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DETERMINATION OF THE COEFFICIENT OF A SEMILINEAR PARABOLIC EQUATION IN THE CASE OF BOUNDARY-VALUE PROBLEM WITH NONLINEAR BOUNDARY CONDITION

ВИЗНАЧЕННЯ КОЕФІЦІЄНТА НАПІВЛІНІЙНОГО ПАРАБОЛІЧНОГО РІВНЯННЯ ДЛЯ ГРАНИЧНОЇ ЗАДАЧІ З НЕЛІНІЙНОЮ ГРАНИЧНОЮ УМОВОЮ

The goal of this paper is to investigate the well-posedness of the inverse problem of determination of the coefficient of a minor term of a semilinear parabolic equation in the case of nonlinear boundary condition. The additional condition is given in the nonlocal integral form. A uniqueness theorem and a "conditional" stability theorem are proved.

Досліджено коректність оберненої задачі про визначення коефіцієнта молодшого члена напівлінійного параболічного рівняння за наявності нелінійної граничної умови. Додаткову умову наведено в нелокальній інтегральній формі. Доведено теорему про єдиність та теорему про „умовну” стійкість.

Let R^n be an n-dimensional real Euclidean space, $x = (x_1, \ldots, x_n)$ be an arbitrary point of the bounded domain $D \subset R^n$ with a sufficiently smooth boundary ∂D, $\Omega = D \times (0;T]$, $S = \partial D \times [0;T]$, $0 < T$ be a fixed number.

The spaces $C^l(\cdot)$, $C^{l+\alpha}(\cdot)$, $C^{l,l/2}(\cdot)$, $C^{l+\alpha,(l+\alpha)/2}(\cdot)$, $l = 0,1,2$, $\alpha \in (0,1)$, and the norms in these spaces are defined as in [1, p. 12 – 20]

$$\|\cdot\|_l = \|\cdot\|_{C^l}, \quad \|g(x,t,u)\|_0 = \sup_{\Omega} |g(x,t,u(x,t))|,$$

$$u_t = \frac{\partial u}{\partial t}, \quad u_{x_i} = \frac{\partial u}{\partial x_i}, \quad i = \overline{1,n},$$

$\Delta u = \sum_{i=1}^n \dfrac{\partial^2 u}{\partial x_i^2}$ is a Laplacian, $\dfrac{\partial u}{\partial \nu}$ is an internal conormal derivative.

We consider an inverse problem on determining a pair of functions $\{u(x,t), c(x)\}$ from the conditions

$$u_t - \Delta u + c(x)u = f(x,t,u), \quad (x,t) \in \Omega, \tag{1}$$

$$u(x,0) = \varphi(x), \quad x \in \overline{D} = D \cup \partial D, \tag{2}$$

$$\frac{\partial u}{\partial \nu} = \psi(x,t,u), \quad (x,t) \in S, \tag{3}$$

$$\int_0^T u(x,t)\,dt = h(x), \quad x \in \overline{D}, \tag{4}$$

here $f(x,t,p)$, $\varphi(x)$, $\psi(x,t,p)$, $h(x)$ are the given functions.

The coefficiental inverse problems were studied in the papers [2–4] (see also the references therein).

For the input data of problem (1)–(4), we take the following assumptions:

1^0) for the function $f = f(x,t,p)$, we shall assume the following:

the function $f(x,t,p)$ is defined and continuous on the set

$$A = \left\{(x,t,p)|(x,t) \in \overline{\Omega}, \quad p \in R^1\right\},$$

for each $m_1 > 0$ and for $|p| < m_1$, the function $f(x,t,p)$ is uniformly Hölder continuous in x and t of orders α and $\alpha/2$, respectively, for each compact subset of A,

there exists a constant $m_2 > 0$ such that

$$|f(x,t,p_1) - f(x,t,p_2)| \le m_2 |p_1 - p_2|,$$

holds for all p_1, p_2 and $(x,t) \in \overline{\Omega}$;

2^0) $\varphi(x) \in C^{2+\alpha}\left(\overline{D}\right)$;

3^0) for the function $\psi = \psi(x,t,p)$, we shall assume the following:

the function $\psi(x,t,p)$ is defined and continuous on the set

$$B = \left\{(x,t,p)|(x,t) \in S, \quad p \in R^1\right\},$$

for each $m_3 > 0$ and for $|p| < m_3$, the function $\psi(x,t,p)$ is uniformly Hölder continuous in x and t of orders α and $\alpha/2$, respectively, for each compact subset of B,

there exists a constant $m_4 > 0$ such that

$$|\psi(x,t,p_1) - \psi(x,t,p_2)| \le m_4 |p_1 - p_2|,$$

holds for all p_1, p_2 and $(x,t) \in S$;

4^0) $h(x) \in C^{2+\alpha}\left(\overline{D}\right)$.

