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ATOMS IN THE p-LOCALIZATION OF STABLE HOMOTOPY CATEGORY
АТОМИ В p-ЛОКАЛІЗАЦІЇ СТАБІЛЬНОЇ ГОМОТОПІЧНОЇ КАТЕГОРІЇ

We study p-localizations, where p is an odd prime, of the full subcategories $\mathcal{S}^n$ of stable homotopy category consisting of CW-complexes having cells in n successive dimensions. Using the technique of triangulated categories and matrix problems, we classify atoms (indecomposable objects) in $\mathcal{S}_p^n$ for $n \leq 4(p-1)$ and show that, for $n > 4(p-1)$, this classification is wild in the sense of the representation theory.

Вивчаються p-локалізації (де p — непарне просте число) повних підкатегорій $\mathcal{S}^n$ стабільної гомотопічної категорії, що складається з CW-комплексів із клітинами в n послідовних розмірностях. Застосовуючи техніку триангульованих категорій та матричні задачі, ми наводимо класифікацію атомів (нерозкладних об'єктів) у $\mathcal{S}_p^n$ для $n \leq 4(p-1)$ і показуємо, що для $n > 4(p-1)$ ця класифікація є дикою у сенсі теорії зображень.

Introduction. Classification of homotopy types of polyhedra (finite CW-complexes) is an old problem. It is well-known that it becomes essentially simpler if we consider the stable situation, i.e., identify two polyhedra having homotopy equivalent (iterated) suspensions. It leads to the notion of stable homotopy category and stable homotopy equivalence. Such a classification has been made for polyhedra of low dimensions by several authors; a good survey of these results is the paper of Baues [2]. Unfortunately, it cannot be done for higher dimensions, since the problem becomes extremely complicated. Actually, it results in "*wild problems*" of the representation theory, i.e., problems containing classification of representations of all finitely generated algebras over a field (cf. [3, 10, 11]; for generalities about wild problems see the survey [9]).

In the survey [10] the first author proposed a new approach to the stable homotopy classification which seems more "algebraic" and simpler for calculations. It is based on the triangulated structure of the stable homotopy category and uses the technique of "matrix problems", more exactly, *bimodule categories* in the sense of [9]. In particular, it gave simplified proofs of the results of [3 – 5]. In [11] this technique gave new results on classification of polyhedra with torsion free homologies.

The main difficulties in the stable homotopy classification are related to the 2-components of homotopy groups. That is why it is natural to study *p-local polyhedra*, where p is an odd prime; then we only use the p-parts of homotopy groups. In this paper we use the technique of [10, 11] to classify p-local polyhedra that only have cells in n successive dimensions for $n \leq 4(p-1)$. Analogous results have been obtained by Henn [13], who used a different approach. Our description seems more straightforward and more visual. It gives explicit construction of polyhedra by successive attaching simpler polyhedra to each other. We also show that for $n > 4(p-1)$ the stable classification of p-local polyhedra becomes a wild problem, so the obtained results are in some sense closing.

Section 1 covers the main notions from the stable homotopy theory, bimodule categories and their relations. In Section 2 we calculate morphisms between Moore polyhedra and their products. In Section 3 we describe polyhedra in the case $n = 2p-1$. This classification happens to be "*essentially finite*" in the sense that there is an upper bound for the number of cells in indecomposable polyhedra (*atoms*); actually, atoms have at most 4 cells. Section 4 is the main one. Here we describe polyhedra for $2p \leq n \leq 4(p-1)$. The result is presented in terms of *strings and bands*, which is usual in the modern representation theory. String and band polyhedra are defined by some combinatorial invariant (a *word*) and, in band case, an irreducible polynomial over the residue field $\mathbb{Z}/p$. In the representation theory such description is said to be *tame*. Finally, in Section 5 we prove that the classification becomes wild if $n > 4(p-1)$.

The description obtained by matrix methods is *local*, just as that of [13]. Using the results of [12] we also obtain a global description of p-primary polyhedra. Fortunately, it almost coincides with the local one, except rare special cases, when one local object gives rise to $(p-1)/2$ global ones.

The first author expresses his thanks to H.-J. Baues, who introduced him into the world of algebraic topology and was his coauthor in several first papers on this topic.

1. Stable homotopy category and bimodule categories. We use basic definitions and facts concerning stable homotopy from [8]. We denote by $\mathcal{S}$ the stable homotopy category of *polyhedra*, i.e., finite CW-complexes. It is an additive category and the morphism groups in it are $\mathrm{Hos}(X,Y) = \varinjlim_k \mathrm{Hot}\big(X[k],Y[k]\big)$, where $X[k]$ denotes the k-fold suspension of X and $\mathrm{Hot}(X,Y)$ denotes the set of homotopy classes of continuous maps $X \to Y$. Note that the direct sum in this category is the wedge (bouquet, or one-point gluing) $X \vee Y$ and the natural map $\mathrm{Hos}(X,Y) \to \mathrm{Hos}\big(X[k],Y[k]\big)$ is an isomorphism. In what follows, we always deal with polyhedra as the objects of this category. In particular, *isomorphism* means stable homotopy equivalence. Note that all groups $\mathrm{Hos}(X,Y)$ are finitely generated and the stable homotopy groups $\pi_n^S(X) = \mathrm{Hos}(S^n,X)$ are torsion if $n > \dim X$. It is convenient to formally add to $\mathcal{S}$ the "negative shifts" $X[-k]$ $k \in \mathbb{N}$ of polyhedra with the natural sets of morphisms, so that $X[k][l] \simeq X[k+l]$ and $\mathrm{Hos}(X[k],Y[k]) \simeq \mathrm{Hos}(X,Y)$ for all $k \in \mathbb{Z}$. Then $\mathcal{S}$ becomes a *triangulated category*, where the suspension plays role of the shift and the exact triangles are cofibre sequences $X \to Y \to Z \to X[1]$ (in $\mathcal{S}$ they are the same as fibre sequences). From now on we consider $\mathcal{S}$ with these additional objects. Actually, the category obtained in this way is equivalent to the category of finite S-spectra [8, 15].

We denote by $\mathcal{S}^n$ the full subcategory of $\mathcal{S}$ whose objects are the shifts $X[k]$, $k \in \mathbb{Z}$, of polyhedra only having cells in at most n successive dimensions, or, the same, $(m-1)$-connected and of dimension at most $n+m$ for some m. The Freudenthal theorem [8] (Theorem 1.21) implies that every object of $\mathcal{S}^n$ is a shift (iterated suspension) of an n-connected polyhedron of dimension at most $2n-1$. We denote the full subcategory of $\mathcal{S}^n$ consisting of such polyhedra by $\overline{\mathcal{S}}^n$. Moreover, if two such polyhedra are isomorphic in $\mathcal{S}$, they are homotopy equivalent. Following Baues [2], we call an object from $\mathcal{S}^n$ an *atom* if it belongs to $\overline{\mathcal{S}}^n$, does not belong to $\mathcal{S}^{n-1}$ and is indecomposable (into a wedge of non-contractible polyhedra).

Recall that the *p-localization* of an additive category $\mathcal{C}$ is the category $\mathcal{C}_p$ such that $\mathrm{Ob}\,\mathcal{C}_p = \mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}_p}(A,B) = \mathbb{Z}_p \otimes \mathrm{Hom}_{\mathcal{C}}(A,B)$, where $\mathbb{Z}_p \subset \mathbb{Q}$ is the subring $\left\{ \dfrac{a}{b} \,\middle|\, a,b \in \mathbb{Z},\ p \nmid b \right\}$. We consider the localized categories $\mathcal{S}_p$ and $\mathcal{S}_p^n$ and denote their groups of morphisms $X \to Y$ by $\mathrm{Hos}_p(X,Y)$. Actually, $\mathcal{S}_p$ coincides with the stable homotopy category of finite *p-local CW-complexes* in the sense of [14]. Every such space can be considered an image in $\mathcal{S}_p$ of a *p-primary polyhedron*, i.e., such polyhedron X that the map $p^k 1_X$ for some k can be factored through a wedge of spheres [8].

To study the categories $\mathcal{S}_p^n$ we use the technique of *bimodule categories*, like in [11]. We recall the corresponding notions.

