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A CLASS OF p-VALENT MEROMORPHIC FUNCTIONS DEFINED BY THE LUI – SRIVASTAVA OPERATOR

ПРО КЛАС p-ВАЛЕНТНИХ МЕРОМОРФНИХ ФУНКЦIЙ, ЩО ВИЗНАЧЕНI ОПЕРАТОРОМ ЛУI – ШРIВАСТАВИ

In this paper, we introduce a subclass of p-valent meromorphic functions involving the Lui – Srivastava operator and investigate various properties of this subclass. We also indicate the relationships between various results presented in the paper with the results obtained in earlier works.

Введено пiдклас p-валентних мероморфних функцiй, що визначаються оператором Луi – Шрiвастави та вивчено рiзнманiтнi властивостi цього пiдкласу. Ми також вказуємо спiввiдношення мiж рiзнманiтними результатами, що отриманi в роботi та результатами, що отриманi ранiше.

1. Introduction. Let Σ_p denote the class of all meromorphic functions f of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p}, \quad p \in \mathbb{N} = 1, 2, \ldots, \tag{1.1}$$

which are analytic and p-valent in the punctured disc $U^* = \{z \in \mathbb{C} \colon 0 < |z| < 1\} = U \backslash \{0\}$. Let $\Sigma\mathcal{S}_p^*(\lambda)$ denote the class of all meromorphic p-valent starlike of order $\lambda\,(0 \leq \lambda < p)$ in U.

For functions $f \in \Sigma_p$, given by (1.1), and $g \in \Sigma_p$ defined by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p}, \quad p \in \mathbb{N},$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z).$$

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$; $j = 1, 2, \ldots, s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by (see, for example, [9, p.19])

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k} \frac{z^k}{k!},$$

$$q \leq s + 1, \qquad q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad z \in U,$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1, & \nu = 0, \quad \theta \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}, \\ \theta(\theta - 1) \ldots (\theta + \nu - 1), & \nu \in \mathbb{N}, \quad \theta \in \mathbb{C}. \end{cases}$$

Corresponding to the function $h_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$, defined by

$$h_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)=z^{-p}\,{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z),$$

we consider a linear operator

$$H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)\colon \Sigma_p\to\Sigma_p,$$

which is defined by the following Hadamard product (or convolution):

$$H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)=h_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)*f(z).$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)=z^{-p}+\sum_{k=1}^{\infty}\Gamma_{p,q,s}\left(\alpha_1\right)\,a_kz^{k-p}, \tag{1.2}$$

where

$$\Gamma_{p,q,s}\left(\alpha_1\right)=\frac{(\alpha_1)_k\ldots(\alpha_q)_k}{(\beta_1)_k\ldots(\beta_s)_k\;k!}. \tag{1.3}$$

If, for convenience, we write

$$H_{p,q,s}[\alpha_1]=H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)\,,$$

then one can easily verify from the definition (1.2) that (see [5])

$$z(H_{p,q,s}[\alpha_1]f(z))'=\alpha_1H_{p,q,s}[\alpha_1+1]f(z)-(\alpha_1+p)H_{p,q,s}\left[\alpha_1\right]f(z). \tag{1.4}$$

The linear operator $H_{p,q,s}[\alpha_1]$ was investigated recently by Liu and Srivastava [5] and Aouf [2].

In particular, for $q=2$, $s=1$, $\alpha_1>0$, $\beta_1>0$ and $\alpha_2=1$, we obtain the linear operator

$$H_p(\alpha_1,1;\beta_1)\,f(z)=\ell_p(\alpha_1,\beta_1)f(z),$$

which was introduced and studied by Liu and Srivastava [4] .

We note that,

$$H_{p,2,1}(n+p,1;1)f(z)=D^{n+p-1}f(z)=\frac{1}{z^p(1-z)^{n+p}}*f(z),\quad n>-p,$$

where D^{n+p-1} is the differential operator studied by Uralegaddi and Somanatha [10] and Aouf [1].

