UDC 517.9

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YETTER-DRINFEL'D HOPF ALGEBRAS ON BASIC CYCLE*

ХОПФОВІ АЛГЕБРИ ЄТТЕРА-ДРІНФЕЛЬДА НА БАЗОВОМУ ЦИКЛІ

A class of Yetter-Drinfel'd Hopf algebras on basic cycle are constructed.

Побудовано клас хопфових алгебр Єттера-Дрінфельда на базовому циклі.

1. Introduction. Let H be a Hopf algebra. A Yetter-Drinfel'd module over H is a $\mathbb{K}$-linear space V such that V is both an H-module and an H-comodule and satisfies a compatibility condition. Yetter-Drinfel'd Hopf algebras are Hopf algebras in Yetter-Drinfel'd module category. It is a class of braided Hopf algebras. Nichols algebras [11], (G,χ)-Hopf algebras [12, p. 206] (10.5.11) and twisted Hopf algebras [10] are important examples of Yetter-Drinfel'd Hopf algebras.

Radford's projection theorem [13] leads to a decomposition of the given Hopf algebra into a Radford biproduct of two factors, one is no longer a Hopf algebra, but rather a Yetter-Drinfel'd Hopf algebra over the other factor. After Radford's work, some important advances are the following. Doi considered Hopf modules in Yetter-Drinfel'd module category in [6]. Scharfschwerdt proved Nichols – Zoeller theorem for Yetter-Drinfel'd Hopf algebras, see [15]. Schauenburg proved that a Yetter-Drinfel'd module category is equivalent to a category of the left modules over the Drinfel'd double, and also to a Hopf bimodule category, see [16]. Sommerhäuser studied Yetter-Drinfel'd Hopf algebras over groups of prime order in [17]. Andruskiewitsch and Schneider studied Nichols algebras in [1]. Recently, Grana, Heckenberger and Vendramin classified Nichols algebras of irreducible Yetter-Drinfel'd module over nonabelian groups in [7].

The quiver methods in the representation theory of algebras were considered by Ringel in [14]. The coalgebra structure on quivers were considered by Chin and Montgomery in [4]. Quivers allow one to present algebras or coalgebras in a useful way. For example, Cibils and Rosso constructed Hopf quivers and quiver quantum groups in [3] and [5] respectively. Green and Solberg have investigated the structure of finite dimensional basic Hopf algebras in [8].

One can get a Hopf algebra or a quantum group via quivers. The constructions of braided Hopf algebras via quivers are not numerous. In this paper, we provide such an explicit construction via quivers. Let $C_d(n)$ be a subcoalgebra of the coalgebra $\mathbb{K}Z_n^c$ of paths in the oriented cycle quiver Z_n^c of length n with basis the set of all paths of length strictly less than d. Assume that $G=\{1,g,\ldots,g^{n-1}\}$ is a group and $\mathbb{K}G$ a group Hopf algebra. In this paper, we prove that $C_d(n)$ is a Yetter-Drinfel'd module over $\mathbb{K}G$. Moreover, $C_d(n)$ is a Yetter-Drinfel'd Hopf algebra over $\mathbb{K}G$, see Theorem 5.

Throughout, $\mathbb{K}$ will denote a fixed field. All algebras, coalgebras, (co)modules, $\otimes$ and Hom are over $\mathbb{K}$. For basic definitions and facts about coalgebras, Hopf algebras and (co)modules we refer to Sweedler's book [18]. In particular, the comultiplication of a coalgebra C is denoted by

* Supported by the Chinese NSF (Grant No. 10901098 and No. 11271239) and Basic Academic Discipline Program, the 11 th five year plan of 211 Project for Shanghai University of Finance and Economics.

$\Delta(c) = \sum c_1 \otimes c_2$ for all $c \in C$, and the structure map of a left C-comodule V is denoted by $\rho(v) = \sum v^{-1} \otimes v^0$ for all $v \in V$. For quivers we refer to Auslander – Reiten – Smalφ's book [2].