Definition 1.1 (cf. [9], Section 4)**.** *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be additive categories*, $\mathcal{M}$ *be an* $\mathcal{A}$-$\mathcal{B}$-*bimodule*, *i.e., a biadditive functor* $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Ab}$ (*the category of abelian groups*). *The bimodule category* $\mathcal{E}(\mathcal{M})$ (*or the category of elements of* $\mathcal{M}$) *is defined as follows*:

$\operatorname{Ob}\mathcal{E}(\mathcal{M}) = \bigcup_{\substack{A\in\operatorname{Ob}\mathcal{A}\\ B\in\operatorname{Ob}\mathcal{B}}} \mathcal{M}(A,B)$.

If $u\in\mathcal{M}(A,B)$, $v\in\mathcal{M}(A',B')$, *then*

$$\operatorname{Hom}_{\mathcal{E}(\mathcal{M})}(u,v) = \{\,(f,g)\mid f\colon A\to A',\ g\colon B\to B',\ gu=vf\,\}$$

$\big($*both these elements are from* $\mathcal{M}(A,B')\big)$. $\mathcal{E}(\mathcal{M})$ *is also an additive category. Note that we only consider bipartite bimodules in the sense of* [9].

Usually we choose a set of *additive generators* of $\mathcal{A}$ and $\mathcal{B}$, i.e., sets $\{A_1,A_2,\dots,A_s\}\subset\operatorname{Ob}\mathcal{A}$ and $\{B_1,B_2,\dots,B_r\}\subset\operatorname{Ob}\mathcal{B}$ such that every object from $\mathcal{A}$ (respectively, from $\mathcal{B}$) is isomorphic to a direct sum $\bigoplus_{j=1}^{s}k_jA_j$ (respectively, $\bigoplus_{i=1}^{r}l_iB_i$). Then an object of $\mathcal{E}(\mathcal{M})$ can be presented as a block matrix $F=(F_{ij})$, where F_{ij} is a matrix of size $l_i\times k_j$ with coefficients from $\mathcal{M}(A_j,B_i)$. If we present morphisms in the analogous matrix form, the action of morphisms on elements from $\mathcal{M}$ is presented by the usual matrix multiplication.

We use the following localized version of [11] (Theorem 2.2).

Theorem 1.1. *Let* $n\le m<2n-1$. *Denote by* $\mathcal{A}$ (*respectively, by* $\mathcal{B}$) *the full subcategory of* $\mathcal{S}_p$ *consisting of* $(m-1)$*-connected polyhedra of dimension at most* $2n-2$ (*respectively, of* $(n-1)$*-connected polyhedra of dimension at most* m). *Consider the* $\mathcal{A}$*-*$\mathcal{B}$*-bimodule* $\mathcal{M}$ *such that* $\mathcal{M}(A,B)=\operatorname{Hos}_p(A,B)$. *Let* $\mathcal{I}$ *be the ideal of the category* $\mathcal{E}(\mathcal{M})$ *consisting of all morphisms* $(\alpha,\beta)\colon f\to f'$ *such that* α *factors through* f *and* β *factors through* f'. *Let also* $\mathcal{J}$ *be the ideal of* $\overline{\mathcal{S}}_p^n$ *consisting of all maps* $f\colon X\to Y$ *such that* f *factors both through an object from* $\mathcal{A}[1]$ *and through an object from* $\mathcal{B}$. *The map* $f\mapsto Cf$ (*the* cone *of* f) *induces an equivalence* $\mathcal{E}(\mathcal{M})/\mathcal{I}\simeq\overline{\mathcal{S}}_p^n/\mathcal{J}$. *Moreover,* $\mathcal{J}^2=0$, *hence the isomorphism classes of the categories* $\overline{\mathcal{S}}_p^n$ *and* $\overline{\mathcal{S}}_p^n/\mathcal{J}$ *are the same.*

Note also that all groups $\mathcal{J}(X,Y)$ are finite [12] (Corollary 1.10).

Finally, recall that, for $k<l<k+2p(p-1)-1$, the only nontrivial p-components of the stable homotopy groups $\operatorname{Hos}(S^l,S^k)$ are $\operatorname{Hos}_p(S^{k+q_s},S^k)=\mathbb{Z}/p$, where $1\le s<p$ and $q_s=$ $=2s(p-1)-1$ [16].

2. Moore polyhedra. The only atoms in $\mathcal{S}_p^2$ are *Moore atoms* M_k ($k\in\mathbb{N}$) which are cones of the maps $S^2\xrightarrow{p^k}S^2$. We denote their d-dimensional suspensions $M_k[d-3]$ by M_k^d and call them *Moore polyhedra*. For unification, we denote S^d by M_0^d. We need to know the morphism groups $\mathcal{M}_{kl}^{dr}=\operatorname{Hos}_p(M_l^r,M_k^d)$. We always suppose that $d-1\le r<d+2p-1$. Obviously, $\mathcal{M}_{00}^{dd}=\mathbb{Z}_p$, $\mathcal{M}_{00}^{d,d+2p-3}=\mathbb{Z}/p$ and $\mathcal{M}_{00}^{dr}=0$ if $r\notin\{d,d+2p-3\}$. If $k>0$, from the cofibre sequences

$$S^{d-1}\xrightarrow{p^k}S^{d-1}\to M_k^d\to S^d\xrightarrow{p^k}S^d \qquad (\mathbf{E}_k^d)$$

one easily obtains that $\mathcal{M}_{0k}^{dr}=\mathcal{M}_{k0}^{dr}=0$, except the cases

$$\mathcal{M}_{k0}^{d,d-1}\simeq\mathcal{M}_{0k}^{dd}\simeq\mathbb{Z}/p^k,$$

$$\mathcal{M}_{k0}^{d,d+2p-3}\simeq\mathcal{M}_{0k}^{d,d+2p-3}\simeq\mathcal{M}_{k0}^{d,d+2p-4}\simeq\mathcal{M}_{0k}^{d,d+2p-2}\simeq\mathbb{Z}/p.$$

The values of $\mathcal{M}_{kl}^{dr}$ for $k,l\in\mathbb{N}$, $d-1\le r<d+2p-1$ can be obtained if we apply $\operatorname{Hos}_p(\mathcal{M}_l^r,\_)$ to the cofibre sequences ($\mathbf{E}_k^d$). It gives exact sequences

$$\mathcal{M}_{0l}^{d-1,r}\xrightarrow{p^k}\mathcal{M}_{0l}^{d-1,r}\to\mathcal{M}_{kl}^{dr}\to\mathcal{M}_{0l}^{dr}\xrightarrow{p^k}\mathcal{M}_{0l}^{dr},$$

whence we get

$$\mathcal{M}_{kl}^{dr} = \begin{cases} \mathbb{Z}/p^{\min(k,l)} & \text{if } r \in \{d-1, d\}, \\ \mathbb{Z}/p & \text{if } r \in \{d+2p-2, d+2p-4\}, \\ \mathbb{Z}/p \oplus \mathbb{Z}/p & \text{if } r = d+2p-3, \\ 0 & \text{in other cases.} \end{cases} \tag{2.1}$$

The only nontrivial value here is for $r = d + 2p - 3$: we need to know that the exact sequence

$$0 \to \mathbb{Z}/p \xrightarrow{\alpha} \mathcal{M}_{kl}^{d,d+2p-3} \xrightarrow{\beta} \mathbb{Z}/p \to 0 \tag{2.2}$$

splits. It splits indeed for $k = 1$ since the middle term is a module over $\mathcal{M}_{11}^{dd} = \mathbb{Z}/p$. If $k > 1$, suppose that the sequence for $\mathcal{M}_{k-1,l}^{d,d+2p-3}$ splits. The commutative diagram

$$\begin{array}{ccccccccc} S^{d-1} & \xrightarrow{p^k} & S^{d-1} & \longrightarrow & M_k^d & \longrightarrow & S^d & \xrightarrow{p^k} & S^d \\ {\scriptstyle p}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle 1}\downarrow \\ S^{d-1} & \xrightarrow{p^{k-1}} & S^{d-1} & \longrightarrow & M_{k-1}^d & \longrightarrow & S^d & \xrightarrow{p^{k-1}} & S^d \end{array} \tag{2.3}$$

induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & \mathcal{M}_{kl}^{d,d+2p-3} & \longrightarrow & \mathbb{Z}/p & \longrightarrow & 0 \\ & & {\scriptstyle 1}\downarrow & & \downarrow & & {\scriptstyle 0}\downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & \mathcal{M}_{k-1,l}^{d,d+2p-3} & \longrightarrow & \mathbb{Z}/p & \longrightarrow & 0. \end{array}$$

Since the second row splits, the first one splits as well. Therefore, the sequence (2.2) splits for all values of k and l.

