(N + \operatorname{Im} f)/\operatorname{Im} f$ is a supplement of $M/\operatorname{Im} f$. By [2] (20.5(2)), $N + \operatorname{Im} f$ is a supplement of M because $\operatorname{Im} f$ is a supplement of M. Since $N + \operatorname{Im} f = \operatorname{Im} f \oplus M_1'' \oplus M_2''$, by [2] (20.5(1)), $\operatorname{Im} f \oplus M_2''$ is a supplement of M and $M = \operatorname{Im} f + M_2'' + M_1' + M_1'' \subseteq (\operatorname{Im} f \oplus M_2'') + M_1$. By using Lemma 2.1 again, we have $M = (\operatorname{Im} f \oplus M_2'') \oplus M_1^* \oplus M_2^*$, with $M_1^* \subseteq M_1$ and $M_2^* \subseteq M_2$. Hence $M = \operatorname{Im} f \oplus M_1^* \oplus (M_2^* \oplus M_2'')$, where $M_1^* \subseteq M_1$ and $M_2^* \oplus M_2'' \subseteq M_2$. Therefore $\operatorname{Im} f \leq^{\oplus} M$ and since any direct summand of M is a coclosed epimorphic image of M, $M = M_1 \oplus M_2$ is an exchange decomposition.

Theorem 2.4. *Let M_1 and M_2 be modules and $M = M_1 \oplus M_2$ such that M is M_i-$\mathcal{I}$-lifting for $i = 1, 2$ and let for every $f \in S$ we have $\pi(\operatorname{Im} f) = \operatorname{Im} f \cap \operatorname{Im} \pi$, where π is any projection map of M. If any direct summand of M_1 is generalized M_2-projective and vice versa, then M is $\mathcal{I}$-lifting.*

Proof. Let $f \in S$. Since M is M_1-$\mathcal{I}$-lifting, there exists a decomposition $M = M_1' \oplus M_1''$ such that $M_1' \subseteq \operatorname{Im} f \cap M_1 = \operatorname{Im}(\pi_{M_1} f)$ and $\operatorname{Im}(\pi_{M_1} f) \cap M_1'' \ll M_1''$ (π will denote the obvious projections). Set $K_1 = \operatorname{Im} f \cap (M_1'' \oplus M_2)$. Note that $\operatorname{Im}(\pi_{(M_1'' \oplus M_2)} f) = K_1$. Since M is M_2-$\mathcal{I}$-lifting, there exists a decomposition $M_2 = M_2' \oplus M_2''$ such that $M_2' \subseteq \pi_{M_2}(K)$ and $M_2'' \cap \pi_{M_2}(K_1) \ll M_2''$. Setting $K_2 = (M_1'' \oplus M_2'') \cap \operatorname{Im} f$, we get that $\pi_{M_i''}(K_2) \ll M_i''$ for $i = 1, 2$, $M = \pi_{M_1}(\operatorname{Im} f) + M_1'' + M_2 =$

