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GENERALIZATIONS OF STEFFENSEN'S INEQUALITY BY LIDSTONE'S POLYNOMIAL*

УЗАГАЛЬНЕННЯ НЕРІВНОСТІ СТЕФФЕНСЕНА ДЛЯ ПОТЕНЦІАЛІВ ЛІДСТОУНА

We obtain generalizations of Steffensen's inequality by using Lidstone's polynomial. Furthermore, the functionals associated with the obtained generalizations are used to generate n-exponentially and exponentially convex functions, as well as the new Stolarsky-type means.

Отримано узагальнення нерівності Стеффенсена за допомогою потенціалів Лідстоуна. Крім того, функціонали, що відповідають отриманим узагальненням, також застосовуються для одержання як n-експоненціально та експоненціально опуклих функцій, так і нових середніх Столярського.

1. Introduction. Since its appearance in 1918 Steffensen's inequality is still the subject of the investigation and generalization by many mathematicians. The well-known Steffensen inequality reads [10]:

Theorem 1.1. *Suppose that f is decreasing and g is integrable on $[a,b]$ with $0 \le g \le 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int\limits_{b-\lambda}^{b} f(t)dt \le \int\limits_{a}^{b} f(t)g(t)dt \le \int\limits_{a}^{a+\lambda} f(t)dt.$$

The inequalities are reversed for f increasing.

In 1929 G. J. Lidstone [5] introduced a generalization of Taylor's series, today known as Lidstone series. It approximates a given function in the neighborhood of two points instead of one. Such series have been studied by H. Poritsky [8], J. M. Wittaker [13], I. J. Schoenberg [9], R. P. Boas [3] and others.

Definition 1.1. *Let $f \in C^\infty([0,1])$, then Lidstone series has the form*

$$\sum_{k=0}^{\infty} \Big(f^{(2k)}(0)\Lambda_k(1-x) + f^{(2k)}(1)\Lambda_k(x) \Big),$$

where Λ_n is Lidstone polynomial of degree $2n+1$ defined by the relations

$$\Lambda_0(t) = t,$$

$$\Lambda_n''(t) = \Lambda_{n-1}(t),$$

$$\Lambda_n(0) = \Lambda_n(1) = 0, \qquad n \ge 1.$$

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Another explicit representations of Lidstone polynomial are given in [1] and [13]. Some of those representations are given by

$$\Lambda_n(t) = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \sin k\pi t, \quad n \geq 1,$$

$$\Lambda_n(t) = \frac{1}{6}\left[\frac{6t^{2n+1}}{(2n+1)!} - \frac{t^{2n-1}}{(2n-1)!}\right] -$$

$$-\sum_{k=0}^{n-2} \frac{2(2^{2k+3}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-3}}{(2n-2k-3)!}, \quad n = 1, 2, \ldots,$$

$$\Lambda_n(t) = \frac{2^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{1+t}{2}\right), \quad n = 1, 2, \ldots,$$

where B_{2k+4} is the $(2k+4)$th Bernoulli number and $B_{2n+1}\left(\frac{1+t}{2}\right)$ is a Bernoulli polynomial.

In [12] Widder proved the following fundamental lemma:

Lemma 1.1. *If $f \in C^{2n}([0,1])$, then*

$$f(t) = \sum_{k=0}^{n-1} \left[f^{(2k)}(0)\Lambda_k(1-t) + f^{(2k)}(1)\Lambda_k(t)\right] + \int_0^1 G_n(t,s) f^{(2n)}(s)ds,$$

where

$$G_1(t,s) = G(t,s) = \begin{cases} (t-1)s, & \text{if} \quad s < t, \\ (s-1)t, & \text{if} \quad t \leq s, \end{cases}$$

is the homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ on $[0,1]$, and with the successive iterates of $G(t,s)$

$$G_n(t,s) = \int_0^1 G_1(t,p) G_{n-1}(p,s)dp, \quad n \geq 2.$$

The aim of this paper is to generalize Steffensen's inequality using Lidstone's polynomial. In Section 2 we obtain difference of integrals on two intervals from which we obtain some general inequality. This general inequality is used in Section 3 to obtain new generalizations of Steffensen's inequality for $(2n)$-convex functions. In Section 4 we give estimation of the difference of the left-hand and right-hand sides of obtained generalizations. In Section 5 we consider three functionals associated with new generalizations and use them to generate n-exponentially and exponentially convex functions. In Section 6 we apply results from Section 5 to some families of functions to obtain new Stolarsky-type means related to these functionals.