Making use of the operator $H_{p,q,s}[\alpha_1]$, we now introduce a subclass of the function class Σ_p as follows:

we say that a function $f\in\Sigma_p$ is in the class $\Omega_{p,q,s}(\alpha_1;\lambda)$, if it satisfies the following inequality:

$$\operatorname{Re}\left\{\alpha_1\frac{H_{p,q,s}[\alpha_1+1]f(z)}{H_{p,q,s}[\alpha_1]f(z)}-(\alpha_1+p)\right\}<-\lambda,\qquad 0\le\lambda<p,\quad p\in\mathbb{N},$$

or, in view of (1.4), if it satisfies the following inequality:

$$\operatorname{Re}\left\{\frac{z\big(H_{p,q,s}[\alpha_1]f(z)\big)'}{H_{p,q,s}[\alpha_1]f(z)}\right\}<-\lambda,\qquad 0\le\lambda<p,\quad p\in\mathbb{N}.$$

2. Main results. In order to establish our main results, we need the following lemma.

Lemma 2.1 [3]. *Let $w(z)$ be a non-constan analytic in U with $w(0) = 0$. If $\big|w(z)\big|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then we have*

$$z_0 w'(z_0) = \zeta w(z_0), \tag{2.1}$$

where $\zeta \geq 1$ is a real number.

Theorem 2.1. *Let $\alpha_1 \geq 0$ and $0 \leq \lambda < p$, then*

$$\Omega_{p,q,s}(\alpha_1 + 1; \lambda) \subset \Omega_{p,q,s}(\alpha_1; \lambda).$$

Proof. Let $f \in \Omega_{p,q,s}(\alpha_1 + 1; \lambda)$, then

$$\mathrm{Re}\left\{\frac{z\big(H_{p,q,s}[\alpha_1 + 1]f(z)\big)'}{H_{p,q,s}[\alpha_1 + 1]f(z)}\right\} < -\lambda, \quad z \in U. \tag{2.2}$$

We have to show that implies the inequality

$$\mathrm{Re}\left\{\frac{z\big(H_{p,q,s}[\alpha_1]f(z)\big)'}{H_{p,q,s}[\alpha_1]f(z)}\right\} < -\lambda, \quad z \in U. \tag{2.3}$$

Define the function $w(z)$ in U by

$$\frac{z\big(H_{p,q,s}[\alpha_1]f(z)\big)'}{H_{p,q,s}[\alpha_1]f(z)} = -\frac{p + (p - 2\lambda)w(z)}{1 - w(z)}. \tag{2.4}$$

Clearly, $w(z)$ is analytic in U and $w(0) = 0$. Using the identity (1.4), (2.4) may be written as

$$\alpha_1 \frac{H_{p,q,s}[\alpha_1 + 1]f(z)}{H_{p,q,s}[\alpha_1]f(z)} = \frac{\alpha_1 - [\alpha_1 + 2(p - \lambda)]w(z)}{1 - w(z)}. \tag{2.5}$$

Differentiating (2.5) logarithmically with respect to z and using (1.4), we obtain

$$\frac{z\big(H_{p,q,s}[\alpha_1 + 1]f(z)\big)'}{H_{p,q,s}[\alpha_1 + 1]f(z)} + \lambda =$$

$$= -(p - \lambda)\left\{\frac{1 + w(z)}{1 - w(z)} + \frac{2zw'(z)}{\big(1 - w(z)\big)\big(\alpha_1 - [\alpha_1 + 2(p - \lambda)]w(z)\big)}\right\}. \tag{2.6}$$

We claim that $|w(z)| < 1$ in U. For otherwise, there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} \big|w(z)\big| = \big|w(z_0)\big| = 1.$$

Applying Lemma 2.1 to $w(z)$ at the point z_0, we have $z_0 w'(z_0) = \zeta w(z_0)$ where $\zeta \geq 1$. So, (2.6) yields

$$\mathrm{Re}\left\{\frac{z\big(H_{p,q,s}[\alpha_1 + 1]f(z_0)\big)'}{H_{p,q,s}[\alpha_1 + 1]f(z_0)} + \lambda\right\} =$$

$$= -(p-\lambda)\mathrm{Re}\left\{\frac{1+w(z_0)}{1-w(z_0)} + \frac{2\zeta w(z_0)}{\big(1-w(z_0)\big)\big(\alpha_1 - [\alpha_1 + 2(p-\lambda)]\, w(z_0)\big)}\right\} \geq$$

$$\geq \frac{p-\lambda}{2(\alpha_1 + p - \lambda)} > 0,$$

which contradicts the inequality (2.2). Hence, $\left|w(z)\right| < 1$ in $U$and it follows that $f \in \Omega_{p,q,s}(\alpha_1; \lambda)$.

Theorem 2.1 is proved.