Definition 2.1. *We fix generators of the groups $\mathcal{M}_{kl}^{dr}$ and denote, for $r = d + 2p - 3$,*
by $\alpha_{kl}^{d_*^*}$ $(k, l \in \mathbb{N})$ the generator of $\mathcal{M}_{kl}^{d+1,r+1}$ which is in the image of the map α from (2.2);
by α_{kl}^{d} $(k, l \in \mathbb{N} \cup \{0\})$ the generator of $\mathcal{M}_{kl}^{dr}$ which is not in $\operatorname{Im}\alpha$;
by $\alpha_{kl}^{d^*}$ $(k \in \mathbb{N} \cup \{0\}, l \in \mathbb{N})$ the generator of $\mathcal{M}_{kl}^{d,r+1}$;
by $\alpha_{kl}^{d_*}$ $(k \in \mathbb{N}, l \in \mathbb{N} \cup \{0\})$ the generator of $\mathcal{M}_{kl}^{d+1,r}$;
by γ_{kl}^{d} $(k, l \in \mathbb{N} \cup \{0\})$ the generator of $\mathcal{M}_{kl}^{dd}$;
by γ_{kl}^{d*} $(k \in \mathbb{N}, l \in \mathbb{N} \cup \{0\})$ the generator of $\mathcal{M}_{kl}^{d+1,d}$.

Note that all these morphisms are actually induced by maps $S^r \to S^d$. Using diagrams of the sort (2.3), one easily verifies that these generators can be so chosen that

$$\alpha_{kl}^{d_*^*}\gamma_{ll'}^{r+1} = \begin{cases} \alpha_{kl'}^{d_*^*} & \text{if } l \le l', \\ 0 & \text{if } l > l', \end{cases} \qquad \alpha_{kl}^{d^*}\gamma_{ll'}^{r+1} = \begin{cases} \alpha_{kl'}^{d^*} & \text{if } l \le l', \\ 0 & \text{if } l > l', \end{cases}$$

$$\alpha_{kl}^{d}\gamma_{ll'}^{r} = \begin{cases} \alpha_{kl'}^{d} & \text{if } l \ge l' \text{ or } l = 0, \\ 0 & \text{if } 0 < l < l', \end{cases} \qquad \alpha_{kl}^{d_*}\gamma_{ll'}^{r} = \begin{cases} \alpha_{kl'}^{d_*} & \text{if } l \ge l' \text{ or } l = 0, \\ 0 & \text{if } 0 < l < l', \end{cases}$$

$$\begin{gathered}\alpha^{d^*}_{kl}\gamma^{r*}_{lk'} = \alpha^{d}_{kk'}, \qquad \alpha^{d^*_*}_{kl}\gamma^{r*}_{lk'} = \alpha^{d_*}_{kk'},\\
\gamma^{d+1}_{k'k}\alpha^{d^*_*}_{kl} = \begin{cases}\alpha^{d^*_*}_{k'l} & \text{if } k \ge k',\\ 0 & \text{if } k < k',\end{cases} \qquad \gamma^{d+1}_{k'k}\alpha^{d_*}_{kl} = \begin{cases}\alpha^{d_*}_{k'l} & \text{if } k \ge k',\\ 0 & \text{if } k < k',\end{cases}\\
\gamma^{d}_{k'k}\alpha^{d}_{kl} = \begin{cases}\alpha^{d}_{k'l} & \text{if } k \le k',\\ 0 & \text{if } k > k',\end{cases} \qquad \gamma^{d}_{k'k}\alpha^{d^*}_{kl} = \begin{cases}\alpha^{d^*}_{k'l} & \text{if } k \le k',\\ 0 & \text{if } k > k',\end{cases}\\
\gamma^{d*}_{k'k}\alpha^{d}_{kl} = \alpha^{d_*}_{k'l}, \qquad \gamma^{d*}_{k'k}\alpha^{d^*}_{kl} = \alpha^{d^*_*}_{k'l}\end{gathered} \tag{2.4}$$

(always $r = d + 2p - 3$).

3. Atoms in $\mathcal{S}^{2p-1}_p$. For $n \le 2p-1$ the description of the category $\mathcal{S}^n_p$ is very simple. First, the next fact is rather obvious.

Proposition 3.1. *If $n < 2p-1$, all indecomposable polyhedra in $\mathcal{S}^n_p$ are Moore spaces M^d_k. In particular, M^2_k are atoms in $\mathcal{S}^2_p$ and there are no atoms in $\mathcal{S}^n_p$ if $2 < n < 2p-1$.*

Proof is an easy induction. For $n = 2$ it is known. Suppose that $2 < n < 2p-1$ and the claim is true for $\mathcal{S}^{n-1}_p$. We use Theorem 1.1 with $m = 2n-2$. Then $\mathcal{A}$ consists of wedges of the sphere S^{2n-2}, while the spheres S^d $(n \le d \le 2n-2)$ and the Moore atoms M^d_k $(n < d \le 2n-2)$ form a set of additive generators of $\mathcal{B}$. Note that in our case $\mathcal{M}^{dr}_{k0} = 0$ for $n < d \le r \le 2n-2$, except $\mathcal{M}^{2n-2,2n-2}_{00}$. Therefore, the only new indecomposable polyhedra in $\mathcal{S}^n_p$ are the Moore spaces M^{2n-1}_k, which are not atoms.

Proposition 3.1 is proved.

Consider the category $\mathcal{S}^{2p-1}_p$. Again we use Theorem 1.1 with $m = 2n-3 = 4p-5$. Now a set of additive generators of $\mathcal{A}$ is

$$\mathbf{A} = \Big\{S^{4p-4} = M^{4p-4}_0,\ S^{4p-5} = M^{4p-5}_0,\ M^{4p-5}_k\Big\},$$

and a set of additive generators of $\mathcal{B}$ is

$$\mathbf{B} = \Big\{S^d = M^d_0\ (2p-1 \le d \le 4p-5),\ M^d_k\ (2p-1 < d \le 4p-5)\Big\}.$$

The only nonzero values of $\mathrm{Hos}_p(A,B)$, where $A \in \mathbf{A}$, $B \in \mathbf{B}$, are

$\mathcal{M}^{2p,4(p-1)}_{kl} \simeq \mathbb{Z}/p$, with generators $\alpha^{(2p-1)*}_{kl}$, $k \in \mathbb{N}$, $l \in \mathbb{N} \cup \{0\}$,

$\mathcal{M}^{2p-1,4(p-1)}_{0l} \simeq \mathbb{Z}/p$ with generators α^{2p-1}_{0l}, $l \in \mathbb{N} \cup \{0\}$,

$\mathcal{M}^{4p-5,4p-5}_{00} = \mathbb{Z}_p$ with generator γ^{4p-5}_{00}.