Definition 1. *The pair of functions* $\{c(x), u(x,t)\}$ *is called the solution of problem* (1)–(4) *if*:

1) $c(x) \in C\left(\overline{D}\right)$;

2) $u(x,t) \in C^{2,1}(\Omega) \cap C^{1,0}\left(\overline{\Omega}\right)$;

3) *the conditions* (1)–(4) *hold for these functions, here the condition* (3) *is defined in the following sense*:

$$\frac{\partial u(x,t)}{\partial \nu(x,t)} = \lim_{\substack{y \to x \\ y \in \sigma}} \frac{\partial u(y,t)}{\partial \nu(x,t)},$$

here σ *is any closed cone with a vertex* x, *contained in* $D \cup \{x\}$.

The uniqueness theorem and the estimation of stability of the solutions of inverse problems occupy a central place in investigation of their well-posedness. In the paper, the uniqueness of the solution of problem (1)–(4) is proved under more general assumptions and the estimation characterizing the „conditional" stability of the problem is established.

Define the set K_α as

$$K_\alpha = \{(u,c)|u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}), \ c(x) \in C^\alpha(\overline{D}),$$

$$|u(x,t)|,\ |u_{x_i}(x,t)|,\ |u_{x_ix_j}(x,t)| \le m_5,\ i,j=\overline{1,n}, (x,t)\in\overline{\Omega},\ |c(x)|\le m_6,\ x\in\overline{D}\}.$$

Let $\{u_i(x,t), c_i(x)\}$ be the solutions (1)–(4) corresponding to the given functions.

Definition 2. *A solution of problem* (1)–(4) *is called stable if for any* $\varepsilon>0$ *there is a* $\delta(\varepsilon)>0$ *such that for* $\|f_1-f_2\|_0<\delta$, $\|\varphi_1-\varphi_2\|_2<\delta$, $\|\psi_1-\psi_2\|_0<\delta$, $\|h_1-h_2\|_2<\delta$ *the inequality* $\|u_1-u_2\|_0+\|c_1-c_2\|_0\le\varepsilon$ *is fulfilled.*

Theorem 1. *Let*:

1) $f_i,\ \varphi_i,\ \psi_i,\ h_i,\ i=1,2,$ *satisfy conditions* 1^0–4^0, *respectively*;

2) *there exist the solutions* $\{u_i(x,t), c_i(x)\}$, $i=1,2,$ *of problem* (1)–(4) *in the sense of Definition* 1, *and let they belong to the set* K_α.

Then there exists a $T^*>0$ *such that for* $(x,t)\in\overline{D}\times[0,T^*]$ *the solution of problem* (1)–(4) *is unique, and the stability estimation*

$$\|u_1-u_2\|_0+\|c_1-c_2\|_0\le$$

$$\le m_7\left[\|f_1-f_2\|_0+\|\varphi_1-\varphi_2\|_2+\|\psi_1-\psi_2\|_0+\|h_1-h_2\|_2\right] \tag{5}$$

is valid, here $m_7>0$ *depends on the data of the problem* (1)–(4) *and on the set* K_α.

Proof. First, we prove the validity of the estimation (5). Taking into account (2) and the conditions of the theorem, from equation (1) for the function $c(x)$ we have

$$c(x)=-\left[u(x,T)-\varphi(x)-\Delta h(x)-\int\limits_0^T f(x,t,u)\,dt\right](h(x))^{-1}. \tag{6}$$