Theorem 2.2. *Let* $\delta > 0$ *and* $f(z) \in \Sigma_p$ *satisfy the following inequality*:

$$\mathrm{Re}\left\{\frac{z\big(H_{p,q,s}[\alpha_1]f(z)\big)'}{H_{p,q,s}[\alpha_1]f(z)}\right\} < -\lambda + \frac{p-\lambda}{2(\delta + \lambda p - \lambda)}, \quad z \in U. \tag{2.7}$$

Then the function $\mathcal{F}_{\delta,p}(f)$ *defined by*

$$\mathcal{F}_{\delta,p}(f)(z) = \frac{\delta}{z^{\delta+p}} \int\limits_0^z t^{\delta+p-1} f(t)dt, \quad \delta > 0, \tag{2.8}$$

belongs to $\Omega_{p,q,s}(\alpha_1; \lambda)$.

Proof. From (2.8), we readily have

$$z\Big(H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)\Big)' = \delta H_{p,q,s}[\alpha_1]f(z) - (\delta + p)H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z). \tag{2.9}$$

Using the identity (1.9) and (2.9), condition (2.7) may be written as

$$\mathrm{Re}\left\{\alpha_1 \frac{H_{p,q,s}[\alpha_1+1]\mathcal{F}_{\delta,p}(f)(z)}{H_{p,q,s}\,[\alpha_1]\,\mathcal{F}_{\delta,p}(f)(z)} \frac{\dfrac{z\big(H_{p,q,s}[\alpha_1+1]\mathcal{F}_{\delta,p}(f)(z)\big)'}{H_{p,q,s}[\alpha_1+1]\mathcal{F}_{\delta,p}(f)(z)} + \delta + p}{\dfrac{z\big(H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)\big)'}{H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)} + \delta + p} - \alpha_1 - p\right\} <$$

$$< -\lambda + \frac{p-\lambda}{2\,(\delta + \lambda p - \lambda)}. \tag{2.10}$$

We have to prove that $\mathcal{F}_{\delta,p}(f) \in \Omega_{p,q,s}(\alpha_1; \lambda)$ implies the inequality

$$\mathrm{Re}\left\{\frac{z\,(H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z))'}{H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)}\right\} < -\lambda, \qquad 0 \leq \lambda < p, \quad z \in U. \tag{2.11}$$

Consider the function $w(z)$ in U defined by

$$\frac{z\big(H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)\big)'}{H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)} = -\frac{p + (p-2\lambda)w(z)}{1-w(z)}, \qquad 0 \leq \lambda < p, \quad z \in U. \tag{2.12}$$

Clearly, $w(z)$ is analytic and $w(0) = 0$. (2.12) may be written as

$$\alpha_1 \frac{H_{p,q,s}[\alpha_1+1]\mathcal{F}_{\delta,p}(f)(z)}{H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)} = \frac{\alpha_1 - \big[\alpha_1 + 2(p-\lambda)\big]w(z)}{1-w(z)}. \tag{2.13}$$

Differentiating (2.13) logarithmically with respect to z and using (2.12), we obtain

$$\frac{z\big(H_{p,q,s}[\alpha_1+1]\mathcal{F}_{\delta,p}(f)(z)\big)'}{H_{p,q,s}[\alpha_1+1]\mathcal{F}_{\delta,p}(f)(z)}=$$

$$=-\frac{p+(p-2\lambda)w(z)}{1-w(z)}-\frac{2(p-\lambda)zw'(z)}{(1-w(z))\big(\alpha_1-[\alpha_1+2(p-\lambda)]w(z)\big)}. \tag{2.14}$$

Using (2.12) – (2.14) and (2.10), we get

$$\alpha_1\frac{H_{p,q,s}[\alpha_1+1]\mathcal{F}_{\delta,p}(f)(z)}{H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)}\frac{\dfrac{z\big(H_{p,q,s}[\alpha_1+1]\mathcal{F}_{\delta,p}(f)(z)\big)'}{H_{p,q,s}[\alpha_1+1]\mathcal{F}_{\delta,p}(f)(z)}+\delta+p}{\dfrac{z\big(H_{p,q,s}[\alpha_1]\mathcal{F}_{\delta,p}(f)(z)\big)'}{H_{p,q,s}\left[\alpha_1\right]\mathcal{F}_{\delta,p}(f)(z)}+\delta+p}-\alpha_1-p+\lambda=$$

$$=-(p-\lambda)\left\{\frac{1+w(z)}{1-w\left(z\right)}+\frac{2zw'(z)}{\big(1-w(z)\big)\big(\delta-[\delta+2(p-\lambda)]w(z)\big)}\right\}. \tag{2.15}$$

The remaining part of the proof is similar to that of Theorem 2.1.