$= \operatorname{Im} f + M_1'' + M_2 = \operatorname{Im} f + M_1'' + \pi_{M_2}(K_1) + M_2'' = \operatorname{Im} f + M_1'' + K_1 + M_2'' = \operatorname{Im} f + (M_1'' \oplus M_2'')$ and $K_2 \subseteq \pi_{M_1''}(K_2) \oplus \pi_{M_2''}(K_2) \ll M_1'' \oplus M_2''$. By hypothesis and [2] (4.43), direct summands of M_1 and M_2 are relatively generalized projective and by Theorem 2.1, direct summands of M and M_i are relatively $\mathcal{I}$-lifting for $i = 1, 2$. Set $N = M_1' \oplus M_2''$. Then $N = M_2'' + [(\operatorname{Im} f + M_1'') \cap N]$. Define $L = (\operatorname{Im} f + M_1'') \cap N$. Then $N = L + M_2''$. Consider the endomorphism $g\colon M_1' \oplus M_1'' \oplus M_2' \oplus M_2'' \to$ $\to M_1' \oplus M_1'' \oplus M_2' \oplus M_2''$ defined by $g(m_1' + m_1'' + m_2' + m_2'') = f(m_1' + m_1'' + m_2' + m_2'') + m_1''$. Note that $\operatorname{Im} g = \operatorname{Im} f + M_1''$ and $L = \operatorname{Im} \pi_N g$. By using the Lemma 2.1, we have $N = U \oplus \widetilde{M_1'} \oplus \widetilde{M_2''}$, where $U \overset{cs}{\hookrightarrow} L$ in N, $\widetilde{M_1'} \subseteq M_1'$ and $\widetilde{M_2''} \subseteq M_2''$. By [2] (3.2(1)), we get $N = U + M_2''$. Let $M_1' = \overline{M_1'} \oplus \widetilde{M_1'}$ and $M_2'' = \overline{M_2''} \oplus \widetilde{M_2''}$. Now, we get $M = M_1'' \oplus M_2' \oplus U \oplus \widetilde{M_1'} \oplus \widetilde{M_2''}$. Next, set $T = U \oplus M_1'' \oplus M_2'$. Then T is a direct summand of M. Now $M_2' \subseteq \pi_{M_2}(K_1) \subseteq \operatorname{Im} f + M_1''$. Then $M = N \oplus M_1'' \oplus M_2'$ and $M_1'' \oplus M_2' \subseteq (\operatorname{Im} f + M_1'')$ imply $\operatorname{Im} f + M_1'' = L \oplus M_1'' \oplus M_2'$. Since $U \overset{cs}{\hookrightarrow} L$ in N and N is a direct summand of M, we have $T \overset{cs}{\hookrightarrow} (\operatorname{Im} f + M_1'')$ in M. Also $T = M_1'' + (\operatorname{Im} f \cap T)$ and $M = T + M_2''$. Hence $M = (\operatorname{Im} f \cap T) + M_1'' + M_2''$. As $\operatorname{Im} f =$ $= (\operatorname{Im} f \cap T) + \operatorname{Im} f \cap (M_1'' \oplus M_2'') = (\operatorname{Im} f \cap T) + K_2$ and $K_2 \ll M$, by [2] (3.2(6)), we get that $(\operatorname{Im} f \cap T) \overset{cs}{\hookrightarrow} \operatorname{Im} f$ in M. Now, set $A = M_1'' \oplus \overline{M_1'}$ and $B = M_2' \oplus \overline{M_2''}$. Then A and B are relatively generalized projective and M is A-$\mathcal{I}$-lifting. Let $\phi = \pi_{A \oplus B}|_T$. Then $\phi\colon T \to A \oplus B$ is an isomorphism and $A \oplus B = \phi(T) = \phi(\operatorname{Im} f \cap T) + M_1'' = \phi(\operatorname{Im} f \cap T) + A = \phi(\pi_T f(M)) + A$. Using the Lemma 2.1 again, then there exists a direct summand $T' \subseteq T$ such that $T' \overset{cs}{\hookrightarrow} (\operatorname{Im} f \cap T)$ in M. Since $(\operatorname{Im} f \cap T) \overset{cs}{\hookrightarrow} \operatorname{Im} f$ in M, we have $T' \overset{cs}{\hookrightarrow} \operatorname{Im} f$ in M by [2] (3.2(2)). Therefore M is an $\mathcal{I}$-lifting module.

***Corollary* 2.6.** *Let M be M_i-$\mathcal{I}$-lifting for $i \in \{1, 2, \ldots, n\}$ and put $M = M_1 \oplus \ldots \oplus M_n$. Assume that M_i' and T are relatively generalized projective for any direct summand M_i' of M_i and any direct summand T of $\oplus_{j \neq i} M_j$, for any $1 \leq i \leq n$ and let for every $f \in S$ we have $\pi(\operatorname{Im} f) = \operatorname{Im} f \cap \operatorname{Im} \pi$, where π is any projection map of M. Then M is $\mathcal{I}$-lifting.*

Proof. It follows from Theorem 2.4 by induction on n.