2. Preliminaries. Let $(H, m, u, \triangle, \epsilon, S)$ be a Hopf algebra with antipode S. A left *Yetter-Drinfel'd module* over H is a $\mathbb{K}$-vector space V such that V is both a left H-module with action $\rightarrow$ and left H-comodule with coaction ρ, and satisfies the compatibility condition:

$$\sum (h \rightarrow v)^{-1} \otimes (h \rightarrow v)^0 = \sum h_1 v^{-1} S(h_3) \otimes h_2 \rightarrow v^0, \tag{1}$$

for all $h \in H$, $v \in V$. The category of left Yetter-Drinfel'd modules over H is denoted by ${}^H_H\mathcal{YD}$. The category is a pre-braided category and the pre-braiding is given by

$$\tau_{V,W} : V \otimes W \longrightarrow W \otimes V, \qquad v \otimes w \longmapsto \sum (v^{-1} \rightarrow w) \otimes v^0.$$

The above map is a braiding when H has a bijective antipode. Denote by $\bar{S}$ the inverse of S. The inverse of $\tau_{V,W}$ is

$$\tau^{-1}_{V,W} : W \otimes V \longrightarrow V \otimes W, \qquad w \otimes v \longmapsto \sum v^0 \otimes \bar{S}(v^{-1}) \rightarrow w.$$

Let A be a Yetter-Drinfel'd module. We call the 6-tuple $(A, m, u, \triangle, \epsilon, S)$ *a Yetter-Drinfel'd Hopf algebra* (or *Hopf algebra in* ${}^H_H\mathcal{YD}$) if A satisfies the following conditions:

(a_1) (A, m, u) is a left H-module algebra, i.e.,

$$h \rightarrow (ab) = \sum (h_1 \rightarrow a)(h_2 \rightarrow b), \qquad h \rightarrow 1_A = \epsilon(h) 1_A.$$

(a_2) (A, m, u) is a left H-comodule algebra, i.e.,

$$\rho(ab) = \sum (ab)^{-1} \otimes (ab)^0 = \sum a^{-1} b^{-1} \otimes a^0 b^0,$$

$$\rho(1_A) = 1_H \otimes 1_A.$$

(a_3) $(A, \triangle, \epsilon)$ is a left H-module coalgebra, i.e.,

$$\triangle(h \rightarrow a) = \sum (h_1 \rightarrow a_1) \otimes (h_2 \rightarrow a_2), \qquad \epsilon_A(h \rightarrow a) = \epsilon_H(h) \epsilon_A(a).$$

(a_4) $(A, \triangle, \epsilon)$ is a left H-comodule coalgebra, i.e.,

$$\sum a^{-1} \otimes (a^0)_1 \otimes (a^0)_2 = \sum {a_1}^{-1} {a_2}^{-1} \otimes {a_1}^0 \otimes {a_2}^0,$$

$$\sum a^{-1} \epsilon_A(a^0) = \epsilon_A(a) 1_H.$$

(a_5) $\triangle$ and ϵ are algebra maps in ${}^H_H\mathcal{YD}$, i.e.,

$$\triangle(ab) = \sum a_1({a_2}^{-1} \rightarrow b_1) \otimes {a_2}^0 b_2,$$

$$\triangle(1) = 1 \otimes 1, \qquad \epsilon(ab) = \epsilon(a)\epsilon(b), \qquad \epsilon(1_A) = 1_k.$$

(a_6) There exists a $\mathbb{K}$-linear map $S\colon A \longrightarrow A$ in ${}^H_H\mathcal{YD}$ such that it is a convolution inverse of identity, i.e., $S * \mathrm{Id} = u\epsilon = \mathrm{Id} * S$.

When the pre-braiding τ is trivial, Yetter-Drinfel'd Hopf algebras are ordinary Hopf algebras, see [18, p. 8] for details. However, generally, Yetter-Drinfel'd Hopf algebras are not ordinary Hopf algebras because the bialgebra axiom asserts that they obey (a5).

Let $q \in \mathbb{K}$. For nonnegative integer l and $0 \le m \le l$, the *Gaussian polynomials* is defined to be

$$\binom{l}{m}_q := \frac{(l)!_q}{m!_q(l-m)!_q}$$

where

$$l!_q := 1_q \dots l_q, \qquad 0!_q := 1, \qquad l_q := 1 + q + \dots + q^{l-1}.$$

Next, we will give several conclusions of Gaussian polynomials. They will be used in next section. Firstly, we recall the *q-Pascal identity*, it can be found in [9] (Proposition IV.2.1).

$$\binom{l}{m}_q = \binom{l-1}{m-1}_q + q^m\binom{l-1}{m}_q = \binom{l-1}{m}_q + q^{l-m}\binom{l-1}{m-1}_q. \tag{2}$$

For any scalar a and a variable element z, for any positive integer l, Kassel proved that

$$(a-z)(a-qz)\dots(a-q^{l-1}z) = \sum_{k=0}^{l}(-1)^k\binom{l}{k}_q q^{\frac{k(k-1)}{2}}a^{l-k}z^k$$

(see [9], IV.2.7). Especially, let $a = 1$ and $z = 1$, we have

$$\sum_{k=0}^{l}(-1)^k q^{\frac{k(k-1)}{2}}\binom{l}{k}_q = 0. \tag{3}$$

Moreover, the following equation also holds.