2. Difference of integrals on two intervals. If $[a,b] \cap [c,d] \neq \varnothing$ we have four possible cases for two intervals $[a,b]$ and $[c,d]$. First case is $[c,d] \subset [a,b]$, second case is $[a,b] \cap [c,d] = [c,b]$ and other two cases are obtained by changing $a \leftrightarrow c$, $b \leftrightarrow d$. Hence, in the following theorem we will only observe first two cases.

In this paper by $T_{w,n}^{[a,b]}$ we will denote

$$T_{w,n}^{[a,b]} = \sum_{k=0}^{n-1} (b-a)^{2k} \int\limits_a^b w(x) \left[f^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + f^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right] dx.$$

Theorem 2.1. *Let $f\colon [a,b] \cup [c,d] \to \mathbb{R}$ be of class C^{2n} on $[a,b] \cup [c,d]$ for some $n \geq 1$. Let $w\colon [a,b] \to [0,\infty\rangle$ and $u\colon [c,d] \to [0,\infty\rangle$. Then if $[a,b] \cap [c,d] \neq \varnothing$ we have*

$$\int\limits_a^b w(t) f(t) dt - \int\limits_c^d u(t) f(t) dt - T_{w,n}^{[a,b]} + T_{u,n}^{[c,d]} = \int\limits_a^{\max\{b,d\}} K_n(s) f^{(2n)}(s) ds, \tag{2.1}$$

where, in case $[c,d] \subseteq [a,b]$,

$$K_n(s) = \begin{cases} (b-a)^{2n-1} \int\limits_a^b w(x) G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) dx, & s \in [a,c], \\ (b-a)^{2n-1} \int\limits_a^b w(x) G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) dx - \\ \qquad -(d-c)^{2n-1} \int\limits_c^d u(x) G_n\left(\frac{x-c}{d-c}, \frac{s-c}{d-c} \right) dx, & s \in \langle c,d], \\ (b-a)^{2n-1} \int\limits_a^b w(x) G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) dx, & s \in \langle d,b], \end{cases} \tag{2.2}$$

and, in case $[a,b] \cap [c,d] = [c,b]$,

$$K_n(s) = \begin{cases} (b-a)^{2n-1} \int\limits_a^b w(x) G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) dx, & s \in [a,c], \\ (b-a)^{2n-1} \int\limits_a^b w(x) G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) dx - \\ \qquad -(d-c)^{2n-1} \int\limits_c^d u(x) G_n\left(\frac{x-c}{d-c}, \frac{s-c}{d-c} \right) dx, & s \in \langle c,b\,], \\ -(d-c)^{2n-1} \int\limits_c^d u(x) G_n\left(\frac{x-c}{d-c}, \frac{s-c}{d-c} \right) dx, & s \in \langle b,d]. \end{cases} \tag{2.3}$$

Proof. From Widder's lemma for $f \in C^{2n}([a,b])$ we have the following identity:

$$f(x) = \sum_{k=0}^{n-1} (b-a)^{2k} \left[f^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + f^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right] +$$

$$+(b-a)^{2n-1} \int\limits_a^b G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) f^{(2n)}(s) ds. \tag{2.4}$$

Multiplying identity (2.4) by $w(x)$, then integrating from a to b and using Fubini's theorem we obtain

$$\int_a^b w(x)f(x)dx = \sum_{k=0}^{n-1}(b-a)^{2k}\int_a^b w(x)\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]dx+$$

$$+(b-a)^{2n-1}\int_a^b f^{(2n)}(s)\left(\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx\right)ds. \tag{2.5}$$

Now subtracting identities (2.5) for interval $[a,b]$ and $[c,d]$ we get (2.1).

Theorem 2.2. *Let $f\colon [a,b]\cup[c,d]\to\mathbb{R}$ be $(2n)$-convex on $[a,b]\cup[c,d]$, $w\colon [a,b]\to[0,\infty\rangle$, $u\colon [c,d]\to[0,\infty\rangle$. Then if $[a,b]\cap[c,d]\neq\varnothing$ and*

$$K_n(s)\geq 0, \tag{2.6}$$

we have

$$\int_a^b w(t)f(t)dt - T_{w,n}^{[a,b]} \geq \int_c^d u(t)f(t)dt - T_{u,n}^{[c,d]}, \tag{2.7}$$

where, in case $[c,d]\subseteq[a,b]$, $K_n(s)$ is defined by (2.2) *and, in case $[a,b]\cap[c,d]=[c,b]$, $K_n(s)$ is defined by* (2.3).

