UDC 517.9

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CRITICAL POINTS APPROACHES TO ELLIPTIC PROBLEMS DRIVEN BY A $p(x)$-LAPLACIAN

ПІДХОДИ ДО ЕЛІПТИЧНИХ ЗАДАЧ З $p(x)$-ЛАПЛАСІАНОМ ОСНОВАНІ НА ТЕОРІЇ КРИТИЧНИХ ТОЧОК

We establish the existence of at least three solutions for elliptic problems driven by a $p(x)$-Laplacian. The existence of at least one nontrivial solution is also proved. The approaches are based on variational methods and critical point theory.

Встановлено існування принаймні трьох розв'язків еліптичних задач з $p(x)$-лапласіаном. Існування щонайменше одного нетривіального розв'язку також продемонстровано. Застосовані підходи грунтуються на варіаційних методах та теорії критичних точок.

1. Introduction. This paper treats the following elliptic problem:

$$\begin{aligned} &-\Delta_{p(x)}u = \lambda f(x,u) \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, $\Omega \subset R^N$, $N \geq 1$, is a nonempty bounded open set with a smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$ satisfies the condition $N < p^- :=$ $:= \inf_{x\in\Omega} p(x) \leq p(x) \leq \sup_{x\in\Omega} p(x) < +\infty$, $\lambda > 0$, $f\colon \Omega \times R \to R$ is an L^1-Carathéodory function.

Recently, the study of differential equations and variational problems with variable exponent has been a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [29, 31]). It also has wide applications in different research fields, such as image processing model (see, e.g., [16, 24]), stationary thermo-rheological viscous flows (see [1]) and the mathematical description of the processes filtration of an idea barotropic gas through a porous medium (see [2]).

Let us point out that when $p(x) = p =$ constant, there is a large literature which deals with problems involving the p-Laplacian with Dirichlet boundary conditions both in the scalar case and elliptic systems in bounded or unbounded domains, which we do not need to cite here since the reader may easily find such papers. Many authors investigated the existence and multiplicity of solutions for problems involving $p(x)$-Laplacian. In recent years there has been an increasing interest in the study of variational problems and elliptic equations with variable exponent. We refer to [19, 21, 27] for the theory of $L^{p(x)}$ and $W^{1,p(x)}(\Omega)$. The case of $p(x)$-Laplacian with Dirichlet conditions on the scalar case has been studied by Fan and Zhang [20]. In fact, in [20], Fan and Zhang first introduced some basic properties of the generalized Lebesgue – Sobolev spaces $W_0^{1,p(x)}(\Omega)$ which can be regarded as a special class of generalized Orlicz – Sobolev spaces, and second they presented several important properties of $p(x)$-Laplace operator, and finally under some appropriate conditions on the nonlinear term, they established some existence results of weak solutions of the problem (1).

* Research of S. Heidarkhani was in part supported by a grant from IPM (No. 91470046).

Bonanno and Chinnì in [7], employing a three critical point theorem for nondifferentiable functionals due to Bonanno and Marano [12] (Theorem 3.6), established the existence of at least three weak solutions for the problem

$$-\Delta_{p(x)}u = \lambda(f(x,u) + \mu g(x,u)) \qquad \text{in} \quad \Omega,$$

$$u = 0 \qquad \text{on} \quad \partial\Omega,$$

where $\Omega \subset R^N$, $N \geq 1$, is a nonempty bounded open set with a smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$, λ and μ are two positive parameters and $f, g \colon \Omega \times \mathbb{R} \to \mathbb{R}$ are two measurable with respect to each variable separately functions and possibly discontinuous with respect to u. Bonanno and Chinnì also in [8] based on a convenient form of the recent three critical points theorem obtained by G. Bonanno and S. A. Marano [12] investigated multiplicity of solutions for the problem (1).

In the present paper, motivated by [7, 8], first employing two related three critical points theorems for differentiable functionals due to Bonanno and Candito [6], we ensure the existence of at least three weak solutions for the problem (1) (see Theorems 4 and 5) and then, using a very recent a local minimum theorem for differentiable functionals due to Bonanno [5], under different assumptions which have been assumed in Theorems 4 and 5 we establish the existence of at least one nontrivial weak solution for the problem (1) (see Theorem 7). Theorems 4 and 5 extend the results of [8].

For a thorough account on the subject, we refer the reader to the papers [9, 10, 13, 14, 22, 23, 26].

2. Preliminaries and basic notations. In this section, we introduce some definitions and results which will be used in the next section. Firstly, we introduce some theories of Lebesgue – Sobolev spaces with variable exponent. The details can be found in [17, 19, 21]. Set

$$L^\infty_+(\Omega) = \left\{ p \in L^\infty(\Omega) \colon \operatorname*{ess\,inf}_{x\in\Omega} p(x) > 1 \right\}.$$

For $p \in L^\infty_+(\Omega)$, denote

$$p^- = p^-(\Omega) = \operatorname*{ess\,inf}_{x\in\Omega} p(x) \qquad \text{and} \qquad p^+ = p^+(\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} p(x)$$

for any $p(x) \in L^\infty_+(\Omega)$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u \colon u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 \colon \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

The space $(L^{p(x)}(\Omega), \|.\|_{p(x)})$ is a Banach space. Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \colon |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. On $W_0^{1,p(x)}(\Omega)$ we consider the norm

$$\|u\| := \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Here we display some facts which will be used later.