***Corollary* 2.7.** *Let M be M_i-$\mathcal{I}$-lifting for $i \in \{1, 2, \ldots, n\}$ and set $M = M_1 \oplus \ldots \oplus M_n$. Suppose that M_i and M_j are relatively projective for each $1 \leq i, j \leq n$, $i \neq j$ and let for every $f \in S$ we have $\pi(\operatorname{Im} f) = \operatorname{Im} f \cap \operatorname{Im} \pi$, where π is any projection map of M. Then M is $\mathcal{I}$-lifting.*

Theorem 2.5. *Let $M = \oplus_{i=1}^n M_i$ be a module and $M_i \trianglelefteq M$ for all $i \in \{1, \ldots, n\}$. Then M is an $\mathcal{I}$-lifting module if and only if M_i is $\mathcal{I}$-lifting for all $i \in \{1, \ldots, n\}$.*

Proof. The necessity follows from Theorem 2.1. Conversely, let M_i be an $\mathcal{I}$-lifting module for all $i \in \{1, \ldots, n\}$. Let $\phi = (\phi_{ij})_{i,j \in \{1,\ldots,n\}} \in \operatorname{End}_R(M)$ be arbitrary, where $\phi_{ij} \in \operatorname{Hom}(M_j, M_i)$. Since $M_i \trianglelefteq M$ for all $i \in \{1, \ldots, n\}$, $\operatorname{Im} \phi = \oplus_{i=1}^n \operatorname{Im} \phi_{ii}$. As M_i is $\mathcal{I}$-lifting, there exists a direct summand X_i of M_i and a submodule Y_i of M_i with $X_i \subseteq \operatorname{Im} \phi_{ii}$, $\operatorname{Im} \phi_{ii} = X_i + Y_i$ and $Y_i \ll M_i$. Set $X = \oplus_{i=1}^n X_i$, then X is a direct summand of M. Moreover, $\operatorname{Im} \phi = \oplus_{i=1}^n \operatorname{Im} \phi_{ii} =$ $= \sum_{i=1}^n X_i + \sum_{i=1}^n Y_i$ and $\oplus_{i=1}^n Y_i \ll \oplus_{i=1}^n M_i = M$. Therefore M is $\mathcal{I}$-lifting.

Proposition 2.8. *Let M be an $\mathcal{I}$-lifting projective module. Then $\operatorname{Rad}(M) \ll M$.*

Proof. Let $N \subseteq M$ be any submodule with $N + \operatorname{Rad}(M) = M$. Then $\operatorname{Rad}(M) \to M \to M/K$ is epimorphism and there exists $f\colon M \to \operatorname{Rad}(M)$ with $M = \operatorname{Im} f + N$. Since M is $\mathcal{I}$-lifting, there exists a decomposition $M = M_1 \oplus M_2$ with $M_1 \subseteq \operatorname{Im} f$ and $M_2 \cap \operatorname{Im} f \ll M_2$. Note that $M_1 \subseteq \operatorname{Rad}(M)$. By [19] (22.3), $M_1 = 0$ and so $\operatorname{Im} f \ll M$. Hence $N = M$. Therefore $\operatorname{Rad}(M) \ll M$.

Theorem 2.6. *Let M be a projective $\mathcal{I}$-lifting module. Then M is a direct sum of cyclic modules.*