Proof. Since f is $(2n)$-convex, withouth loss of generality we can assume that f is $(2n)$-times differentiable and $f^{(2n)}\geq 0$ see [7, p. 16 and 293]. Now we can apply Theorem 2.1 to obtain (2.7).

3. Generalization of Steffensen's inequality by Lidstone's polynomial. For a special choice of weights and intervals in previous section we obtain a generalization of Steffensen's inequality.

Theorem 3.1. *Let $f\colon [a,b]\cup[a,a+\lambda]\to\mathbb{R}$ be $(2n)$-convex on $[a,b]\cup[a,a+\lambda]$ and $w\colon [a,b]\to[0,\infty\rangle$. Then if*

$$K_n(s)\geq 0, \tag{3.1}$$

we have

$$\int_a^b w(t)f(t)dt - T_{w,n}^{[a,b]} \geq \int_a^{a+\lambda} f(t)dt - T_{1,n}^{[a,a+\lambda]}, \tag{3.2}$$

where, in case $a\leq a+\lambda\leq b$,

$$K_n(s)=\begin{cases} (b-a)^{2n-1}\displaystyle\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx- & \\ \qquad -\lambda^{2n-1}\displaystyle\int_a^{a+\lambda} G_n\left(\frac{x-a}{\lambda},\frac{s-a}{\lambda}\right)dx, & s\in[a,a+\lambda], \\ (b-a)^{2n-1}\displaystyle\int_a^b w(x)G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)dx, & s\in\langle a+\lambda,b], \end{cases} \tag{3.3}$$

and, in case $a < b \leq a + \lambda$,

$$K_n(s) = \begin{cases} (b-a)^{2n-1} \int\limits_a^b w(x) G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) dx - \\ \qquad -\lambda^{2n-1} \int\limits_a^{a+\lambda} G_n\left(\frac{x-a}{\lambda}, \frac{s-a}{\lambda}\right) dx, & s \in [a,b], \\ -\lambda^{2n-1} \int\limits_a^{a+\lambda} G_n\left(\frac{x-a}{\lambda}, \frac{s-a}{\lambda}\right) dx, & s \in \langle b, a+\lambda] . \end{cases} \tag{3.4}$$

Proof. We take $c = a$, $d = a + \lambda$ and $u(t) = 1$ in Theorem 2.2.

Theorem 3.2. *Let* $f\colon [a,b] \cup [b-\lambda, b] \to \mathbb{R}$ *be* $(2n)$*-convex on* $[a,b] \cup [b-\lambda,b]$ *and* $w\colon [a,b] \to \to [0,\infty\rangle$. *Then if*

$$K_n(s) \geq 0, \tag{3.5}$$

we have

$$\int\limits_{b-\lambda}^b f(t)dt - T_{1,n}^{[b-\lambda,b]} \geq \int\limits_a^b w(t)f(t)dt - T_{w,n}^{[a,b]}, \tag{3.6}$$

where, in case $a \leq b - \lambda \leq b$,

$$K_n(s) = \begin{cases} -(b-a)^{2n-1} \int\limits_a^b w(x) G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) dx, & s \in [a, b-\lambda], \\ \lambda^{2n-1} \int\limits_{b-\lambda}^b G_n\left(\frac{x-b+\lambda}{\lambda}, \frac{s-b+\lambda}{\lambda}\right) dx - \\ \qquad -(b-a)^{2n-1} \int\limits_a^b w(x) G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) dx, & s \in \langle b-\lambda, b] , \end{cases} \tag{3.7}$$

and, in case $b - \lambda \leq a \leq b$,

$$K_n(s) = \begin{cases} \lambda^{2n-1} \int\limits_{b-\lambda}^b G_n\left(\frac{x-b+\lambda}{\lambda}, \frac{s-b+\lambda}{\lambda}\right) dx, & s \in [b-\lambda, a], \\ \lambda^{2n-1} \int\limits_{b-\lambda}^b G_n\left(\frac{x-b+\lambda}{\lambda}, \frac{s-b+\lambda}{\lambda}\right) dx - \\ \qquad -(b-a)^{2n-1} \int\limits_a^b w(x) G_n\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right) dx, & s \in \langle a, b] . \end{cases} \tag{3.8}$$

Proof. First we change $a \leftrightarrow c$, $b \leftrightarrow d$ and $w \leftrightarrow u$ in Theorem 2.2 and then we take $c = b - \lambda$, $d = b$ and $u(t) = 1$.