Proposition 1 (see [20]). (i) *The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) *There is a constant $c > 0$ such that $\|u\|_{L^{p(x)}(\Omega)} \le c\|\nabla u\|_{L^{p(x)}(\Omega)}$ for all $u \in W_0^{1,p(x)}(\Omega)$.*

Proposition 2 (see [7]). *Set $\rho_p(u) = \int_\Omega |u(x)|^{p(x)} dx$. For $u \in W_0^{1,p(x)}(\Omega)$, we have*

(i) $\|u\| < 1 (= 1; > 1) \Longleftrightarrow \rho_p(|\nabla u|) < 1 (= 1; > 1)$.

(ii) *If* $\|u\| > 1$, *then* $\dfrac{1}{p^+}\|u\|^{p^-} \le \displaystyle\int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx \le \frac{1}{p^-}\|u\|^{p^+}$.

(iii) *If* $\|u\| < 1$, *then* $\dfrac{1}{p^+}\|u\|^{p^+} \le \displaystyle\int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx \le \frac{1}{p^-}\|u\|^{p^-}$.

As pointed in [20, 27], $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ and, since $p^- > N$, $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus $W^{1,p(x)}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. So, in particular, there exists a positive constant c_0 such that

$$\|u\|_{C^0(\overline{\Omega})} \le c_0\|u\| \tag{2}$$

for each $u \in W_0^{1,p(x)}(\Omega)$.

Let X denotes the Sobolev space $W_0^{1,p(x)}(\Omega)$. Let $G(u) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx$ for all $u \in X$. We denote $L = G' : X \to X^*$, then

$$L(u)(v) = \int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x)\, dx$$

for all $u, v \in X$.

Proposition 3 (see [20]). (i) *$L : X \to X^*$ is a continuous, bounded and strictly monotone operator.*

(ii) *L is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} (L(u_n), u_n - u) \le 0$, then $u_n \to u$ in X.*

(iii) *$L : X \to X^*$ is a homeomorphism.*

We say that u is a weak solution to the problem (1) if $u \in X$ and

$$\int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x) dx - \lambda \int_\Omega f(x, u(x)) v(x) dx = 0$$

for every $v \in X$.

Put

$$\delta(x) = \sup\left\{\delta > 0 : S(x, \delta) \subset \Omega\right\}$$

where $S(x,\delta)$ denotes the ball with center at x and radius of δ, for all $x \in \Omega$, one can prove that there exists $x_0 \in \Omega$ such that $S(x_0, D) \subset \Omega$, where $D = \sup_{x\in\Omega} \delta(x)$. Set

$$m := \frac{\pi^{N/2}}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)}$$

where Γ is the Euler function, and for each $r > 0$, set

$$\gamma_r := \max\left\{(p^+ r)^{1/p^-},\ (p^+ r)^{1/p^+}\right\}.$$

Put

$$F(x,t) = \int\limits_0^t f(x,\xi)\, d\xi$$

for all $(x,t) \in \Omega \times R$.

3. Existence of three solutions. In the following section we establish the existence of at least three weak solutions for the problem (1). Our main tools are two three critical points theorems. In the first one the coercivity of the functional $\Phi - \lambda\Psi$ is required, in the second one a suitable sign hypothesis is assumed. The first result has been obtained in [4], the second one in [3]. Here we recall them as given in [6].

Theorem 1 ([6], Theorem 3.2)**.** *Let X be a reflexive real Banach space, $\Phi\colon X \longrightarrow R$ be a coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi\colon X \longrightarrow R$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there is a positive constant r and $\overline{v} \in X$, with $2r < \Phi(\overline{v})$, such that

(a$_1$) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \dfrac{2}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$;

(a$_2$) *for all* $\lambda \in \left]\dfrac{3}{2}\dfrac{\Phi(\overline{v})}{\Psi(\overline{v})}, \dfrac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$, *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \left]\dfrac{3}{2}\dfrac{\Phi(\overline{v})}{\Psi(\overline{v})}, \dfrac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[$ *the functional $\Phi - \lambda\Psi$ has at least three distinct critical points.*

Theorem 2 ([6], Theorem 3.3)**.** *Let X be a reflexive real Banach space, $\Phi\colon X \longrightarrow R$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi\colon X \longrightarrow R$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

2) *for each $\lambda > 0$ and for every u_1, u_2 which are local minimum for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{v} \in X$, with $2r_1 < \Phi(\overline{v}) < \dfrac{r_2}{2}$, such that (b$_1$) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$;

(b$_2$) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$.

Then, for each $\lambda \in \left]\dfrac{3}{2}\dfrac{\Phi(\overline{v})}{\Psi(\overline{v})}, \min\left\{\dfrac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}, \dfrac{r_2/2}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[$, *the functional* $\Phi-\lambda\Psi$ *has at least three distinct critical points which lie in* $\Phi^{-1}(]-\infty,r_2[)$.