Therefore, the matrix F defining a morphism $f\colon A \to B$ $(A \in \mathcal{A},\ B \in \mathcal{B})$ is a direct sum $F' \oplus \oplus F''$, where F'' is with coefficients from $\mathcal{M}^{4p-5,4p-5}_{00}$ and F' is a block matrix $(F_{kl})_{k,l \in \mathbb{N}\cup\{0\}}$, where F_{kl} is with coefficients from $\mathcal{M}^{2p,4(p-1)}_{kl}$ if $k \ne 0$ and F_{0l} is with coefficients from $\mathcal{M}^{2p-1,4(p-1)}_{0l}$. We denote by F_k the horizontal stripe $(F_{kl})_{l \in \mathbb{N}\cup\{0\}}$ with fixed k and by F^l the vertical stripe $(F_{kl})_{k \in \mathbb{N}\cup\{0\}}$ with fixed l. Morphisms between objects from $\mathbf{A}$ and $\mathbf{B}$ act according to the rules (2.4). They imply that two matrices F and G of such structure define isomorphic objects from $\mathcal{E}(\mathcal{M})$ if and only if $G'' = TF''T'$ for some invertible matrices T, T' over $\mathbb{Z}_p$ and F' can be transformed to G' by a sequence of the following transformations:

$F_k \mapsto TF_k$, where T is an invertible matrix over $\mathbb{Z}/p$;

$F^l \mapsto F^l T'$, where T' is an invertible matrix over $\mathbb{Z}/p$;

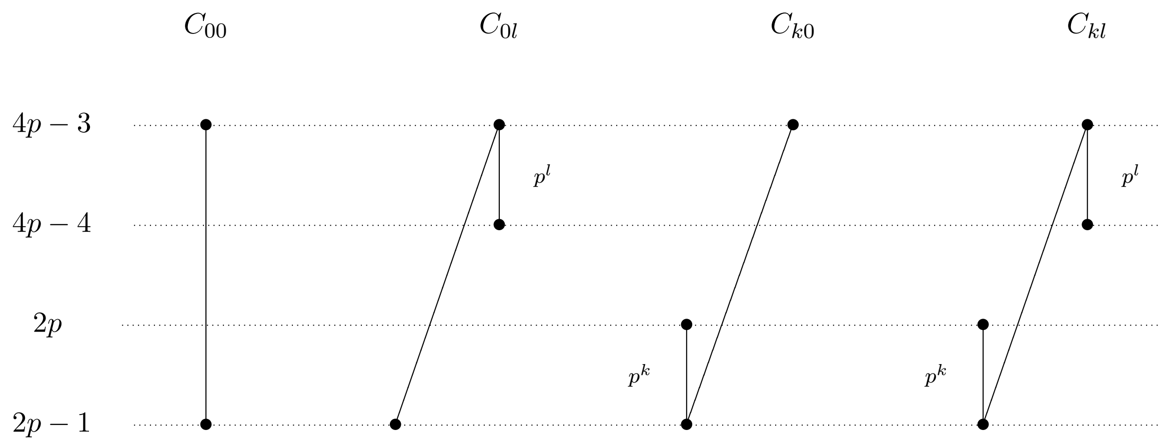
$F_k \mapsto F_k + UF_{k'}$, where $k' > k$ or $k' = 0$, $k \neq 0$ and U is any matrix of appropriate size over $\mathbb{Z}/p$;

$F^l \mapsto F^l + F^{l'}U'$, where $l' < l$ and U' is any matrix of appropriate size over $\mathbb{Z}/p$.

Using these transformations one can easily make the matrix F'' diagonal and reduce F' to a matrix having at most one nonzero element in each row and in each column. Then the corresponding object from $\mathcal{E}(\mathcal{M})$ splits into direct sum of objects given by (1×1)-matrices. The (1×1)-matrices over $\mathcal{M}_{00}^{4p-5,4p-5}$ give Moore polyhedra M_t^{4p-4}, which are not atoms (and belong to $\mathcal{A}$). Therefore, the atoms in $\mathcal{S}_p^{2p-1}$ are C_{kl} $(k, l \in \mathbb{N} \cup \{0\})$ corresponding to the (1×1)-matrices $\left(\alpha_{kl}^{(2p-1)*}\right)$ if $k \neq 0$ and to $\left(\alpha_{0l}^{2p-1}\right)$ if $k = 0$. We call these polyhedra *Chang atoms*, in analogy with [2]. They are defined by the cofibration sequences

$$\begin{aligned} &M_l^{4p-4} \to M_k^{2p} \to C_{kl} \to M_l^{4p-3} \to M_k^{2p+1} \quad \text{if} \quad k \neq 0, \\ &M_l^{4p-4} \to S^{2p-1} \to C_{0l} \to M_l^{4p-3} \to S^{2p} \quad \text{if} \quad k = 0. \end{aligned} \qquad (\mathbf{C}_{kl})$$

We can also present Chang atoms by their *gluing diagrams*, as in [2, 10, 11]:



Here bullets correspond to cells, lines show the attaching maps and these maps are specified if necessary.

Theorem 1.1 and cofibration sequences $(\mathbf{C}_{kl})$ easily give the following values of the endomorphism rings of Chang atoms modulo the ideal $\mathcal{J}$:

$$\Delta = \{\,(a,b) \mid a \equiv b \pmod p\,\} \subset \mathbb{Z}_p \times \mathbb{Z}_p \quad \text{for} \quad C_{00},$$

$$\Delta_k = \{\,(a,b) \mid a \equiv b \pmod p\,\} \subset \mathbb{Z}_p \times \mathbb{Z}/p^k \quad \text{for} \quad C_{0k} \quad \text{and} \quad C_{k0} \quad (k \neq 0),$$

$$\Delta_{kl} = \{\,(a,b) \mid a \equiv b \pmod p\,\} \subset \mathbb{Z}/p^k \times \mathbb{Z}/p^l \quad \text{for} \quad C_{kl} \quad (k \neq 0, l \neq 0).$$

Since all these rings are local and $\mathcal{J}^2 = 0$, the endomorphism rings of Chang atoms are local. Therefore, these polyhedra are indeed indecomposable (hence atoms). Moreover, we can use the unique decomposition theorem of Krull – Schmidt – Azumaya [1] (Theorem I.3.6) and obtain the final result.

Theorem 3.1. *The atoms in* $\mathcal{S}_p^{2p-1}$ *are Chang atoms* C_{kl} $(k, l \in \mathbb{N} \cup \{0\})$. *Every polyhedron from* $\mathcal{S}_p^{2p-1}$ *uniquely decomposes into a wedge of spheres, Moore polyhedra and Chang atoms.*

In Section 5 we will need the whole endomorphism ring of the atom $C = C_{00}$. Applying Hos_p to the sequence $(\mathbf{C}_{00})$ as below, we obtain the commutative diagram with exact columns and rows

$$
\begin{tikzcd}
 & S^{2p} & S^{4p-3} \arrow[l] & C \arrow[l] & S^{2p-1} \arrow[l] & S^{4p-4} \arrow[l] \\
S^{4p-4} \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[d, "s"'] \\
S^{2p-1} \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & p\mathbb{Z}_p \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[r, "s"] \arrow[d, "1"'] & \mathbb{Z}/p \arrow[d] \\
C \arrow[d] & 0 \arrow[r] \arrow[d] & p\mathbb{Z}_p \arrow[r] \arrow[d] & \mathrm{Hos}_p(C,C) \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[r] \arrow[d] & 0 \arrow[d] \\
S^{4p-3} \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[r, "1"] \arrow[d, "s"'] & \mathbb{Z}_p \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
S^{2p} & \mathbb{Z}_p \arrow[r, "s"] & \mathbb{Z}/p \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
\tag{3.1}
$$

where s marks surjections. The central row and the central column, corresponding to the polyhedron C, are easily calculated from all other values. It shows that $\mathrm{Hos}_p(C, C)$ has no torsion, hence coincides with Δ. Analogous calculations show that $\mathcal{J}(C_{kl}, C_{kl})$ equals $\mathbb{Z}/p$ if $k = 0$ or $l = 0$ (but not both) and $(\mathbb{Z}/p)^2$ if both $k \neq 0$ and $l \neq 0$.

Theorem 3.1 also gives a description of *genera* of p-primary polyhedra in $\mathcal{S}^{2p-1}$. Recall that a *genus* is a class of polyhedra such that all their localizations are isomorphic (in the corresponding localized categories). Certainly, if these polyhedra are p-primary, we only need to compare their p-localizations. Equivalently, two polyhedra X, Y are in the same genus if and only if there is a wedge of spheres W such that $X \vee W \simeq Y \vee W$ in $\mathcal{S}$ [12] (Theorem 2.5). Let $g(X)$ be the number of isomorphism classes of polyhedra in the genus of X. If $\Lambda = \mathrm{Hos}(X, X)/\operatorname{tors}(X)$, where $\operatorname{tors}(X)$ is the torsion part of $\mathrm{Hos}(X, X)$, then $\mathbb{Q} \otimes \Lambda$ is a semisimple $\mathbb{Q}$-algebra, so there is a maximal order $\Gamma \supseteq \Lambda$ in this algebra. Then $\Lambda \supseteq m\Gamma$ for some positive integer m and $g(X) = g(\Lambda)$ equals the number of cosets

$$\operatorname{Im}\gamma \backslash (\Gamma/m\Gamma)^\times/(\Lambda/m\Lambda)^\times,$$

where $R^\times$ denotes the group of invertible elements of a ring R and γ is the natural map $\Gamma^\times \to (\Gamma/m\Gamma)^\times$ [12] (Section 3). If $X = C_{0k}$ or $X = C_{k0}$, then $\Lambda = \mathbb{Z}$; if $X = C_{kt}$, then $\Lambda = 0$. So $g(X) = 1$ for all these cases. For $X = C$ this formula implies that $g(C) = (p-1)/2$. If $\nu \in \mathrm{Hos}_p(S^{4p-4}, S^{2p-1})$ is an element of order p, the polyhedra from the genus of C can be realized as the cones $C(c)$ of the maps $S^{4p-4} \xrightarrow{c\nu} S^{2p-1}$ for $1 \leq c \leq (p-1)/2$.