Lemma 1. *Let l and k be nonnegative integers. For any integer s, where $0 \le s \le l+k$, we have*

$$\sum_{\substack{m+p=s\\ 0\le m\le l,\, 0\le p\le k}} q^{m(k-p)}\binom{l+k-s}{l-m}_q\binom{s}{m}_q = \binom{l+k}{l}_q. \tag{4}$$

3. Construction. Let Z_n^c denote the basic cycle of length n, i.e., an oriented graph with n vertices $e_0, \dots, e_{n-1}$, and a unique arrow a_i from e_i to e_{i+1} for each $0 \le i \le n-1$. The indices are taken modulo n. Set $\gamma_i^m := a_{i+m-1}\dots a_{i+1}a_i$ to be the path of length m starting at the vertex e_i. Note that $\gamma_i^0 = e_i$ and $\gamma_i^1 = a_i$.

Let $C_d(n)$ be the subcoalgebra of $\mathbb{K}Z_n^c$ with basis the set of all paths of length strictly less than d. Observe that if the order of q is d, then $\binom{d}{l}_q = 0$ for $1 \le l \le d-1$. Then $C_d(n)$ is a path coalgebra with comultiplication $\triangle(\gamma_i^l) = \sum_{k=0}^{l}\gamma_{i+k}^{l-k}\otimes\gamma_i^k$, and counit $\epsilon(\gamma_i^l) = \delta_{l,0}$. Here, $\delta_{l,0}$ is the Kronecker symbol.

Define a multiplication on $C_d(n)$ by

$$\gamma_i^l \gamma_j^s = \binom{l+s}{l}_q \gamma_{i+j}^{l+s}, \tag{5}$$

where $l+s<d$. Observe that if $l+s \geq d$, then $\gamma_i^l \gamma_j^s = 0$ since $q^d = 1$. It is easy to see that the unit element of $C_d(n)$ is $1 = \gamma_0^0$.

Definition 1. *Let A be a vector space. We call A a pre-bialgebra if A is an algebra and a coalgebra.*

From Definition 1, we know that a pre-bialgebra is a bialgebra if and only if $\triangle$ and ϵ are algebra morphisms.

The following lemma is routine, we omit the proof.

Lemma 2. *Coalgebra $C_d(n)$ is a pre-bialgebra with multiplication* (5).

Let $G = \{1, g, g^2, \ldots, g^{n-1}\}$ be a group. Then $\mathbb{K}G$ is a Hopf algebra, see [12] (1.5.3). It is clear that $C_d(n)$ becomes a left $\mathbb{K}G$-module with the left module structure

$$g^s \rightarrow \gamma_i^l = q^{sl} \gamma_i^l \tag{6}$$

and $C_d(n)$ is also a left $\mathbb{K}G$-comodule with comodule structure

$$\rho(\gamma_i^l) = \sum g^l \otimes \gamma_i^l. \tag{7}$$

Then we have the following lemma.

Lemma 3. *Coalgebra $C_d(n)$ is a Yetter-Drinfel'd module over $\mathbb{K}G$ with module* (6) *and comodule* (7).

Proof. Take $g^s \in \mathbb{K}G$ and $\gamma_i^l \in C_d(n)$. Recall that

$$\sum (g^s \rightarrow \gamma_i^l)^{-1} \otimes (g^s \rightarrow \gamma_i^l)^0 = q^{sl} g^l \otimes \gamma_i^l.$$

Moreover, we have

$$\sum (g^s)_1 (\gamma_i^l)^{-1} S((g^s)_3) \otimes (g^s)_2 \rightarrow \gamma_i^l = g^s g^l S(g^s) \otimes g^s \rightarrow \gamma_i^l = g^l \otimes q^{sl} \gamma_i^l.$$

This means that (1) holds. Thus $C_d(n)$ is a Yetter-Drinfel'd module over $\mathbb{K}G$.

Next, we will give the main theorem.

Theorem 1. *Coalgebra $C_d(n)$ is a Yetter-Drinfel'd Hopf algebra over $\mathbb{K}G$.*

Proof. We divide the proof into six steps as the definition of Yetter-Drinfel'd Hopf algebras. In the following, we take $\gamma_i^l, \gamma_j^k \in C_d(n)$ and $g^s \in G$.

It is easy to check that (a_1)$-$(a_4) hold. We only need to show (a_5) and (a_6).