A special case of our main results is the following theorem.

Theorem 3. *Let* $\Omega\subseteq R^2$ *be a nonempty bounded open set with a smooth boundary* $\partial\Omega$. *Let* $f: R\to R$ *be a continuous function and put* $F(t)=\int_0^t f(\xi)\,d\xi$ *for all* $t\in R$ *such that* $F(h)>0$ *for some* $h>0$ *and* $F(\xi)\geq 0$ *in* $[0,h]$. *Fix* $p(x)=p>2$ *and assume that*

$$\liminf_{\xi\to 0}\frac{F(\xi)}{|\xi|^p}=\limsup_{|\xi|\to+\infty}\frac{F(\xi)}{|\xi|^p}=0.$$

Then, there is $\lambda^*>0$ *such that for each* $\lambda>\lambda^*$ *the problem*

$$-\Delta_p u=\lambda f(u) \quad in \quad \Omega,$$

$$u=0 \quad on \quad \partial\Omega$$

admits at least three weak solutions.

***Remark* 1.** Similar results to Theorem 3 have been obtained in [11] (Theorem 0) in which a class of Dirichlet quasilinear elliptic systems driven by a (p,q)-Laplacian operator has been considered, and also in [25] (Theorem 1) in which a quasilinear second-order differential equation has been studied.

We formulate the existence results as follows:

Theorem 4. *Let* $f:\Omega\times R\to R$ *be an* L^1*-Carathéodory function such that* $\operatorname{ess\,inf}_{x\in\Omega}F(x,\xi)\geq \geq 0$ *for all* $\xi\in R$. *Assume that there exist two positive constants* r *and* h *such that*

(A$_1$) $\dfrac{1}{p^+}\min\left\{\left(\dfrac{2h}{D}\right)^{p^-},\left(\dfrac{2h}{D}\right)^{p^+}\right\}mD^N\dfrac{2^N-1}{2^N}>2r$;

(A$_2$) $\dfrac{\int_\Omega\sup_{|t|\leq c_0\gamma_r}F(x,t)dx}{r}<\dfrac{2\operatorname{ess\,inf}_{x\in\Omega}F(x,h)}{\dfrac{3}{p^-}\max\left\{\left(\dfrac{2h}{D}\right)^{p^-},\left(\dfrac{2h}{D}\right)^{p^+}\right\}(2^N-1)}$;

(A$_3$) $\limsup_{|t|\to+\infty}\dfrac{F(x,t)}{|t|^{p^-}/p^+}<\dfrac{\int_\Omega\sup_{|t|\leq c_0\gamma_r}F(x,t)dx}{r}$.

Then, for each

$$\lambda\in\left]\frac{3}{2}\frac{\frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\}(2^N-1)}{\operatorname{ess\,inf}_{x\in\Omega}F(x,h)},\ \frac{r}{\int_\Omega\sup_{|t|\leq c_0\gamma_r}F(x,t)dx}\right[$$

the problem (1) *admits at least three weak solutions.*

Proof. In order to apply Theorem 1 to our problem, we introduce the functionals Φ, $\Psi\colon X\to R$ for each $u\in X$, as follows:

$$\Phi(u)=\int\limits_{\Omega}\frac{1}{p(x)}\left|\nabla u(x)\right|^{p(x)}dx$$

and

$$\Psi(u)=\int\limits_{\Omega}F(x,u(x))dx.$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u\in X$ are the functionals $\Phi'(u),\Psi'(u)\in X^*$, given by

$$\Phi'(u)(v)=\int\limits_{\Omega}\left|\nabla u(x)\right|^{p(x)-2}\nabla u(x)\nabla v(x)dx$$

and

$$\Psi'(u)(v)=\int\limits_{\Omega}f(x,u(x))v(x)dx$$

for every $v\in X$, respectively, as well as Ψ is sequentially weakly upper semicontinuous. Moreover, Φ is sequentially weakly lower semicontinuous and Φ' admits a continuous inverse on X^*. Furthermore, $\Psi'\colon X\to X^*$ is a compact operator. Set

$$w(x)=\begin{cases}0 & \text{if}\quad x\in\Omega\setminus S(x_0,D),\\ h & \text{if}\quad x\in S\left(x_0,\dfrac{D}{2}\right),\\ \dfrac{2h}{D}\left(D-\sqrt{\sum_{i=1}^{N}(x_i-x_{0i})^2}\right) & \text{if}\quad x\in S(x_0,D)\setminus S\left(x_0,\dfrac{D}{2}\right).\end{cases}\tag{3}$$

It is easy to see that $w\in X$ and, in particular, one has

$$\frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\}mD^N\frac{2^N-1}{2^N}\leq\Phi(w)\leq$$

$$\leq\frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\}mD^N\frac{2^N-1}{2^N}\tag{4}$$

and

$$\Psi(w)\geq\int\limits_{S(x_0,D/2)}F(x,w(x))dx\geq\operatorname*{ess\,inf}_{x\in\Omega}F(x,h)m\left(\frac{D}{2}\right)^N.\tag{5}$$