Proof. If M is a projective module, then, by Kaplansky's Theorem [19] (8.10), M is a direct sum of countably generated module. Hence it is enough to prove the assertion for countably generated modules. We prove by induction. First, consider $M = \sum_{i\in\mathbb{N}} Rm_i$, $m_i \in M$. Since the canonical map $f\colon \oplus_{\mathbb{N}} Rm_i \to M$ splits, there exists $g\colon M \to \oplus_{\mathbb{N}} Rm_i$ with $gf = id_M$. Consider the morphisms $g_i = \pi_i g \in \operatorname{End}_R(M)$, where π_i is the canonical projection. Then $M = \sum_{\mathbb{N}} g_i(M)$. Since M is $\mathcal{I}$-lifting, there exists a decomposition $M = P_1 \oplus Q_1$ with $P_1 \subseteq \operatorname{Im} g_1$ and $K_1 = \operatorname{Im} g_1 \cap Q_1 \ll M$. So $\operatorname{Im} g_1 = P_1 + K_1$. Note that P_1 is cyclic because P_1 is a direct summand of Rm_1. Suppose, for $k \in \mathbb{N}$, we have found cyclic modules $P_i \subseteq M$ with $M = \left(\sum_{i\le k} P_i\right) \oplus Q_k$ and $\sum_{i\le k} \operatorname{Im} g_i = = (\oplus_{i\le k} P_i) + K_k$, $K_k \ll M$. Since M is $\mathcal{I}$-lifting, by Theorem 2.1, M is Q_k-$\mathcal{I}$-lifting. Hence there exists a decomposition $Q_k = P_{k+1} \oplus Q_{k+1}$, with $P_{k+1} \subseteq \operatorname{Im} g_{k+1}$, $\operatorname{Im} g_{k+1} = P_{k+1} + K'_{k+1}$, and $K'_{k+1} = \operatorname{Im} g_{k+1} \cap Q_{k+1} \ll M$. Thus we have $M = (\oplus_{i\le k+1} P_i) \oplus Q_{k+1}$ and $\sum_{i\le k+1} \operatorname{Im} g_i = = (\oplus_{i\le k+1} P_i) + K_{k+1}$ with $K_{k+1} = K'_{k+1} + K_k \ll M$. By Proposition 2.8, $\operatorname{Rad}(M) \ll M$ and so $\sum_{i\in\mathbb{N}} K_i \subseteq \operatorname{Rad}(M) \ll M$. Therefore $M = \sum_{i\in\mathbb{N}} \operatorname{Im} g_i = (\oplus_{i\in\mathbb{N}} P_i) + \sum_{i\in\mathbb{N}} K_i = \oplus_{i\in\mathbb{N}} P_i$.

A ring R is called *f-semiperfect* if, every finitely generated R-module has a projective cover. A module M is said to be *principally lifting* if, for every cyclic submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ and $N \cap M_2 \ll M_2$.

The following theorem gives a characterization of f-semiperfect rings.

Theorem 2.7. *The following are equivalent for a ring R:*

(1) *R is f-semiperfect*;

(2) *R_R is finitely supplemented*;

(3) *every cyclic right ideal has a supplement in R_R*;

(4) *R_R is $\mathcal{I}$-supplemented*;

(5) *R_R is principally lifting*;

(6) *R_R is $\mathcal{I}$-lifting.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) by [19] (42.11).

(3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6) are clear because $\operatorname{Im}\phi$ is cyclic for every $\phi \in \operatorname{End}_R(R_R)$.

(4) $\Rightarrow$ (3). Assume that $I = aR$ is any cyclic right ideal of R. Consider the R-homomorphism $\phi\colon R_R \to R_R$ defined by $\phi(r) = ar$, where $r \in R$. Then $\operatorname{Im}\phi = I$. By (4), $\operatorname{Im}\phi = I$ has a supplement in R_R.

(6) $\Rightarrow$ (5). It is similar to the proof of (4) $\Rightarrow$ (3).

3. Relation between dual Rickart modules and $\mathcal{I}$-lifting modules. It is clear that if M is a dual Rickart module, then M is $\mathcal{I}$-lifting, while the converse is not true (the $\mathbb{Z}$-module $\mathbb{Z}_4$ is $\mathcal{I}$-lifting but it is not dual Rickart).

