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A NOTE ON SOLYMOSI'S SUM – PRODUCT ESTIMATE FOR ORDERED FIELDS

ПРО ОЦІНКУ ШОЛІМОШІ ТИПУ СУМА – ДОБУТОК ДЛЯ ВПОРЯДКОВАНИХ ПОЛІВ

It is proved that Solymosi's sum – product estimate $\max\{|A+A|, |A\cdot A|\} \gg |A|^{4/3}/(\log|A|)^{1/3}$ holds for any finite set A in an ordered field F.

Доведено, що оцінка шолімоші типу сума – добуток $\max\{|A+A|, |A\cdot A|\} \gg |A|^{4/3}/(\log|A|)^{1/3}$ справедлива для будь-якої скінченної множини A у впорядкованому полі F.

1. Introduction. For a set A of a given ring $(R, +, \cdot)$, define the sum-set and the product-set to be

$$A + A = \{a + a' : a,\, a' \in A\},$$

$$A \cdot A = \{a \cdot a' : a,\, a' \in A\}.$$

When R is a field and $0 \notin A$, we also apply similar definition for A/A.

Since $\mathbb{Z}$ and $\mathbb{R}$ do not have zero divisors and finite subrings, it is expected that the sum-set and the product-set can not be relatively small simultaneously. Erdős and Szemerédi [2] conjectured that for any finite set $A \subseteq \mathbb{Z}$, the estimate (here $\ll$ and $\gg$ are Vinogradov notations)

$$\max\big\{|A+A|, |A\cdot A|\big\} \gg |A|^{2-\varepsilon}$$

holds, where $\varepsilon \to 0$ when $|A| \to \infty$. And they proved that

$$\max\big\{|A+A|, |A\cdot A|\big\} \gg |A|^{1+\delta}$$

for some $\delta > 0$. Later Nathanson [6] showed that $\delta \geq 1/31$ and Ford [3] improved this bound to $\delta \geq 1/15$. For finite sets of reals (also correct for finite sets of integers), bounds were given by Elekes [1] ($\delta \geq 1/4$), Solymosi [7] ($\delta \geq 3/11 - \varepsilon$) and Solymosi [8] ($\delta \geq 1/3 - \varepsilon$). The proofs in [1] and [8] are quite beautiful. Geometry is taken use of in these two papers.

For sum-product estimates for the finite fields and the complex numbers, we refer the reader to [4, 9, 10].

In this note, Solymosi's bound is extended to finite sets of any ordered rings. The geometry proof is transferred to a type of elementary linear algebra.

Definition. *An ordered field (or ring) is a field (or ring, respectively)* $(F, +, \cdot)$ *with a total order* $\leq$ *such that for all* a, b *and* c *in* F, *the following two properties hold*:

(i) *If* $a \leq b$, *then* $a + c \leq b + c$,

(ii) *If* $0 \leq a$ *and* $0 \leq b$, *then* $0 \leq ab$.

Examples of ordered fields include $\mathbb{Q}$, $\mathbb{R}$, the field of fractions of $R[x]$ with R an ordered ring, computable numbers, superreal numbers, hyperreal numbers and so on. One can found details on Wikipedia.

Theorem. *Supose F is an ordered field. Let $A \subseteq F$ be any finite set with sufficiently large cardinality. Then*

$$|A + A|^2 |A \cdot A| \gg \frac{|A|^4}{\log |A|}.$$

From the theorem one can deduce the follow sum-product estimate.