From (A_1), taking (4) into account, we get $\Phi(w)>2r$. Thanks to the embedding $X\hookrightarrow C^0(\overline{\Omega})$, we have

$$\Phi^{-1}(]-\infty,r[)=\{u\in X;\Phi(u)<r\}=$$

$$= \left\{ u \in X; \int\limits_{\Omega} \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx < r \right\} \subseteq$$

$$\subseteq \big\{ u \in X; |u(x)| \leq c_0 \gamma_r \ \text{ for all } \ x \in \Omega \big\},$$

and it follows that

$$\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) = \sup_{u \in \Phi^{-1}(]-\infty, r[)} \int\limits_{\Omega} F(x, u(x)) dx \leq$$

$$\leq \int\limits_{\Omega} \sup_{|t| \leq c_0 \gamma_r} F(x, t)\, dx.$$

Therefore, owing to the assumption (A_2), (4) and (5), we get

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} = \frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \int_{\Omega} F(x, u(x)) dx}{r} \leq$$

$$\leq \frac{\int_{\Omega} \sup_{|t| \leq c_0 \gamma_r} F(x, t) dx}{r} <$$

$$< \frac{2}{3} \frac{\operatorname{ess\,inf}_{x \in \Omega} F(x, h)}{\dfrac{1}{p^-} \max\left\{ \left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+} \right\} (2^N - 1)} \leq \frac{2}{3} \frac{\Psi(w)}{\Phi(w)}.$$

Furthermore, from (A_3) there exist two constants $\eta, \vartheta \in R$ with

$$\eta < \frac{\int_{\Omega} \sup_{|t| \leq c_0 \gamma_r} F(x, t) dx}{r}$$

such that

$$|\Omega| c_0 F(x, t) \leq \eta \frac{|t|^{p^-}}{p^+} + \vartheta \qquad \text{for all} \quad x \in \Omega \quad \text{and for all} \quad t \in R^n.$$

Fix $u \in X$. Then

$$F(x, u(x)) \leq \frac{1}{|\Omega| c_0} \left(\eta \frac{|u(x)|^{p^-}}{p^+} + \vartheta \right) \qquad \text{for all} \quad x \in \Omega. \tag{6}$$

Now, in order to prove the coercivity of the functional $\Phi - \lambda\Psi$, first we assume that $\eta > 0$. So, if $\|u\| \geq 1$, for any fixed

$$\lambda \in \left] \frac{3}{2} \frac{\dfrac{1}{p^+} \min\left\{ \left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+} \right\} (2^N - 1)}{\operatorname{ess\,inf}_{x \in \Omega} F(x, h)}, \frac{r}{\int_{\Omega} \sup_{|t| \leq c_0 \gamma_r} F(x, t) dx} \right[,$$

bearing (2) in mind, from Proposition 2 and (6) we obtain

$$\Phi(u) - \lambda\Psi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx - \lambda \int_\Omega F(x, u(x)) dx \geq$$

$$\geq \frac{1}{p^+} \|u\|^{p^-} - \frac{\lambda\eta}{|\Omega| c_0} \frac{\int_\Omega |u(x)|^{p^-} dx}{p^+} - \frac{\lambda\vartheta}{c_0} \geq$$

$$\frac{1}{p^+} \|u\|^{p^-} - \frac{\lambda\eta}{|\Omega| c_0} \frac{|\Omega| c_0 \|u\|^{p^-}}{p^+} - \frac{\lambda\vartheta}{c_0} \geq$$

$$\geq \left(1 - \eta \frac{r}{\int_\Omega \sup_{|t| \leq c_0 \gamma_r} F(x,t) dx} \right) \frac{1}{p^+} \|u\|^{p^-} - \frac{\lambda\vartheta}{c_0},$$

and thus

$$\lim_{\|u\| \to +\infty} \big(\Phi(u) - \lambda\Psi(u) \big) = +\infty.$$

On the other hand, if $\eta \leq 0$. Clearly, we get $\lim_{\|u\| \to +\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty$. Both cases lead to the coercivity of functional $\Phi - \lambda\Psi$.

So, the assumptions (a_1) and (a_2) in Theorem 1 are satisfied. Hence, by using Theorem 1, taking into account that the weak solutions of the problem (1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$, we have the conclusion.

Theorem 4 is proved.