Lemma 3.1. *Let M be a module. Then M is a dual Rickart module if and only if for every $g \in S$, there exists an idempotent $e \in S$ such that $D_S(\operatorname{Im} g) = eS$.*

Proof. Let M be a dual Rickart module and $g \in S$. Then there exists an idempotent $e \in S$ such that $\operatorname{Im} g = eM$. Hence $e \in D_S(\operatorname{Im} g)$ and so $eS \subseteq D_S(\operatorname{Im} g)$. Now if $f \in D_S(\operatorname{Im} g)$, then $\operatorname{Im} f \subseteq \operatorname{Im} g = eM$. Moreover, since $S = eS \oplus (1-e)S$, we have $f = es_1 + (1-e)s_2$ for some $s_1, s_2 \in S$. Since $\operatorname{Im} f \subseteq eM$, $f = es_1$. Therefore $f \in eS$ and so $eS = D_S(\operatorname{Im} g)$. Conversely, let for every $g \in S$, there exists an idempotent $e \in S$ such that $D_S(\operatorname{Im} g) = eS$. Then for $g \in S$, $eM \subseteq \operatorname{Im} g$. On the other hand, we have $g \in D_S(\operatorname{Im} g) = eS$. Thus there exists $s \in S$ such that $g = eS$. It follows that $\operatorname{Im} g \subseteq eM$. Therefore $\operatorname{Im} g = eM$.

Lemma 3.2. *An $\mathcal{T}$-noncosingular $\mathcal{I}$-lifting module M is a dual Rickart module.*

Proof. Let $g \in S$. Since M is $\mathcal{I}$-lifting, $\operatorname{Im} g = eM \oplus B$, where $e^2 = e \in S$ and $B \ll M$. Hence $eS = D_S(eM) \subseteq D_S(\operatorname{Im} g)$. Now, let $\phi \in D_S(\operatorname{Im} g)$. We want to show that $\phi \in eS$. Note that $M = eM \oplus (1-e)M$ and $\operatorname{Im} g \cap (1-e)M = (eM \oplus B) \cap (1-e)M \subseteq (1-e)B$. Since $B \ll M$, we have $(1-e)B \ll M$. Thus $\operatorname{Im} g \cap (1-e)M \ll M$. As $S = eS \oplus (1-e)S$, there exists s_1 and s_2 in S such that $\phi = es_1 + (1-e)s_2$. Thus $\operatorname{Im}(1-e)s_2 \leq \operatorname{Im} g \cap (1-e)M \ll M$. By hypothesis, we have $(1-e)s_2 = 0$ and so $\phi = es_1 \in eS$. Thus $D_S(\operatorname{Im} g) = eS$. By Lemma 3.1, M is dual Rickart.

The following results exhibits the connection between dual Rickart modules and $\mathcal{I}$-lifting modules.

***Corollary* 3.1.** *Let M be a module. Then M is dual Rickart if and only if M is $\mathcal{I}$-lifting and $\mathcal{T}$-noncosingular.*

Proposition 3.1. *Let M be a module with $Rad(M) = 0$. Then M is $\mathcal{I}$-lifting if and only if M is dual Rickart.*

Proof. Let M be an $\mathcal{I}$-lifting module and let $\phi \in S$. Then there exists a direct summand X of M and a submodule Y of M such that $\operatorname{Im} \phi = X \oplus Y$ and $Y \ll M$. Hence $Y \subseteq \operatorname{Rad}(M) = 0$ and so $\operatorname{Im} \phi$ is a direct summand of M, this means that M is dual Rickart. The converse is clear.

Recall that a ring R is said to be a *right V-ring* if every simple right R-module is injective.

***Corollary* 3.2.** *Let R be a right V-ring and M be an R-module. Then M is dual Rickart if and only if M is $\mathcal{I}$-lifting.*

Proof. By [19] (23.1), $\operatorname{Rad}(M) = 0$ for right R-module M. Thus, by Proposition 3.1, every $\mathcal{I}$-lifting R-module is dual Rickart.

***Corollary* 3.3.** *Let R be a commutative regular ring and M be an R-module. Then M is $\mathcal{I}$-lifting if and only if M is dual Rickart.*

Proof. It is clear by Corollary 3.2 and [19] (23.5(2)).

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