Denote by

$$z(x,t)=u_1(x,t)-u_2(x,t),\quad \lambda(x)=c_1(x)-c_2(x),$$

$$\delta_1(x,t,p)=f_1(x,t,p)-f_2(x,t,p),\quad \delta_2(x)=\varphi_1(x)-\varphi_2(x),$$

$$\delta_3(x,t,p)=\psi_1(x,t,p)-\psi_2(x,t,p),\quad \delta_4(x)=h_1(x)-h_2(x).$$

One can verify that the functions $\lambda(x)$, $w(x,t)=z(x,t)-\delta_2(x)$ satisfy the following conditions:

$$w_t-\Delta w=F(x,t),\quad (x,t)\in\Omega, \tag{7}$$

$$w(x,0)=0,\quad x\in\overline{D};\quad \frac{\partial w}{\partial\nu}(x,t)=\Psi(x,t),\quad (x,t)\in S, \tag{8}$$

$$\lambda(x)=z(x,T)(h_1(x))^{-1}-H(x),\quad x\in\overline{D}, \tag{9}$$

where

$$F(x,t)=\delta_1(x,t,u_1)+\Delta\delta_2(x)-c_1(x)z(x,t)-$$

$$-\lambda(x)u_2(x,t)+f_2(x,t,u_1)-f_2(x,t,u_2),$$

$$\Psi\left(x,t\right)=\delta_3\left(x,t,u_1\right)-\frac{\partial\delta_2(x)}{\partial\nu}+\psi_2\left(x,t,u_1\right)-\psi_2\left(x,t,u_2\right),$$

$$H\left(x\right)=\left\{\left[\delta_2\left(x\right)+\Delta\delta_4\left(x\right)+\int\limits_0^T\delta_1\left(x,t,u_1\right)dt\ +\right.\right.$$

$$\left.+\int\limits_0^T\left[f_2\left(x,t,u_1\right)-f_2\left(x,t,u_2\right)\right]dt\right]h_2\left(x\right)+$$

$$\left.+\left[u_2\left(x,T\right)-\varphi_2\left(x\right)-\Delta h_2\left(x\right)-\int\limits_0^T f_2\left(x,t,u_2\right)dt\right]\delta_4\left(x\right)\right\}\left[h_1\left(x\right)h_2\left(x\right)\right]^{-1}.$$

Under the conditions of the theorem, if follows that there exists a classic solution of problem (7), (8) on determination of $w\left(x,t\right)$ and it may be represented in the following form [5, p. 182]:

$$w\left(x,t\right)=\int\limits_0^t\int\limits_D\Gamma\left(x,t;\xi,\tau\right)F\left(\xi,\tau\right)d\xi d\tau+\int\limits_0^t\int\limits_{\partial D}\Gamma\left(x,t;\xi,\tau\right)\rho\left(\xi,\tau\right)d\xi_0 d\tau, \qquad (10)$$

here $\Gamma\left(x,t;\xi,\tau\right)$ is a fundamental solution of the equation $w_t-\Delta w=0$, $d\xi=d\xi_1\ldots d\xi_n, d\xi_0$ is an element of the surface ∂D, $\rho\left(x,t\right)$is a continuous bounded solution of the following integral equation [5, p. 182]:

$$\rho\left(x,t\right)=2\int\limits_0^t\int\limits_D\frac{\partial\Gamma\left(x,t;\xi,\tau\right)}{\partial\nu\left(x,t\right)}F\left(\xi,\tau\right)d\xi d\tau+$$

$$+2\int\limits_0^t\int\limits_{\partial D}\frac{\partial\Gamma\left(x,t;\xi,\tau\right)}{\partial\nu\left(x,t\right)}\rho\left(\xi,\tau\right)d\xi_0 d\tau-2\Psi\left(x,t\right). \qquad (11)$$

Assume, that

$$\chi=\left\|u_1-u_2\right\|_0+\left\|c_1-c_2\right\|_0.$$

Estimate the function $\left|z\left(x,t\right)\right|$. Taking into account that $z\left(x,t\right)=w\left(x,t\right)+\delta_2\left(x\right)$, from (10) we get

$$\left|z\left(x,t\right)\right|\leq\left|w\left(x,t\right)\right|+\left|\delta_2\left(x\right)\right|\leq\left|\delta_2\left(x\right)\right|+\int\limits_0^t\int\limits_D\Gamma\left(x,t;\xi,\tau\right)\left|F\left(\xi,\tau\right)\right|d\xi d\tau+$$

$$+\int\limits_0^t\int\limits_{\partial D}\Gamma\left(x,t;\xi,\tau\right)\left|\rho\left(\xi,\tau\right)\right|d\xi_0 d\tau. \qquad (12)$$