Theorem 5. *Let $f : \Omega \times R \to R$ be an L^1-Carathéodory function such that* $\operatorname{ess\,inf}_{x \in \Omega} F(x, \xi) \geq$ ≥ 0 *for all $\xi \in R$ and satisfies the condition $f(x,t) \geq 0$ for all $(x,t) \in \Omega \times (R^+ \cup \{0\})$. Assume that there exist three positive constants r_1, r_2 and h with*

$$2r_1 < \frac{1}{p^+} \min\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\} m D^N \frac{2^N - 1}{2^N}$$

and

$$\frac{1}{p^-} \max\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\} m D^N \frac{2^N - 1}{2^N} < \frac{r_2}{2}$$

such that

(B_1) $$\frac{\int_\Omega \sup_{|t| \leq c_0 \gamma_r} F(x,t) dx}{r_1} < \frac{2 \operatorname{ess\,inf}_{x \in \Omega} F(x, h)}{3 \frac{1}{p^-} \max\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\} (2^N - 1)};$$

(B_2) $$\frac{\int_\Omega \sup_{|t| \leq c_0 \gamma_r} F(x,t) dx}{r_2} < \frac{\operatorname{ess\,inf}_{x \in \Omega} F(x, h)}{3 \frac{1}{p^-} \max\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\} (2^N - 1)}.$$

Then, for each

$$\lambda \in \left] \frac{3}{2} \frac{\frac{1}{p^+} \min\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\} (2^N - 1)}{\operatorname{ess\,inf}_{x\in\Omega} F(x,h)}, \right.$$

$$\left. \min\left\{ \frac{r_1}{\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx}, \frac{\frac{r^2}{2}}{\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx} \right\} \right[$$

the problem (1) *admits at least three nonnegative weak solutions* v^1, v^2, v^3 *such that*

$$\int_\Omega \frac{1}{p(x)} |\nabla v^j(x)|^{p(x)} dx \le r_2 \qquad \text{for each} \quad x \in \Omega, \quad j = 1, 2, 3.$$

Proof. Let Φ and Ψ be as in the proof of Theorem 4. Let us employ Theorem 2 to our functionals. Obviously, Φ and Ψ satisfy the condition 1 of Theorem 2. Now, we verify that the functional $\Phi - \lambda\Psi$ satisfies the assumption 2 of Theorem 2. Let $u^\star$ and $u^{\star\star}$ be two local minima for $\Phi - \lambda\Psi$. Then $u^\star$ and $u^{\star\star}$ are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem (1). Since $f(x,t) \ge 0$ for all $(x,t) \in \Omega \times (R^+ \cup \{0\})$, from the Weak Maximum Principle (see for instance [15]) we deduce $u^\star(x) \ge 0$ and $u^{\star\star}(x) \ge 0$ for every $x \in \Omega$. So, it follows that $su^\star + (1-s)u^{\star\star} \ge 0$ for all $s \in [0,1]$, and that $f(su^\star + (1-s)u^{\star\star}, t) \ge 0$, and consequently, $\Psi(su^\star + (1-s)u^{\star\star}) \ge 0$ for all $s \in [0,1]$. Moreover, from the conditions

$$2r_1 < \frac{1}{p^+} \min\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\} mD^N \frac{2^N - 1}{2^N}$$

and

$$\frac{1}{p^-} \max\left\{ \left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+} \right\} mD^N \frac{2^N - 1}{2^N} < \frac{r_2}{2},$$

we observe $2r_1 < \Phi(w) < \frac{r_2}{2}$. Next, thanks to the embedding $X \hookrightarrow C^0(\overline{\Omega})$, we have

$$\Phi^{-1}(]-\infty, r_1[) = \{u \in X; \Phi(u) < r_1\} =$$

$$= \left\{ u \in X; \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx < r_1 \right\} \subseteq$$

$$\subseteq \big\{ u \in X; |u(x)| \le c_0\gamma_{r_1} \text{ for all } x \in \Omega \big\},$$

and it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)=\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\int\limits_\Omega F(x,u(x))dx\le\int\limits_\Omega\sup_{|t|\le c_0\gamma_r}F(x,t)dx.$$

Therefore, owing to the assumption (B_1), we get

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1}=\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\int_\Omega F(x,u(x))dx}{r_1}\le$$

$$\le\frac{\int_\Omega\sup_{|t|\le c_0\gamma_{r_1}}F(x,t)dx}{r_1}<$$

$$<\frac{2}{3}\frac{\operatorname{ess\,inf}_{x\in\Omega}F(x,h)}{\frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\}(2^N-1)}\le\frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.$$

As above, using the assumption (B_2), we obtain

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2}=\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\int_\Omega F(x,u(x))dx}{r_2}\le$$

$$\le\frac{\int_\Omega\sup_{|t|\le c_0\gamma_{r_2}}F(x,t)dx}{r_2}<$$

$$<\frac{1}{3}\frac{\operatorname{ess\,inf}_{x\in\Omega}F(x,h)}{\frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-},\left(\frac{2h}{D}\right)^{p^+}\right\}(2^N-1)}\le\frac{1}{3}\frac{\Psi(w)}{\Phi(w)}.$$

So, the assumptions (b_1) and (b_2) in Theorem 2 are satisfied. Hence, by using Theorem 2, taking into account that the weak solutions of the problem (1) are exactly the solutions of the equation $\Phi'(u)-\lambda\Psi'(u)=0$, the problem (1) admits at least three distinct weak solutions in X.

Theorem 5 is proved.

We end this section by proving Theorem 3.