4. Atoms in $\mathcal{S}_p^n$ for $2p \leq n \leq 4(p-1)$. Let now $2p \leq n \leq 4(p-1)$. We use Theorem 1.1 with $m = n + 2p - 3$. Then $\mathcal{A}$ has a set of additive generators

$$\mathbf{A} = \{S^r\ (m \le r < 2n-1),\ M_l^r\ (m < r < 2n-1,\, l \in \mathbb{N}\},$$

and $\mathcal{B}$ has a set of additive generators

$$\mathbf{B} = \Big\{S^d\ (n \le d \le m),\ M_k^d\ (n < d \le m,\, k \in \mathbb{N})\Big\}.$$

Morphisms $\varphi\colon A \to B$, where $A \in \mathcal{A}$, $B \in \mathcal{B}$, are given by block matrices such that their blocks have coefficients from $\mathcal{M}_{kl}^{dr}$. Taking into consideration Definition 2.1, it is convenient to denote these blocks as follows.

Definition 4.1. *We introduce sets*

$$\mathfrak{E}^\circ = \Big\{e_k^d\ (n < d \le 2(n-p)+1, k \in \mathbb{N}\cup\{0\}),\ e_k^{d*}\ (n \le d \le 2(n-p), k \in \mathbb{N}),\ e_0^n, e_0^m\Big\},$$

$$\mathfrak{F}^\circ = \Big\{f_l^d\ (n < d \le 2(n-p)+1, l \in \mathbb{N}\cup\{0\}),\ f_l^{d*}\ (n \le d \le 2(n-p), l \in \mathbb{N}),\ f_0^n\Big\},$$

and consider a morphism $\varphi\colon A \to B$, where $A \in \mathcal{A}$, $B \in \mathcal{B}$, as a block matrix $(\Phi_{ef})_{e\in\mathfrak{E}^\circ, f\in\mathfrak{F}^\circ}$. Namely,

the block $\Phi_{e_k^d, f_l^d}$ consists of coefficients at α_{kl}^d;
the block $\Phi_{e_k^{d*}, f_l^d}$ consists of coefficients at $\alpha_{kl}^{d_*}$;
the block $\Phi_{e_k^d, f_l^{d*}}$ consists of coefficients at $\alpha_{kl}^{d^*}$;
the block $\Phi_{e_k^{d*}, f_l^{d*}}$ consists of coefficients at $\alpha_{kl}^{d^*_*}$;
the block $\Phi_{e_0^m, f_0^n}$ consists of coefficients at γ_{00}^m.

Note that for $n = 4(p-1)$ we need not specially add e_0^m to $\mathfrak{E}^\circ$, since $m = 2(n-p)+1$ in this case.

We also denote by Φ_e for a fixed $e \in \mathfrak{E}^\circ$ the horizontal stripe $(\Phi_{ef})_{f\in\mathfrak{F}^\circ}$ and by Φ^f for a fixed $f \in \mathfrak{F}^\circ$ the vertical stripe $(\Phi_{ef})_{e\in\mathfrak{E}^\circ}$.

Note that the horizontal stripes $\Phi_{e_k^d}$ and $\Phi_{e_k^{(d+1)*}}$ have the same number of rows and the vertical stripes $\Phi^{f_l^d}$ and $\Phi^{f_l^{(d+1)*}}$ have the same number of columns. All blocks Φ_{ef} defined above have coefficients from $\mathbb{Z}/p$, except $\Phi_{e_0^m, f_0^n}$ which has coefficients from $\mathbb{Z}_p$.

Using automorphisms of S^m we can make the block $\Phi_{e_0^m, f_0^n}$ diagonal with powers of p or zero on diagonal. So we always suppose that it is of this shape and exclude this block from the matrix Φ. Then we have to split the remaining part of the vertical stripe $\Phi^{f_0^n}$ and, if $n = 4(p-1)$, of the horizontal stripe $\Phi_{e_0^m}$ into several stripes, respectively, $\Phi^{f_0^{n,s}}$ and $\Phi_{e_0^{m,s}}$, where the indices $s \in \mathbb{N}\cup\{\infty\}$ correspond to diagonal entries p^s (setting $p^\infty = 0$). Respectively, we modify the sets $\mathfrak{E}^\circ$ and $\mathfrak{F}^\circ$. Namely, we denote

$$\mathfrak{F} = (\mathfrak{F}^\circ \setminus \{f_0^n\}) \cup \{\, f_0^{n,s} \mid s \in \mathbb{N}\cup\{\infty\}\,\},$$

$$\mathfrak{E} = \mathfrak{E}^\circ \setminus \{e_0^m\} \quad \text{if} \quad n < 4(p-1), \tag{4.1}$$

$$\mathfrak{E} = (\mathfrak{E}^\circ \setminus \{e_0^m\}) \cup \{\, e_0^{m,s} \mid s \in \mathbb{N}\cup\{\infty\}\,\} \quad \text{if} \quad n = 4(p-1).$$

Note that, if $n = 4(p-1)$, the number of rows in the horizontal stripe $\Phi_{e_0^{d,s}}$ with $s \ne \infty$ equals the number of columns in the vertical stripe $\Phi^{f_0^{d,s}}$. We split the sets $\mathfrak{E}$ and $\mathfrak{F}$ according to the upper indices. Namely, $\mathfrak{E}_d$ consists of all elements from $\mathfrak{E}$ with the upper index d, d^* or, if $d = m$, (m, s); $\mathfrak{F}_d$ consists of all elements from $\mathfrak{F}$ with the upper index d, d^* or, if $d = n$, (n, s). We define a linear order on each $\mathfrak{E}_d$ and $\mathfrak{F}_d$ setting

$e^d_k < e^d_{k'}$ and $e^{d*}_k > e^{d*}_{k'}$ if $k < k'$, and $e^d_k < e^{d*}_{k'}$ for all k, k';

if $n = 4(p-1)$, then $e^{m,s}_0 < e^{m,s'}_0 < e^m_k$ for $s > s'$ and any $k \in \mathbb{N}$;

$f^d_k < f^d_{k'}$ and $f^{d*}_k > f^{d*}_{k'}$ if $k < k'$ or $k > k' = 0$, and $f^d_k < f^{d*}_{k'}$ for all k, k';

$f^m_k < f^{m,s}_0 < f^{m,s'}_0$ for $s < s'$ and any $k \in \mathbb{N}$.

The formulae (2.4) imply that two such block matrices Φ and Φ' define isomorphic objects from $\mathcal{E}(\mathcal{M})$ if and only if Φ can be transformed to Φ' by a sequence of the following transformations:

$\Phi_e \mapsto T_e\Phi_e$, where T_e are invertible matrices and $T_{e^{d*}_k} = T_{e^{d+1}_k}$ for all possible values of d, k;

$\Phi^f \mapsto \Phi^f T^f$, where T^f are invertible matrices and $T^{f^{d*}_k} = T^{f^{d+1}_k}$ for all possible values of d, k;

if $n = 4(p-1)$, then, moreover, $T_{e^{m,s}_0} = T^{f^{n,s}_0}$ for all $s \in \mathbb{N}$ (not for $s = \infty$);

$\Phi_e \mapsto U_{ee'}\Phi_{e'}$ if $e' < e$, where $U_{ee'}$ is an arbitrary matrix of the appropriate size;

$\Phi^f \mapsto \Phi^{f'} U^{f'f}$ if $f' > f$, where $U^{f'f}$ is an arbitrary matrix of the appropriate size.