Theorem 2.2 is proved.

According to Theorem 2.2, we have the following corollary.

***Corollary* 2.1.** *If $f\in\Omega_{p,q,s}(\alpha_1;\lambda)$, then the function $\mathcal{F}_{\delta,p}(f)$ defined* (2.8) *also belongs to $\Omega_{p,q,s}(\alpha_1;\lambda)$.*

Theorem 2.3. *If $f\in\Omega_{p,q,s}(\alpha_1;\lambda)$ if and only if the function g defined by*

$$g(z)=\frac{\alpha_1}{z^{\alpha_1+p}}\int\limits_0^z t^{\alpha_1+p-1}f\left(t\right)dt, \quad \alpha_1>0, \tag{2.16}$$

belongs to $\Omega_{p,q,s}(\alpha_1+1;\lambda)$.

Proof. From (2.16), we have

$$z\left(\mathcal{F}_{\delta,p}(f)(z)\right)'=\alpha_1 f(z)-(\alpha_1+p)\,\mathcal{F}_{\delta,p}(f)(z). \tag{2.17}$$

Using identity (1.9) and (2.17), hence

$$H_{p,q,s}[\alpha_1]f(z)=H_{p,q,s}\left[\alpha_1\right]\mathcal{F}_{\delta,p}(f)(z) \tag{2.18}$$

and the result follows.

To prove Theorem 2.4, we need the following lemmas.

Lemma 2.2 [8]. *The function $(1-z)^\gamma=\exp\big(\gamma\log(1-z)\big)$, $\gamma\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}$, is univalent in U if and only if γ is either in the closed disk $|\gamma-1|\le 1$ or in the closed disc $|\gamma+1|\le 1$.*

Lemma 2.3 [7]. *Let $q(z)$ be univalent in U and let $Q(w)$ and $\phi(w)$ be analytic in a domain D containing $q(U)$, with $\phi(w)\neq 0$ when $w\in q(U)$. Set $Q(z)=zq'(z)\phi(q(z))$, $h(z)=\theta(q(z))+Q(z)$ and suppose that*

(1) *$Q(z)$ is starlike (univalent) in U,*

(2) $\operatorname{Re}\left\{\dfrac{zh'(z)}{Q\left(z\right)}\right\}>0$ *$(z\in U)$.*

If $g(z)$ is analytic in U, with $p(0)=q(0)$, $p(U)\subset D$, and

$$\theta\big(g(z)\big)+zg'(z)\phi(g(z))\prec\theta\big(q(z)\big)+zq'(z)\phi\big(q(z)\big)=h(z), \tag{2.19}$$

then $g(z)\prec q(z)$, and $q(z)$ is the best dominant of (2.19).

Theorem 2.4. *Let $f\in\Omega_{p,q,s}(\alpha_1;\lambda)$ and let $\gamma\in\mathbb{C}^*$ and satisfy either*

$$\big|2\gamma(p-\lambda)-1\big|\le 1 \qquad \textit{or} \qquad \big|2\gamma(p-\lambda)+1\big|\le 1. \tag{2.20}$$

Then

$$\big(z^pH_{p,q,s}[\alpha_1]f(z)\big)^\gamma\prec(1-z)^{2\gamma(p-\lambda)}=q(z), \tag{2.21}$$

and $q(z)$ is the best dominant.

Proof. Set

$$g(z)=\big(z^pH_{p,q,s}[\alpha_1]f(z)\big)^\gamma, \quad z\in U, \tag{2.22}$$

then $g(z)$ is analytic in U with $g(0)=1$. Differentiating (2.22) logarithmically with respect to z, we obtain

$$\frac{zg'(z)}{g(z)}=\gamma\left[\frac{z\big(H_{p,q,s}[\alpha_1]f(z)\big)'}{H_{p,q,s}[_1]f(z)}+p\right]. \tag{2.23}$$

Since $f\in\Omega_{p,q,s}(\alpha_1;\lambda)$, this is equivalent to

$$\frac{z\big(H_{p,q,s}[\alpha_1]f(z)\big)'}{H_{p,q,s}[\alpha_1]f(z)}\prec-\frac{p+(p-2\lambda)z}{1-z}, \tag{2.24}$$

from (2.23), (2.24) can be rewritten as

$$-p+\frac{zg'(z)}{\gamma g(z)}\prec-p+\frac{z\left((1-z)^{2\gamma(p-\lambda)}\right)'}{(1-z)^{2\gamma(p-\lambda)}}. \tag{2.25}$$