***Proof of Theorem* 3.** Fix

$$\lambda>\lambda^*:=\frac{9}{2}\frac{\frac{1}{p}\left(\frac{2h}{D}\right)^p}{F(h)}.$$

Taking into account that $\liminf_{\xi\to 0}\dfrac{F(\xi)}{|\xi|^p}=0$ there is $\{r_n\}_{n\in N}\subseteq]0,+\infty[$ such that $\lim_{n\to+\infty}r_n=$ $=0$ and

$$\lim_{n\to+\infty}\frac{|\Omega|\max_{|t|\le c_0(pr_n)^{1/p}}F(t)}{r_n}=0.$$

Hence, there is $\overline{r}>0$ such that

$$\frac{|\Omega| \max_{|t|\le c_0(p\overline{r})^{1/p}} F(t)}{\overline{r}} < \min\left\{\frac{2p}{9}\frac{F(h)}{\left(\dfrac{2h}{D}\right)^p};\ \frac{1}{\lambda}\right\}$$

and $2\overline{r} < \dfrac{3D^2\pi}{4p}\left(\dfrac{2h}{D}\right)^p$.

From Theorem 4 the conclusion follows.

4. Existence of a nontrivial solution. First we here recall for the reader's convenience [28] (Theorem 2.5) as given in [5] (Theorem 5.1) (see also [5] (Proposition 2.1) for related results) which is our main tool to prove the main result.

For a given nonempty set X, and two functionals $\Phi, \Psi\colon X \to R$, we define the following functions:

$$\vartheta(r_1, r_2) = \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) = \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in R$, $r_1 < r_2$.

Theorem 6 ([5], Theorem 5.1)**.** *Let X be a reflexive real Banach space, $\Phi\colon X \to R$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi\colon X \to R$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in R$, $r_1 < r_2$, such that*

$$\vartheta(r_1, r_2) < \rho(r_1, r_2).$$

Then, for each $\lambda \in \left]\dfrac{1}{\rho(r_1,r_2)}, \dfrac{1}{\vartheta(r_1,r_2)}\right[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ $\forall u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

We formulate the main result of this section as follows:

Theorem 7. *Let $f\colon \Omega\times R \to R$ be an L^1-Carathéodory function such that* $\operatorname{ess\,inf}_{x\in\Omega} F(x,\xi) \ge$ ≥ 0 *for all $\xi \in R$. Assume that there exist a nonnegative constant r_1 and two positive constants r_2 and h with*

$$r_1 < \frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} mD^N\frac{2^N-1}{2^N}$$

and

$$\frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} mD^N\frac{2^N-1}{2^N} < r_2$$

such that

$$(\mathrm{C}_1)\qquad \frac{\displaystyle\int_\Omega \sup_{|t|\le c_0\gamma_{r_2}} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h)m\left(\frac{D}{2}\right)^N}{r_2 - \dfrac{1}{p^-}\max\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}} <$$

$$< \frac{\int_\Omega \sup_{|t|\le c_0\gamma_{r_1}} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\frac{D}{2}\right)^N}{r_1 - \frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}}. \tag{7}$$

Then, for each

$$\lambda \in \left] \frac{r_1 - \frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}}{\int_\Omega \sup_{|t|\le c_0\gamma_{r_1}} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\frac{D}{2}\right)^N}, \right.$$

$$\left. \frac{r_2 - \frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}}{\int_\Omega \sup_{|t|\le c_0\gamma_{r_2}} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\frac{D}{2}\right)^N} \right[$$

the problem (1) *admits at least one nontrivial weak solution* $u_0 \in X$ *such that*

$$r_1 < \int_\Omega \frac{1}{p(x)} |\nabla u_0(x)|^{p(x)} dx < r_2.$$

Proof. In order to apply Theorem 6 to our problem, let the functionals $\Phi, \Psi : X \to R$ be as in the proof of Theorem 4. As seen in the proof of Theorem 4, Φ and Ψ satisfy the regularity assumptions of Theorem 6. Choose w as given in (3). Taking (4) into account, from the conditions

$$r_1 < \frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} m D^N \frac{2^N-1}{2^N}$$

and

$$\frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} m D^N \frac{2^N-1}{2^N} < r_2$$

we get

$$r_1 < \Phi(w) < r_2.$$

Thanks to the embedding $X \hookrightarrow C^0(\overline{\Omega})$, we have

$$\Phi^{-1}(]-\infty, r_2[) = \{u \in X; \Phi(u) < r_2\} =$$

$$= \left\{u \in X; \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx < r_2\right\} \subseteq$$

$$\subseteq \big\{u \in X;\ |u(x)| \le c_0\gamma_{r_2} \ \text{ for all } \ x \in \Omega\big\},$$

and it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)=\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\int\limits_{\Omega}F(x,u(x))dx\le\int\limits_{\Omega}\sup_{|t|\le c_0\gamma_{r_2}}F(x,t)dx.$$