(a_5) It is obvious that $\triangle(1) = 1 \otimes 1$, $\epsilon(\gamma_i^l \gamma_j^k) = \binom{l+k}{l}_q \delta_{l+k,0} = \delta_{l,0}\delta_{k,0} = \epsilon(\gamma_i^l)\epsilon(\gamma_j^l)$ and $\epsilon(1) = 1$. Next, we will prove the comultiplication $\triangle$ is an algebra map in Yetter-Drinfel'd category. On one hand, we have

$$\triangle(\gamma_i^l \gamma_j^k) = \binom{l+k}{l}_q \triangle(\gamma_{i+j}^{l+k}) = \binom{l+k}{l}_q \sum_{s=0}^{l+k} \gamma_{i+j+s}^{l+k-s} \otimes \gamma_{i+j}^s. \tag{8}$$

On the other hand, we obtain

$$\sum (\gamma_i^l)_1 ((\gamma_i^l)_2{}^{-1} \rightarrow (\gamma_j^k)_1) \otimes (\gamma_i^l)_2{}^0 (\gamma_j^k)_2 =$$

$$= \sum_{m=0}^{l}\sum_{p=0}^{k} \gamma_{i+m}^{l-m}((\gamma_i^m)^{-1} \rightarrow \gamma_{j+p}^{k-p}) \otimes (\gamma_i^m)^0(\gamma_j^p) =$$

$$= \sum_{m=0}^{l}\sum_{p=0}^{k} \gamma_{i+m}^{l-m}(g^m \rightarrow \gamma_{j+p}^{k-p}) \otimes (\gamma_i^m \gamma_j^p) =$$

$$= \sum_{m=0}^{l}\sum_{p=0}^{k} q^{m(k-p)} \binom{l-m+k-p}{l-m}_q \binom{m+p}{m}_q \gamma_{i+j+m+p}^{l+k-m-p} \otimes \gamma_{i+j}^{m+p}. \tag{9}$$

For $s = 0, 1, \ldots, l+k$, comparing the coefficient of $\gamma_{i+j+s}^{l+k-s} \otimes \gamma_{i+j}^{s}$ in equation (8) and equation (9), we get

$$\binom{l+k}{l}_q \gamma_{i+j+s}^{l+k-s} \otimes \gamma_{i+j}^{s} = \sum_{\substack{m+p=s \\ 0\le m\le l, 0\le p\le k}} q^{m(k-p)} \binom{l+k-s}{l-m}_q \binom{s}{m}_q \gamma_{i+j+s}^{l+k-s} \otimes \gamma_{i+j}^{s}$$

by (4). Thus

$$\binom{l+k}{l}_q \sum_{s=0}^{l+k} \gamma_{i+j+s}^{l+k-s} \otimes \gamma_{i+j}^{s} = \sum_{m=0}^{l}\sum_{p=0}^{k} q^{m(k-p)} \binom{l-m+k-p}{l-m}_q \binom{m+p}{m}_q \gamma_{i+j+m+p}^{l+k-m-p} \otimes \gamma_{i+j}^{m+p}.$$

That means

$$\triangle(\gamma_i^l \gamma_j^k) = \sum (\gamma_i^l)_1 ((\gamma_i^l)_2{}^{-1} \rightarrow (\gamma_j^k)_1) \otimes (\gamma_i^l)_2{}^0 (\gamma_j^k)_2.$$

Hence $\triangle$ is an algebra map in Yetter-Drinfel'd category.

(a_6) Define $S\colon A \longrightarrow A$ by

$$S(\gamma_i^l) = (-1)^l q^{\frac{l(l-1)}{2}} \gamma_{-i-l}^l.$$

Then S is a convolution inverse of identity, since

$$(S * Id)(\gamma_i^l) = \sum_{m=0}^{l} S(\gamma_{i+m}^{l-m})\gamma_i^m = \sum_{m=0}^{l} (-1)^{l-m} q^{\frac{(l-m)(l-m-1)}{2}} \gamma_{-i-l}^{l-m}\gamma_i^m =$$

$$= \sum_{m=0}^{l} (-1)^{l-m} q^{\frac{(l-m)(l-m-1)}{2}} \binom{l}{l-m}_q \gamma_{-l}^l.$$

If $l = 0$, we have $(S*Id)(\gamma_i^0) = \gamma_0^0$. If $l \neq 0$, we have $\sum_{m=0}^{l} (-1)^{l-m} q^{\frac{(l-m)(l-m-1)}{2}} \binom{l}{l-m}_q \gamma_{-l}^l =$ $= 0$ by (3). In a word, $(S * Id)(\gamma_i^l) = 0$. Similarly, $(Id * S)(\gamma_i^l) = 0$. So S is the convolution inverse of identity.

Thus $C_d(n)$ is a Yetter-Drinfel'd Hopf algebra over the group algebra $\mathbb{K}G$. This completes the proof.

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Received 11.09.12,
after revision — 25.11.13