On the other hand, if we take

$$q(z)=(1-z)^{2\gamma(p-\lambda)}, \qquad \theta(z)=-p, \qquad \phi(w)=\frac{1}{\gamma w}$$

in Lemma 2.3, then $q(z)$ is univalent by the condition (2.20) and Lemma 2.2. It is easy to see that $q(z)$, $\theta(w)$, and $\phi(w)$ satisfy the conditions of Lemma 2,3. Since

$$Q(z)=zq'(z)\phi(q(z))=-\frac{2(p-\lambda)z}{1-z}$$

is univalent starlike in U and

$$h(z)=\theta(q(z))+Q(z)=-\frac{p+(p-2\lambda)z}{1-z},$$

from (2.25) and Lemma 2.3, then

$$g(z)\prec(1-z)^{2\gamma(p-\lambda)}=q(z)$$

and the function $(1-z)^{2\gamma(p-\lambda)}$ is the best dominant.

***Corollary* 2.2.** *Let $f \in \Omega_{p,q,s}(\alpha_1;\lambda)$. Then*

$$\mathrm{Re}\left\{\left(z^p H_{p,q,s}[\alpha_1]f(z)\right)^\gamma\right\} > 2^{2\gamma(p-\lambda)}, \quad z \in U,$$

where γ is a real number and $\gamma \in \left[-\frac{1}{2(p-\lambda)}, 0\right)$. The result is sharp.

Proof. From Theorem 2.4, we have

$$\mathrm{Re}\left\{\left(z^p H_{p,q,s}[\alpha_1]f(z)\right)^\gamma\right\} = \mathrm{Re}\left\{\left(1 - w(z)\right)^{2\gamma(p-\lambda)}\right\}, \quad z \in U, \tag{2.26}$$

where $w(z)$ is analytic in U, $w(0) = 0$, and $|w(z)| < 1$ for $z \in U$. In view of

$$\mathrm{Re}\,(t^b) \geq (\mathrm{Re}\,t)^b, \qquad \mathrm{Re}(t) > 0, \quad 0 < b \leq 1,$$

(2.29) yields

$$\mathrm{Re}\left\{\left(z^p H_{p,q,s}[\alpha_1]f(z)\right)^\gamma\right\} \geq \left\{\mathrm{Re}\left(\frac{1}{1-w(z)}\right)^{-2\gamma(p-\lambda)}\right\} > 2^{2\gamma(p-\lambda)}, \quad z \in U,$$

for $-1 \leq 2\gamma(p-\lambda) < 0$. To see that the bound $2^{2\gamma(p-\lambda)}$ cannot be increased, we consider the function $f(z)$ which satisfies

$$z^p H_{p,q,s}[\alpha_1]f(z) = (1-z)^{2(p-\lambda)}, \qquad 0 \leq \lambda < p, \quad z \in U.$$

We easily have $f \in \Omega_{p,q,s}(\alpha_1;\lambda)$ and

$$\mathrm{Re}\left\{\left(z^p H_{p,q,s}[\alpha_1]f(z)\right)^\gamma\right\} \to 2^{2\gamma(p-\lambda)} \qquad \text{as} \quad \mathrm{Re}(z) \to -1.$$

Corollary 2.2 is proved.

3. Convolution conditions. We give some necessary and sufficient condition in terms of convolution operator for meromorphic functions to be in the classes $\mathcal{S}_p^*(\lambda)$ and $\Omega_{p,q,s}(\alpha_1;\lambda)$.

Lemma 3.1. *The function $f(z) \in \Sigma_p$ belongs to the class $\Sigma\mathcal{S}_p^*(\lambda)$ $(0 \leq \lambda < p)$ if and only if*

$$z^p\left[f(z) * \frac{\left(1 - \dfrac{1 - e^{-i\theta} + 2(p-\lambda)}{2(p-\lambda)}z\right)}{z^p(1-z)^2}\right] \neq 0, \qquad 0 < \theta < 2\pi, \quad z \in U.$$

Proof. A function $f(z) \in \Sigma\mathcal{S}_p^*(\lambda)$ if and only if

$$\frac{zf'(z)}{f(z)} \neq -\frac{p + (p-2\lambda)\,e^{i\theta}}{1 - e^{i\theta}}, \quad 0 < \theta < 2\pi, \quad z \in U.$$

which is equivalent to

$$z^p\left[\left(1 - e^{i\theta}\right)zf'(z) + \left[p + (p-2\lambda)\,e^{i\theta}\right]f(z)\right] \neq 0, \qquad 0 < \theta < 2\pi, \quad z \in U. \tag{3.1}$$