Therefore, one has

$$\vartheta(r_1,r_2)\le\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)-\Psi(w)}{r_2-\Phi(w)}\le$$

$$\le\frac{\int_\Omega\sup_{|t|\le c_0\gamma_{r_2}}F(x,t)dx-\Psi(w)}{r_2-\Phi(w)}\le$$

$$\le\frac{\int_\Omega\sup_{|t|\le c_0\gamma_{r_2}}F(x,t)dx-\operatorname{ess\,inf}_{x\in\Omega}F(x,h)m\left(\frac{D}{2}\right)^N}{r_2-\dfrac{1}{p^-}\max\left\{\left(\dfrac{2h}{D}\right)^{p^-},\left(\dfrac{2h}{D}\right)^{p^+}\right\}}.$$

On the other hand, arguing as before, one has

$$\rho(r_1,r_2)\ge\frac{\Psi(w)-\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{\Phi(w)-r_1}\ge$$

$$\ge\frac{\Psi(w)-\int_\Omega\sup_{|t|\le c_0\gamma_{r_1}}F(x,t)dx}{\Phi(w)-r_1}\ge$$

$$\ge\frac{\int_\Omega\sup_{|t|\le c_0\gamma_{r_1}}F(x,t)dx-\operatorname{ess\,inf}_{x\in\Omega}F(x,h)m\left(\frac{D}{2}\right)^N}{r_1-\dfrac{1}{p^+}\min\left\{\left(\dfrac{2h}{D}\right)^{p^-},\left(\dfrac{2h}{D}\right)^{p^+}\right\}}.$$

Hence, from Assumption (C$_1$), one has $\vartheta(r_1,r_2)<\rho(r_1,r_2)$. Therefore, applying Theorem 6, for each

$$\lambda\in\left]\frac{r_1-\dfrac{1}{p^+}\min\left\{\left(\dfrac{2h}{D}\right)^{p^-},\left(\dfrac{2h}{D}\right)^{p^+}\right\}}{\int_\Omega\sup_{|t|\le c_0\gamma_{r_1}}F(x,t)dx-\operatorname{ess\,inf}_{x\in\Omega}F(x,h)m\left(\frac{D}{2}\right)^N},\right.$$

$$\left.\frac{r_2-\dfrac{1}{p^-}\max\left\{\left(\dfrac{2h}{D}\right)^{p^-},\left(\dfrac{2h}{D}\right)^{p^+}\right\}}{\int_\Omega\sup_{|t|\le c_0\gamma_{r_2}}F(x,t)dx-\operatorname{ess\,inf}_{x\in\Omega}F(x,h)m\left(\frac{D}{2}\right)^N}\right[,$$

the functional $\Phi - \lambda\Psi$ has admits at least one critical point $u_0 \in X$ such that $r_1 < \Phi(u_0) < r_2$, that is

$$r_1 < \int\limits_\Omega \frac{1}{p(x)} |\nabla u_0(x)|^{p(x)} dx < r_2.$$

Hence, taking into account that the weak solutions of the problem (1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$, we achieve the stated assertion.

Theorem 7 is proved.

Now we point out the following consequence of Theorem 7.

Theorem 8. *Suppose that*

$$\frac{1}{p^+} \min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} \le \frac{1}{p^-} \max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\}.$$

Let $f\colon \Omega \times R \to R$ be an L^1-Carathéodory function such that $\operatorname{ess\,inf}_{x\in\Omega} F(x,\xi) \ge 0$ *for all $\xi \in R$. Assume that there exist two positive constants r and h with*

$$\frac{1}{p^-} \max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} m D^N \frac{2^N - 1}{2^N} < r$$

such that

$$(\mathrm{C}_2) \qquad \frac{\displaystyle\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx}{r} < \frac{\operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\dfrac{D}{2}\right)^N}{\dfrac{1}{p^-} \max\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}}.$$

Then, for each

$$\lambda \in \left] \frac{\dfrac{1}{p^+} \min\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}}{\operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\dfrac{D}{2}\right)^N}, \frac{r}{\displaystyle\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx} \right[$$

the problem (1) *admits at least one nontrivial weak solution $u_0 \in X$ such that*

$$r_1 < \int\limits_\Omega \frac{1}{p(x)} |\nabla u_0(x)|^{p(x)} dx < r_2.$$

Proof. The conclusion follows from Theorem 7 by taking $r_1 = 0$ and $r_2 = r$. Indeed, owing to our assumptions, one has

$$\frac{\displaystyle\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h) m \left(\dfrac{D}{2}\right)^N}{r - \dfrac{1}{p^-} \max\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}} <$$

$$< \frac{\left(1 - \dfrac{\dfrac{1}{p^-}\max\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}}{r}\right)\displaystyle\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx}{r - \dfrac{1}{p^-}\max\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}} =$$

$$= \frac{\displaystyle\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx}{r} < \frac{\operatorname{ess\,inf}_{x\in\Omega} F(x,h)m\left(\dfrac{D}{2}\right)^N}{\dfrac{1}{p^-}\max\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}} \le$$

$$\le \frac{\operatorname{ess\,inf}_{x\in\Omega} F(x,h)m\left(\dfrac{D}{2}\right)^N}{\dfrac{1}{p^+}\min\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}}.$$

In particular, one has

$$\frac{\displaystyle\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx - \operatorname{ess\,inf}_{x\in\Omega} F(x,h)m\left(\dfrac{D}{2}\right)^N}{r - \dfrac{1}{p^-}\max\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}} < \frac{\displaystyle\int_\Omega \sup_{|t|\le c_0\gamma_r} F(x,t)dx}{r}.$$

Hence, Theorem 7 concludes the result.