These rules show that the classification of polyhedra in $\mathcal{S}^n_p$ actually coincides with the classification of representations of the *bunch of chains* $\mathfrak{X} = \{\mathfrak{E}_d, \mathfrak{F}_d, <, \sim \mid n \le d \le m\}$ (cf. [6] or [7] (Appendix B)), where the relation $\sim$ is defined by the exclusive rules:

$$e^{d*}_k \sim e^{d+1}_k \quad \text{and} \quad f^{d*}_k \sim f^{d+1}_k \quad \text{for} \quad n < d \le 2(n-p), \quad k \in \mathbb{N},$$

and, if $n = 4(p-1)$,

$$e^{m,s}_0 \sim f^{n,s}_0 \quad \text{for} \quad s \in \mathbb{N} \quad (\text{not for} \quad s = \infty).$$

Thus the description of indecomposable representations given in [6, 7] implies a description of indecomposable polyhedra from $\mathcal{S}^n_p$. Recall the necessary combinatorics. We write $e - f$ and $f - e$ if $e \in \mathfrak{E}_d$ and $f \in \mathfrak{F}_d$ (with the same d) and set $|\mathfrak{X}| = \mathfrak{E} \cup \mathfrak{F}$.

Definition 4.2. (1) *A word is a sequence* $w = x_1r_1x_2r_2\ldots x_{l-1}r_{l-1}x_l$, *where* $x_i \in |\mathfrak{X}|$, $r_i \in \{-, \sim\}$ *such that*

a) $r_i \ne r_{i+1}$ *for all* $1 \le i < l-1$;

b) $x_ir_ix_{i+1}$, $1 \le i < l$, *according to the definition of the relations* $\sim$ *and* $-$ *given above*;

c) *if* $r_1 = -$ $(r_{l-1} = -)$, *then* $x_1 \nsim y$ *for all* $y \in |\mathfrak{X}|$ (*respectively,* $x_l \nsim y$ *for all* $y \in |\mathfrak{X}|$).

We say that l *is the length of the word* w *and write* $l = \ln w$.

(2) *For a word* w *as above we denote by* $\mathrm{E}(w) = \{i \mid 1 \le i \le l,\ x_i \in \mathfrak{E}\}$ *and* $\mathrm{F}(w) = \{i \mid 1 \le i \le l,\ x_i \in \mathfrak{F}\}$.

(3) *The inverse word* w^* *of the word* w *is the word* $x_lr_{l-1}x_{l-1}\ldots r_2x_2r_1x_1$.

(4) *A word* w *is said to be a cycle if* $r_1 = r_{l-1} = \sim$ *and* $x_l - x_1$. *Then we set* $r_l = -$, $x_{i+ql} = x_i$ *and* $r_{i+ql} = r_i$ *for all* $q \in \mathbb{Z}$ (*in particular*, $r_0 = -$).

(5) *The* k *th shift of a cycle* w, *where* k *is an even integer, is the cycle* $w^{[k]} = x_{k+1}r_{k+1}\ldots r_{k-1}x_k$ (*obviously, it is enough to consider* $0 \le k < l$).

(6) *A cycle* w *is said to be non periodic if* $w \ne w^{[k]}$ *for* $0 < k < l$.

(7) *For a cycle* w *and an integer* $0 < k < l$ *we denote by* $\nu(k, w)$ *the number of even integers* $0 < i < k$ *such that both* x_i *and* x_{i-1} *belong either to* $\mathfrak{E}$ *or to* $\mathfrak{F}$.

Note that, since $x \nsim x$ for all $x \in |\mathfrak{X}|$, there are no symmetric words and symmetric cycles in the sense of [7] (Appendix B).

To words and cycles correspond indecomposable representations of the bunch of chains $\mathfrak{X}$ called *strings* and *bands*. We describe the corresponding matrices Φ (recall that we have already excluded the part $\Phi_{e_m f_n}$).

Definition 4.3. (1) *If w is a word, the corresponding string matrix $\Phi(w)$ is constructed as follows*:

its rows are labelled by the set $\mathrm{E}(w)$ *and its columns are labelled by the set* $\mathrm{F}(w)$;

the only nonzero entries are those at the places $(i, i+1)$ *if* $r_i = -$ *and* $i \in \mathrm{E}(w)$ *and* $(i+1, i)$ *if* $r_i = -$ *and* $x_i \in \mathrm{F}(w)$; *they equal* 1.

We denote the corresponding polyhedron by $A(w)$ and call it a string polyhedron whenever it does not coincide with a sphere, a Moore or a Chang polyhedron[1].

(2) *If w is a non periodic cycle, $z \in \mathbb{N}$ and $\pi \neq t$ is a unital irreducible polynomial of degree v from $(\mathbb{Z}/p)[t]$, the band matrix $\Phi(w, z, \pi)$ is a block matrix, where all blocks are of size $zv \times zv$, constructed as follows*:

its horizontal stripes are labelled by the set $\mathrm{E}(w)$ *and its vertical stripes are labelled by the set* $\mathrm{F}(w)$;

the only nonzero blocks are those at the places $(i, i+1)$ *if* $r_i = -$ *and* $i \in \mathrm{E}(w)$ *and* $(i+1, i)$ *if* $r_i = -$ *and* $i \in \mathrm{F}(w)$ (*note that here* $i = l$ *is also possible*);

these nonzero blocks equal I_{zv} (*the identity* $zv \times zv$ *matrix*), *except the block at the place* $(l1)$ $\big($*if* $l \in \mathrm{E}(w)\big)$ *or* $(1l)$ $\big($*if* $l \in \mathrm{F}(w)\big)$ *which is the Frobenius matrix with the characteristic polynomial* π^v. *If* $\pi = t - c$ *is linear*, *we replace the Frobenius matrix by the Jordan* $z \times z$ *block with the eigenvalue* c.

We denote the corresponding polyhedron by $A(w, z, \pi)$ and call it a band polyhedron[2].

Using these notions, we obtain the following description of polyhedra in the category $\mathcal{S}_p^n$.

Theorem 4.1. (1) *All string and band polyhedra are indecomposable and every indecomposable polyhedron from* $\mathcal{S}_p^n$, *except spheres*, *Moore and Chang polyhedra*, *is isomorphic to a string or band polyhedron.*

(2) *The only isomorphisms between string and band polyhedra are the following*:

$A(w) \simeq A(w^*)$;

$A(w, z, \pi) \simeq A(w^*, z, \pi)$;

$A(w, z, \pi) \simeq A(w^{[k]}, z, \pi^*)$, *where* $\pi^* = \pi$ *if* $\nu(k, w)$ *is even and* $\pi^*(t) = t^z \pi(0)^{-1} \pi(1/t)$ *if* $\nu(k, w)$ *is odd*[3].

(3) *Endomorphism rings of string and band polyhedra are local*, *hence every polyhedron from* $\mathcal{S}_p^n$ *uniquely decomposes into a wedge of spheres*, *Moore and Chang polyhedra*, *and string and band polyhedra.*

(4) *A string or band polyhedron is an* atom *in* $\mathcal{S}_p^n$ *if and only if the corresponding word contains at least one letter from* $\mathfrak{E}_d$ *and at least one letter from* $\mathfrak{F}_{2(n-p)+1}$.

Note that in this case we can simplify the writing of the words, since for every $x \in |\mathfrak{X}|$ there is at most one element $y \in |\mathfrak{X}|$ such that $x \sim y$ and then $x - y$ is impossible. Hence we can omit all symbols $-$ and write x instead of $x \sim y$. For instance, $e_k^d f_l^{d-1} e_{k'}^{(d-2)*} f_{l'}^{d-1}$ means $e_k^d \sim$ $\sim e_k^{(d-1)*} - f_l^{d-1} \sim f_l^{(d-2)*} - e_{k'}^{d-2} \sim e_{k'}^{(d-1)*} - f_{l'}^{d-1} \sim f_{l'}^{(d-2)*}$. One can prove that there can be

[1] The words consisting of one letter x correspond to spheres, the words of the form $x \sim y$ correspond to Moore polyhedra, the words that only have one symbol '$-$' correspond to Chang polyhedra, and these are all exceptions.