Since

$$f(z) = f(z) * \frac{1}{z^p(1-z)} \qquad \text{and} \qquad zf'(z) = f(z) * \left(\frac{(p+1)\,z - p}{z^p(1-z)^2}\right).$$

Therefor, we may write (3.1) as

$$z^p\left[\left(1 - e^{i\theta}\right) zf'(z) + \left[p + (p-2\lambda)\,e^{i\theta}\right] f(z)\right] =$$

$$= z^p\left\{f(z) * \left[\frac{\left(1-e^{i\theta}\right)}{z^p(1-z)^2} + \frac{\left[p+(p-2\lambda)\,e^{i\theta}\right]}{z^p(1-z)}\right]\right\} =$$

$$= 2(p-\lambda)e^{i\theta}\; z^p\left[f(z) * \frac{\left(1 - \dfrac{1-e^{-i\theta}+2(p-\lambda)}{2\,(p-\lambda)}z\right)}{z^p(1-z)^2}\right] \neq 0.$$

Lemma 3.1 is proved.

Theorem 3.1. *A necessary and sufficient condition for the function* $f(z)$ *defined by* (1.1) *to be in the class* $\Omega_{p,q,s}(\alpha_1;\lambda)$ *is that*

$$1 - \sum_{k=1}^{\infty} \frac{\left(1-e^{-i\theta}\right)k - 2\,(p-\lambda)}{2(p-\lambda)}\Gamma_{p,q,s}\left(\alpha_1\right), \qquad a_k z^k \neq 0 \quad \left(0 < \theta < 2\pi; z \in U\right),$$

where $\Gamma_{p,q,s}\left(\alpha_1\right)$ *is given by* (1.8).

Proof. From Lemma 3.1, we find that $f \in \Omega_{p,q,s}(\alpha_1;\lambda)$ if and only if

$$z^p\left[H_{p,q,s}[\alpha_1]f(z) * \frac{\left(1 - \dfrac{1-e^{-i\theta}+2(p-\lambda)}{2\,(p-\lambda)}z\right)}{z^p(1-z)^2}\right] \neq 0, \qquad 0<\theta<2\pi, \quad 0 \le \lambda < p, \quad z \in U. \tag{3.2}$$

From (1.7), the left hand side of (3.5) may be written as

$$z^p\left[H_{p,q,s}[\alpha_1]f(z) * \frac{\left(1 - \dfrac{1-e^{-i\theta}+2(p-\lambda)}{2\,(p-\lambda)}z\right)}{z^p(1-z)^2}\right] =$$

$$= 1 - \sum_{k=1}^{\infty} \frac{\left(1-e^{-i\theta}\right)k - 2(p-\lambda)}{2(p-\lambda)}\Gamma_{p,q,s}\left(\alpha_1\right)\; a_k z^k \neq 0\,.$$

Theorem 3.1 is proved.

Remarks. 3.1. Putting $q = 2$, $s = \alpha_2 = \beta_1 = 1$ and $\alpha_1 = n + p(n > -p)$ in Theorem 2.1, we obtain the results obtained by Aouf [1] (Theorem 1).

3.2. Putting $q = 2$, $s = \alpha_2 = \beta_1 = 1$, $\delta = c - p + 1(c > p-1)$ and $\alpha_1 = n + p(n > -p)$ in Theorem 2, we obtain the results obtained by Aouf [1] (Theorem 2).

3.3. Putting $q = 2$, $s = \alpha_2 = \beta_1 = 1$ and $\alpha_1 = n + p(n > -p)$ in Theorem 3, we improve the result obtained by Aouf [1] (Theorem 3).

3.4. Taking $q = 2$, $s = \alpha_2 = 1$ and $\alpha_1, \beta_1 > 0$ in Theorems 2.1 – 2.4 Corollaries 2.1 and 2.2, respectively, we obtain the results obtained by Liu and Owa [6] [Theorems 2.2, 2.3, 2.5, 2.9, Corollaries 2.4 and 2.10].

3.5. Taking $e^{-i\theta} = -x$ in Lemma 3.1, we obtain the results obtained by Liu and Owa [6] (Lemma 3.1).

3.6. Taking $q = 2$, $s = \alpha_2 = 1$, α_1, $\beta_1 > 0$ and $e^{-i\theta} = -x$ in Theorem 3.1, we obtain the result obtained by Liu and Owa [6] (Theorem 3.2).

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