Theorem 8 is proved.

Let $f\colon R \to R$ be a continuous function, and put $F(t) = \int_0^t f(\xi)\,d\xi$ for all $t \in R$. We have the following result as a direct consequence of Theorem 7.

Theorem 9. *Let $f\colon R \to R$ be a nonnegative continuous function. Assume that there exist a nonnegative constant r_1 and two positive constants r_2 and h with*

$$r_1 < \frac{1}{p^+}\min\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} mD^N \frac{2^N - 1}{2^N}$$

and

$$\frac{1}{p^-}\max\left\{\left(\frac{2h}{D}\right)^{p^-}, \left(\frac{2h}{D}\right)^{p^+}\right\} mD^N \frac{2^N - 1}{2^N} < r_2$$

such that

$$(\mathrm{C}_3) \qquad \frac{|\Omega|F(c_0\gamma_{r_2}) - F(h)m\left(\dfrac{D}{2}\right)^N}{r_2 - \dfrac{1}{p^-}\max\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}} < \frac{|\Omega|F(c_0\gamma_{r_1}) - F(h)m\left(\dfrac{D}{2}\right)^N}{r_1 - \dfrac{1}{p^+}\min\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}}.$$

Then, for each

$$\lambda \in \left] \frac{r_1 - \dfrac{1}{p^+}\min\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}}{|\Omega|F(c_0\gamma_{r_1}) - F(h)m\left(\dfrac{D}{2}\right)^N}, \frac{r_2 - \dfrac{1}{p^-}\max\left\{\left(\dfrac{2h}{D}\right)^{p^-}, \left(\dfrac{2h}{D}\right)^{p^+}\right\}}{|\Omega|F(c_0\gamma_{r_2}) - F(h)m\left(\dfrac{D}{2}\right)^N} \right[$$

the problem

$$-\Delta_{p(x)}u = \lambda f(u) \quad \textit{in} \quad \Omega,$$

$$u = 0 \quad \textit{on} \quad \partial\Omega$$

admits at least one nontrivial weak solution $u_0 \in X$ such that

$$r_1 < \int\limits_\Omega \frac{1}{p(x)}|\nabla u_0(x)|^{p(x)}dx < r_2.$$

We end this paper by giving the following special case of our main result of this section.

Theorem 10. *Let $p(x) = p > N$. Let $h\colon \Omega \to R$ be a positive and essentially bounded function and $g\colon R \to R$ be a nonnegative function such that*

$$\lim_{t\to 0^+}\frac{g(t)}{t^{p-1}} = +\infty.$$

Then, for each $\lambda \in \left] 0, \left(\dfrac{1}{\displaystyle\int_\Omega h(x)dx}\right)\sup_{r>0}\dfrac{r}{\displaystyle\int_0^{c_0(pr)^{1/p}} g(\xi)\,d\xi}\right[$, *the problem*

$$-\Delta_p u = \lambda h(x)g(u) \quad \textit{in} \quad \Omega,$$

$$u = 0 \quad \textit{on} \quad \partial\Omega$$

admits at least one nontrivial weak solution in X.

Proof. For fixed $\lambda \in \left] 0, \left(\dfrac{1}{\displaystyle\int_\Omega h(x)dx}\right)\sup_{r>0}\dfrac{r}{\displaystyle\int_0^{c_0(pr)^{1/p}} g(\xi)\,d\xi}\right[$, there exists positive constant r such that

$$\lambda < \left(\frac{1}{\displaystyle\int_\Omega h(x)dx}\right)\frac{r}{\displaystyle\int_0^{c_0(pr)^{1/p}} g(\xi)d\xi}.$$

Moreover, the condition $\lim_{t\to 0^+}\dfrac{g(t)}{t^{p-1}} = +\infty$ implies $\lim_{t\to 0^+}\dfrac{\displaystyle\int_0^t g(\xi)d\xi}{t^p} = +\infty$. Therefore, we can choose positive constant h satisfying $\dfrac{1}{p}\left(\dfrac{2h}{D}\right)^p mD^N\dfrac{2^N-1}{2^N} < r$ such that

$$\left(\frac{1}{\lambda}\right)\frac{2^p}{pD^p \operatorname{ess\,inf}_{x\in\Omega} h(x) m\left(\dfrac{D}{2}\right)^N} < \frac{\displaystyle\int_0^h g(\xi)d\xi}{h^p}.$$

Hence, Theorem 8 ensures the conclusion.

***Remark* 2.** All proofs in this paper are based on the computation of value of the corresponding functionals on the function $w(x)$ introduced in (3), which has been taken as the same as in [7].

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Received 04.10.12,
after revision — 26.06.14