Corollary. *Supose F is an ordered field. Let $A \subseteq F$ be any finite set with sufficiently large cardinality. Then*

$$\max\{|A + A|, |A \cdot A|\} \gg \frac{|A|^{4/3}}{(\log |A|)^{1/3}}.$$

For a nontrivial ordered ring R, one can find a nonempty set $P \subseteq R$ such that

(i) If $a, b \in P$, then $a + b \in P$ and $ab \in P$,

(ii) For all $r \in R$, exactly one of the following conditions holds:

$$r \in P, \qquad r = 0, \qquad -r \in P.$$

P is called the positive elements of R and we say r is negative if $-r \in P$. This can be viewed as an alternative definition of an ordered ring. Now we fix an $A \subseteq F$ and begin to prove the theorem. Without loss of generality, we suppose that all the elements in A are positive. (Either the set of positive elements of A or the set of negative ones has cardinality no less than $(|A| - 1)/2 \gg |A|$ and we can substitute it for original A.) Put $S_\lambda = \{(a, b) \in A \times A \colon a/b = \lambda\}$ and $r_{A/A}(\lambda) = |S_\lambda|$. A trivial bound is $r_{A/A} \le |A|$. Define the energy by

$$E_\times(A) = \#\{(a, b, c, d) \in A^4 \colon ab = cd\},$$

$$E_\div(A) = \#\{(a, b, c, d) \in A^4 \colon a/b = c/d\} \quad (0 \notin A).$$

It can be asserted that $E_\times(A) = E_\div(A)$. The energy inequality shows that

$$\frac{|A|^4}{|A \cdot A|} \le E_\times(A) = E_\div(A) = \sum_{\lambda \in A/A} r^2_{A/A}(\lambda).$$

Let $t = \lceil \log |A| / \log 2 \rceil$, where the notation $\lceil x \rceil$ denote the smallest integer larger than or equal to x. For $0 \le j \le t$, denote

$$M_j := \{\lambda \in A/B \colon 2^{j-1} < r_{A/B}(\lambda) \le 2^j\}, \quad m_j := |M_j|.$$

It follows that

$$E_\div(A) = \sum_{j=0}^{t} \sum_{\lambda \in M_j} r^2_{A/A}(\lambda) \le \sum_{j=0}^{t} 2^{2j+2} m_j.$$

Hence

$$\frac{|A|^4}{|A\cdot A|\cdot\log|A|}\le\sup_{0\le j\le t}\{2^{2j+2}m_j\}:=2^{2J+2}m_J. \tag{1}$$

If $m_J=1$, then trivial bound gives

$$2^{2J+2}m_J\ll 2^{2t}\ll|A|^2.$$

By (1), one has $|A\cdot A|\cdot\log|A|\ge|A|^2$. Combining the trivial bound $|A+A|^2\ge|A|^2$, the theorem follows. Now we suppose that $m_J\ge 2$. For $\mu_1,\mu_2\in M_J$, we construct a map $\pi_{\mu_1,\mu_2}: S_{\mu_1}\times S_{\mu_2}\to$ $\to(A+A)\times(A+A)$:

$$\pi_{\mu_1,\mu_2}(a_1,b_1,a_2,b_2)=(a_1+a_2,b_1+b_2).$$

Lemma 1. *When $\mu_1\ne\mu_2$, the map π_{μ_1,μ_2} is an injection.*

Proof. Suppose there exist (a_1,b_1,a_2,b_2) and (a_1',b_1',a_2',b_2') in $S_{\mu_1}\times S_{\mu_2}$ such that

$$\pi_{\mu_1,\mu_2}(a_1,b_1,a_2,b_2)=\pi_{\mu_1,\mu_2}(a_1',b_1',a_2',b_2').$$

Then we have the following linear equations:

$$a_1+a_2=a_1'+a_2', \tag{2}$$

$$b_1+b_2=b_1'+b_2', \tag{3}$$

$$a_1/b_1=a_1'/b_1'=\mu_1, \tag{4}$$

$$a_2/b_2=a_2'/b_2'=\mu_2. \tag{5}$$

Substituting (4) and (5) into (2), we obtain

$$\mu_1b_1+\mu_2b_2=\mu_1b_1'+\mu_2b_2'.$$

Then subtract μ_1 times (3), we get

$$(\mu_2-\mu_1)b_2=(\mu_2-\mu_1)b_2'.$$

Since $\mu_1\ne\mu_2$, it appears that $b_2=b_2'$. Now from (2), (4) and (5), we conclude that

$$(a_1,b_1,a_2,b_2)=(a_1',b_1',a_2',b_2').$$

Lemma 1 is proved.