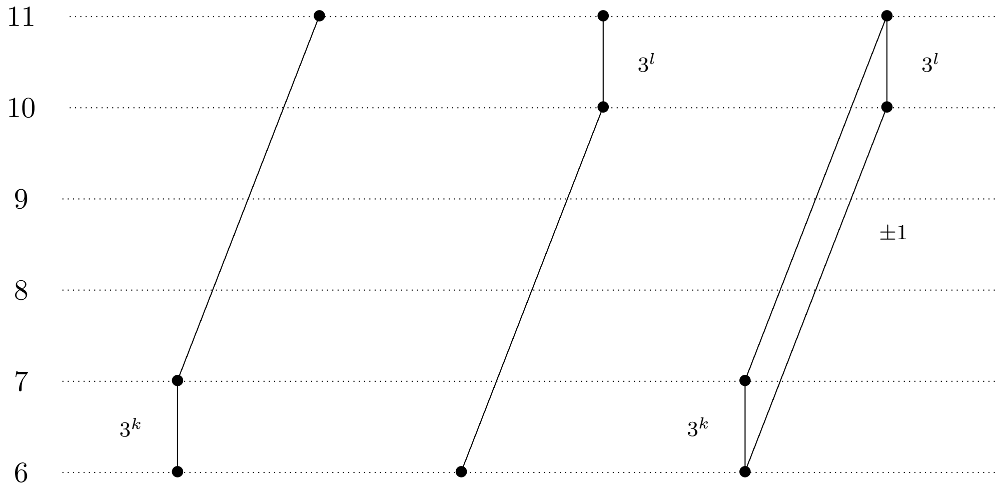
[2] Band polyhedra never coincide with spheres, Moore or Chang polyhedra.

[3] If $\pi = t^v + a_1 t^{v-1} + \ldots + a_{v-1} t + a_v$, then $\pi^* = t^v + a_v^{-1}(a_{v-1} t^{v-1} + \ldots + a_1 t + 1)$.

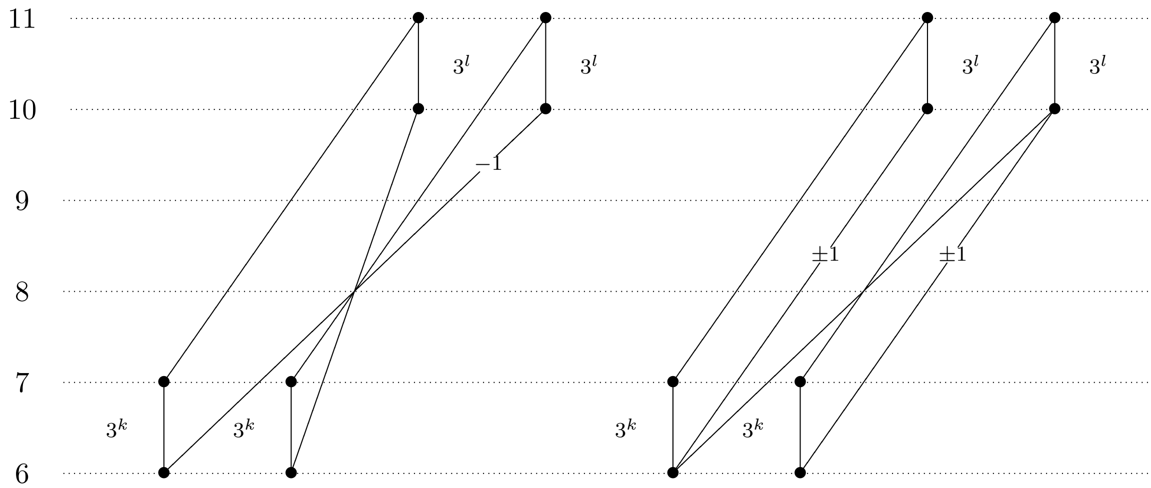
at most one place in a word w where a fragment $e^{m,s} \sim f^{n,s}$ or $f_0^{n,s} \sim e_0^{m,s}$ occurs; moreover, if it occurs, w cannot be a cycle.

***Example* 4.1.** We give several examples of string and band polyhedra and their gluing diagrams. In these examples we suppose that $p = 3$.

(1) The "smallest" possible string atoms are for $n = 6$. They have 3 cells and are given by the words $e_k^{6*} f_0^7$ or $e_0^6 f_l^{6*}$. The smallest band atoms have 4 cells. They are $A(w_0, 1, t \mp 1)$, where $w_0 = e_k^7 f_l^7$. Here are their gluing diagrams:

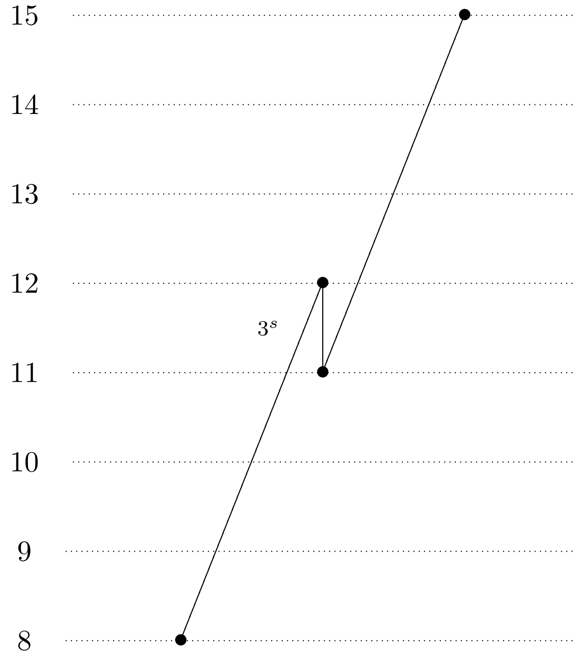


(2) More complicated band atoms are $A(w_0, 1, t^2+1)$ and $A(w_0, 2, t\mp1)$. Their gluing diagrams are

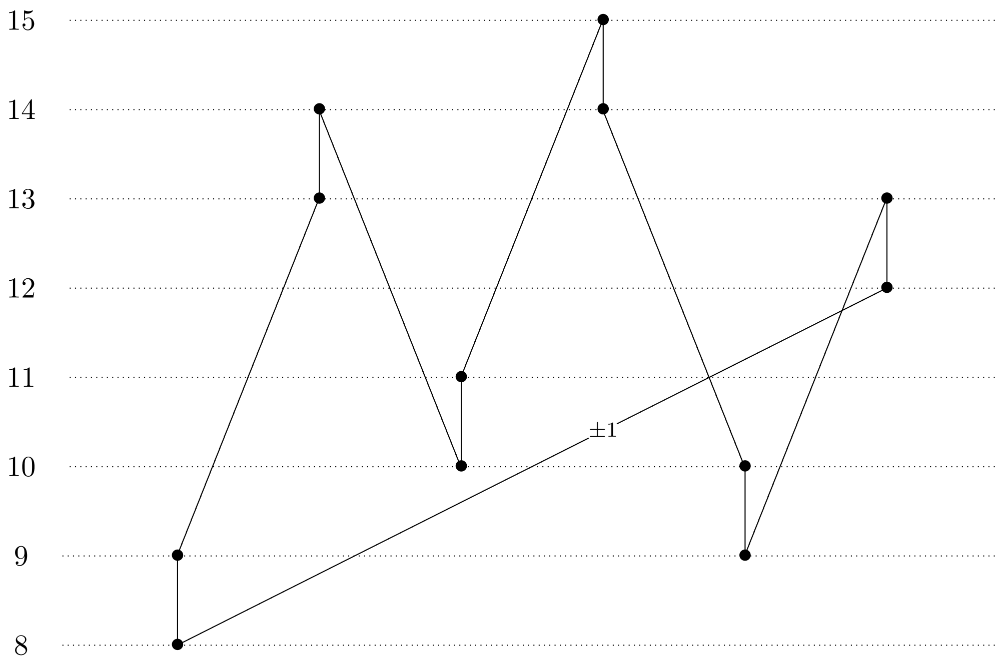


The nontrivial attachments of cells of dimension 10 come, respectively, from the Frobenius matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the Jordan block $\begin{pmatrix} \pm1 & 1 \\ 0 & \pm1 \end{pmatrix}$.