For the fundamental solutions following estimations are true [1, p. 444]:

$$\int\limits_{R^n} \Gamma(x,t;\xi,\tau)d\xi \leq m_8, \tag{13}$$

$$\int\limits_{R_n} \left| D_x^l \Gamma(x,t;\xi,\tau) \right| d\xi \leq m_9 (t-\tau)^{-\frac{l-\alpha}{2}}, \quad l = 1, 2. \tag{14}$$

Due to requirements imposed on the input data and on the set K_α, the integrand function $F(x,t)$ in the second summand of the right-hand side of (12), satisfies the estimation

$$|F(x,t)| \leq |\delta_1(x,t,u_1)| + |\Delta\delta_2(x)| + |c_1(x)|\,|z(x,t)| +$$

$$+|\lambda(x)||u_2(x,t)| + |f_2(x,t,u_1) - f_2(x,t,u_2)| \leq$$

$$\leq \|f_1 - f_2\|_0 + \|\varphi_1 - \varphi_2\|_2 + m_{10}\chi, \quad (x,t) \in \overline{\Omega}, \tag{15}$$

here $m_{10} > 0$ depends on the data of problem (1)–(4) and the set K_α.

From the Gauss–Ostrogradsky formula and (14) for $s = 1$, we have

$$\int\limits_{\partial D} \Gamma(x,t;\xi,\tau)\, d\xi_0 \leq m_{11}(t-\tau)^{-\frac{1-\alpha}{2}}. \tag{16}$$

Taking into account expression (11), (14) for $s = 1$ and $s = 2$, the conditions of the theorem, determination of the set K_α for the function $\rho(x,t)$ we get

$$|\rho(x,t)| \leq m_{12}\left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi\right] + m_{13}\|\rho\|\, t^{\alpha/2}, \quad (x,t) \in S,$$

where $m_{12}, m_{13} > 0$ depend on the data of problem (1)–(4) and on the set K_α.

The last inequality is fulfilled for all $(x,t) \in \partial D \times [0,T]$, therefore the following estimation is true:

$$\|\rho\|_0 \leq m_{12}\left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi\right] + m_{13} t^{\alpha/2}\|\rho\|_0.$$

Let $0 < T_1 \leq T$ be a number such that $m_{13}T_1^{\alpha/2} < 1$. Then for all $(x,t) \in \partial D \times [0,T_1]$ we have

$$\|\rho\|_0 \leq m_{14}\left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi\right], \tag{17}$$

where $m_{14} > 0$ depends on the data of problem (1)–(4) and on the set K_α.

Taking into account inequalities (13), (15), (16) and (17) from (12) for $|z(x,t)|$ we get

$$|z(x,t)| \leq m_{15}\left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0\right] + m_{16}\chi t^{\alpha}, \quad (x,t) \in \overline{\Omega}, \tag{18}$$

where $m_{15},\ m_{16} > 0$ depend on the data of problem (1)–(4) and the set K_α.

Now estimate the function $|\lambda(x)|$. From (9) it follows

$$|\lambda(x)| \leq |z(x,t)| \left| h_1(x)^{-1} \right| + |H(x)|.$$

Taking into account the conditions of the theorem, definitions of the set K_α, inequalities (18) and expressions for $H(x)$, from the last inequality we get

$$|\lambda(x)| \leq m_{17}\left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \|\delta_4\|_2\right] + m_{18}t^\alpha\chi, \quad x \in \overline{\Omega}, \tag{19}$$

where $m_{17},\ m_{18} > 0$ depend on the data of the problem (1)–(4) and the set K_α.

Inequalities (18) and (19) are satisfied for any values of $(x,t) \in \overline{D} \times [0,T]$.

Consequently, combining these inequalities, we obtain

$$\chi \leq m_{19}\left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \|\delta_4\|_2\right] + m_{20}t^\alpha\chi, \tag{20}$$

where $m_{19}, m_{20} > 0$ depend on the data of problem (1)–(4) and the set K_α.

Let $T_2\,(0 < T_2 \leq T)$ be a number such that $m_{20}T_2^\alpha < 1$. Then from (20) we get that for $(x,t) \in \overline{D} \times [0,T^*]$, $T^* = \min(T_1, T_2)$, the stability estimation for the solution of problem (1)–(4) is true.

Uniqueness of the solution of problem (1)–(4) follows from estimation (5) for $f_1(x,t,u) = f_2(x,t,u)$, $\varphi_1(x) = \varphi_2(x)$, $\psi_1(x,t,u) = \psi_2(x,t,u)$, $h_1(x) = h_2(x)$.

The theorem is proved.

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