Lemma 2. *If $\mu_1<\mu_2\le\mu_3<\mu_4$, then*

$$\pi_{\mu_1,\mu_2}(S_{\mu_1}\times S_{\mu_2})\cap\pi_{\mu_3,\mu_4}(S_{\mu_3}\times S_{\mu_4})=\varnothing.$$

Proof. Suppose on the contrary, there exist $(a_1, b_1, a_2, b_2) \in S_{\mu_1} \times S_{\mu_2}$ and $(a_1', b_1', a_2', b_2') \in$ $\in S_{\mu_3} \times S_{\mu_4}$ such that

$$\pi_{\mu_1,\mu_2}(a_1, b_1, a_2, b_2) = \pi_{\mu_3,\mu_4}(a_1', b_1', a_2', b_2').$$

Then we have the following linear equations:

$$a_1 + a_2 = a_1' + a_2', \tag{6}$$

$$b_1 + b_2 = b_1' + b_2', \tag{7}$$

$$a_1/b_1 = \mu_1, \tag{8}$$

$$a_2/b_2 = \mu_2. \tag{9}$$

$$a_1'/b_1' = \mu_3, \tag{10}$$

$$a_2'/b_2' = \mu_4. \tag{11}$$

Substituting (8) – (11) into (6), we obtain

$$\mu_1 b_1 + \mu_2 b_2 = \mu_3 b_1' + \mu_4 b_2'.$$

Combining (7), yields

$$(\mu_2 - \mu_1) b_2 = (\mu_3 - \mu_1) b_1' + (\mu_4 - \mu_1) b_2'.$$

Since $\mu_1 < \mu_2 \leq \mu_3 < \mu_4$, one deduces that

$$(\mu_2 - \mu_1) b_2 > (\mu_2 - \mu_1) b_1' + (\mu_2 - \mu_1) b_2',$$

i.e., $b_2 > b_1' + b_2'$, which is a contradiction to (7) and the fact $b_1 > 0$.

Lemma 2 is proved.

Recall $m_J \geq 2$. Write $M_J := \{\lambda_1, \lambda_2, \ldots, \lambda_{m_J}\}$, where $\lambda_1 < \lambda_2 \ldots < \lambda_{m_J}$. Then

$$\bigcup_{i=1}^{m_J - 1} \pi_{\lambda_i, \lambda_{i+1}} \left(S_{\lambda_i} \times S_{\lambda_{i+1}} \right) \subseteq (A + A) \times (A + A).$$

In view of Lemmas 1 and 2, one has

$$\left| \pi_{\lambda_i, \lambda_{i+1}} (S_{\lambda_i} \times S_{\lambda_{i+1}}) \right| = |S_{\lambda_i}| \cdot |S_{\lambda_{i+1}}| \geq 2^{2J}$$

for $1 \leq i \leq m_J - 1$ and

$$\pi_{\lambda_i, \lambda_{i+1}} \left(S_{\lambda_i} \times S_{\lambda_{i+1}} \right) \cap \pi_{\lambda_j, \lambda_{j+1}} \left(S_{\lambda_j} \times S_{\lambda_{j+1}} \right) = \varnothing.$$

for $1 \leq i < j \leq m_J - 1$. As a result,

$$|A+A|^2 \geq \left| \bigcup_{i=1}^{m_J-1} \pi_{\lambda_i,\lambda_{h+i}} \left(S_{\lambda_i} \times S_{\lambda_{h+i}} \right) \right| =$$

$$= \sum_{i=1}^{m_J-1} \left| \pi_{\lambda_i,\lambda_{m_J-1}} \left(S_{\lambda_i} \times S_{\lambda_{h+i}} \right) \right| = (m_J - 1) \cdot 2^{2J} \gg m_J \cdot 2^{2J}. \tag{12}$$

Combining (1) and (12), gives

$$|A+A|^2 |A \cdot A| \gg \frac{|A|^4}{\log |A|}.$$

Remark. For the sum-division estimate, the $\log |A|$-term in the denominator can be eliminated, using the method from Li [5].

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