(3) For the maximal value $n = 8$ the smallest atoms contain 4 cells. They are given by the words $e_0^8 f_0^{8,s} f_0^{11}$ and have the gluing diagrams

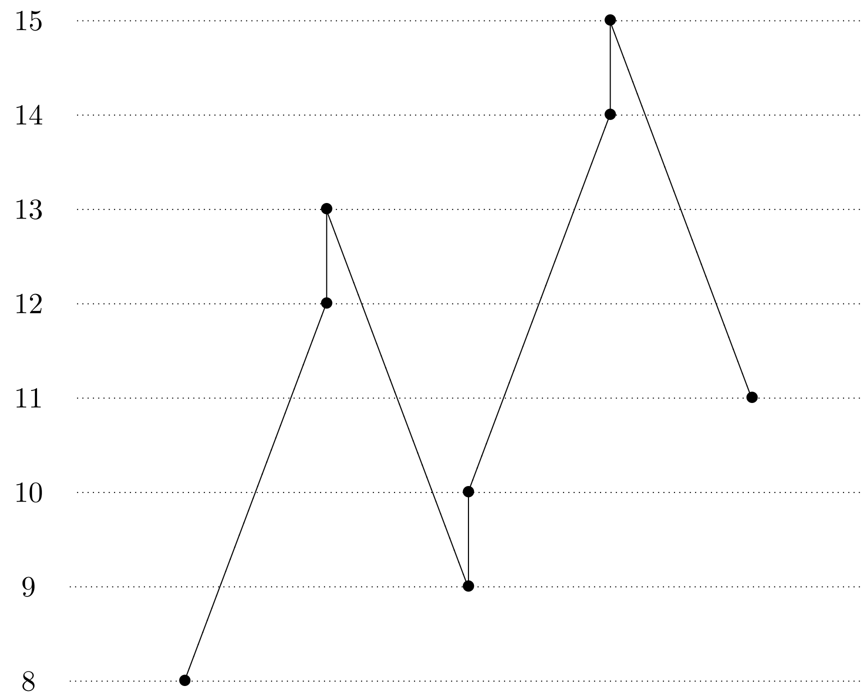


(4) The band atoms for $n = 8$ are rather complicated and cannot be "small". For instance, one of the smallest is $A(w, 1, t \mp 1)$, where $w = e_{k_1}^{8*} f_{l_1}^{9*} e_{k_2}^{10*} f_{l_2}^{11} e_{k_3}^{10} f_{l_3}^{9}$. The gluing diagram for this atom is



(the powers of 3 near vertical lines are omitted).

(5) Finally, we give an example of an atom having exactly one cell of each dimension (we do not precise the corresponding word, since it can be easily restored).



Another atom with this property is the properly shifted S-dual of this one in the sense of [15] (Chapter 14).

One can also calculate genera of p-primary polyhedra for $2p \le n \le 4(p-1)$. Namely, let $\Lambda(X)$ denotes the ring $\mathrm{Hos}(X,X)/\,\mathrm{tors}(X)$. We call the end x_1 or x_l of a word w *spherical* if it of the form e_0^d or f_0^d. Note that these letters can only occur at an end of the word since they are not related by $\sim$ to any letter. It is rather easy to verify that $\Lambda(X) = 0$ if X is a band polyhedron, while for a string polyhedron $X = A(w)$

$$\Lambda(X) = \begin{cases} 0 & \text{if } w \text{ has no spherical ends,} \\ \mathbb{Z} & \text{if one end of } w \text{ is spherical,} \\ \Delta & \text{if both ends of } w \text{ are spherical.} \end{cases}$$

Hence, we obtain the following result.

***Corollary* 4.1.** *If X is a band or string polyhedron, then $g(X) = 1$, except the case when $X = A(w)$ and both ends of the word w are spherical. In the latter case $g(X) = (p-1)/2$.*

5. Case $n > 4(p-1)$. For $n = 4p-3$ we set $m = 6p-5 = n+2p-2$ and $q = 2(n-1) = n+4p-5 = m+2p-3$. Then $\mathcal{A}$ contains Moore polyhedra M_k^q (including $S^q = M_0^q$) and $\mathcal{B}$ contains the shifted Chang polyhedron $C^m = C_{00}[2p-2]$. Let $\mathcal{N}_k = \mathrm{Hos}_p(M_k^q, C^m)$. Applying $\mathrm{Hos}_p(M_k^q, \_)$ to the cofibre sequence

$$0 \to S^{m-1} \to S^n \to C^m \to S^m \to S^{n+1}$$

we get an exact sequence

$$0 \to \mathbb{Z}/p \xrightarrow{\lambda} \mathcal{N}_k \xrightarrow{\mu} \mathbb{Z}/p \to 0.$$

Thus $\#(\mathcal{N}_k) = p^2$. On the other hand, applying $\mathrm{Hos}_p(\_\,, C)$ to the cofibre sequence ($\mathbf{E}^d_k$) of Section 2, we get an exact sequence

$$\mathcal{N}_0 \xrightarrow{p^k} \mathcal{N}_0 \xrightarrow{\eta} \mathcal{N}_k \to 0.$$

Therefore the map η is an isomorphism. Setting $k = 1$, we see that $p\mathcal{N}_0 = 0$, hence $\mathcal{N}_0 \simeq \mathbb{Z}/p \times \mathbb{Z}/p$ and $\mathcal{N}_k \simeq \mathbb{Z}/p \times \mathbb{Z}/p$ for all k. We denote by λ_k a generator of $\mathcal{N}_k$ which is in $\mathrm{Im}\,\lambda$ and by μ_k a generator of $\mathcal{N}_k$ such that $\mu(\mu_k) \neq 0$.

Analogous observations show that the generator of the cyclic group $\mathcal{M}^{qq}_{kl} = \mathrm{Hos}_p(M^q_l, M^q_k)$ induces an isomorphism $\mathcal{N}_k \to \mathcal{N}_l$ if $k \geq l > 0$ and zero map if $0 < k < l$. On the other hand, the diagram (3.1) implies that an element (a, b) of the ring $\Delta = \mathrm{Hos}_p(C, C)$ acts on $\mathcal{N}_k$ as multiplication by a $\big($recall that $a \equiv b \pmod p\big)$. Therefore, a map $\varphi\colon A \to B$, where A is a wedge of Moore polyhedra M^q_k and B is a wedge of Chang polyhedra C^m can be considered as a block matrix $\Phi = (\Phi_{ik})_{\substack{k\in\mathbb{N}\cup\{0\}\\ i=1,2}}$, where all blocks are with coefficients from $\mathbb{Z}/p$ and both horizontal stripes Φ_1, Φ_2 have the same number of rows. Namely, Φ_{1k} consists of coefficients at λ_k and Φ_{2k} consists of coefficients at μ_k. Two such matrices define isomorphic objects from $\mathcal{E}(\mathcal{M})$ if and only if one of them can be transformed to the other by a sequence of the following transformations:

$\Phi_1 \mapsto T\Phi_1$ and $\Phi_2 \mapsto T\Phi_2$ with the same invertible matrix T;

$\Phi^k \mapsto \Phi^k T^k$ for some invertible matrix T^k;

$\Phi^k \mapsto \Phi^k + \Phi^l U_{lk}$ for any matrix U_{lk} of the appropriate size, where $l > k$ or $l = 0 < k$.

It is well-known that this matrix problem is *wild*, i.e., contains the problem of classification of pairs of linear maps in a vector space; hence, a problem of classification of representations of any finitely generated algebra over the field $\mathbb{Z}/p$ (cf. [9], Section 5). Namely, consider the case when the matrix $\Phi = \Phi(F, G)$ is of the form

$$\left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & I & 0 \\ \hline F & I & 0 \\ G & 0 & I \end{array}\right].$$

Here I is a unit matrix of some size, F and G are arbitrary square matrices of the same size; line show the subdivision of Φ into blocks Φ_{ik} (there are only two vertical stripes). One easily checks that $\Phi(F, G)$ and $\Phi(F', G')$ define isomorphic objects if and only if there is an invertible matrix T such that $F' = TFT^{-1}$ and $G' = TGT^{-1}$. So we obtain the following result.

Theorem 5.1. *The classification of p-local polyhedra in $\mathcal{S}^n_p$ for $n > 4(p-1)$ is a wild problem.*

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