4. Estimation of the difference.

Theorem 4.1. *Suppose that all assumptions of Theorem* 2.1 *hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $1/p+1/q=1$. *Let* $\left|f^{(2n)}\right|^p : [a,b] \cup [c,d] \to \mathbb{R}$ *be an R-integrable function for some* $n \geq 1$. *Then we have*

$$\left| \int\limits_a^b w(t)f(t)dt - \int\limits_c^d u(t)f(t)dt - T_{w,n}^{[a,b]} + T_{u,n}^{[c,d]} \right| \leq$$

$$\leq \left\| f^{(2n)} \right\|_p \left(\int\limits_a^{\max\{b,d\}} |K_n(s)|^q \, ds \right)^{\frac{1}{q}}. \tag{4.1}$$

The constant $\left(\int_a^{\max\{b,d\}} |K_n(s)|^q \, ds \right)^{1/q}$ *in the inequality* (4.1) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Proof. Using inequality (2.1) and applying Hölder's inequality we obtain

$$\left| \int\limits_a^b w(t)f(t)dt - \int\limits_c^d u(t)f(t)dt - T_{w,n}^{[a,b]} + T_{u,n}^{[c,d]} \right| =$$

$$= \left| \int\limits_a^{\max\{b,d\}} K_n(s) f^{(2n)}(s) ds \right| \leq \left\| f^{(2n)} \right\|_p \left(\int\limits_a^{\max\{b,d\}} |K_n(s)|^q \, ds \right)^{\frac{1}{q}}.$$

For the proof of the sharpness of the constant $\left(\int_a^{\max\{b,d\}} |K_n(s)|^q \, ds \right)^{1/q}$ we will find a function f for which the equality in (4.1) is obtained.

For $1 < p < \infty$ take f to be such that

$$f^{(2n)}(s) = \operatorname{sgn} K_n(s) \, |K_n(s)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $f^{(2n)}(s) = \operatorname{sgn} K_n(s)$.

For $p = 1$ we will prove that

$$\left| \int\limits_a^{\max\{b,d\}} K_n(s) f^{(2n)}(s) ds \right| \leq \max_{s \in [a, \max\{b,d\}]} |K_n(s)| \left(\int\limits_a^{\max\{b,d\}} \left| f^{(2n)}(s) \right| ds \right) \tag{4.2}$$

is the best possible inequality. Suppose that $|K_n(s)|$ attains its maximum at $s_0 \in [a, \max\{b,d\}]$. First we assume that $K_n(s_0) > 0$. For ε small enough we define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} 0, & a \leq s \leq s_0, \\ \dfrac{1}{\varepsilon\,(2n)!}(s-s_0)^{2n}, & s_0 \leq s \leq s_0 + \varepsilon, \\ \dfrac{1}{(2n)!}(s-s_0)^{2n-1}, & s_0 + \varepsilon \leq s \leq \max\{b,d\}. \end{cases}$$

Then for ε small enough

$$\left|\int\limits_a^{\max\{b,d\}} K_n(s)f^{(2n)}(s)ds\right| = \left|\int\limits_{s_0}^{s_0+\varepsilon} K_n(s)\frac{1}{\varepsilon}ds\right| = \frac{1}{\varepsilon}\int\limits_{s_0}^{s_0+\varepsilon} K_n(s)\,ds.$$

Now from inequality (4.2) we have

$$\frac{1}{\varepsilon}\int\limits_{s_0}^{s_0+\varepsilon} K_n(s)ds \le K_n(s_0)\int\limits_{s_0}^{s_0+\varepsilon}\frac{1}{\varepsilon}ds = K_n(s_0).$$

Since

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_{s_0}^{s_0+\varepsilon} K_n(s)ds = K_n(s_0)$$

the statement follows. In case $K_n(s_0) < 0$ we define

$$f_\varepsilon(s) = \begin{cases} \dfrac{1}{(2n)!}(s-s_0-\varepsilon)^{2n-1}, & a \le s \le s_0, \\ -\dfrac{1}{\varepsilon\,(2n)!}(s-s_0-\varepsilon)^{2n}, & s_0 \le s \le s_0+\varepsilon, \\ 0, & s_0+\varepsilon \le s \le \max\{b,d\}, \end{cases}$$

and the rest of the proof is the same as above.

Theorem 4.2. *Suppose that all assumptions of Theorem* 3.1 *hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \le p, q \le \infty$, $1/p + 1/q = 1$. *Let* $\left|f^{(2n)}\right|^p : [a,b]\cup[a,a+\lambda] \to \mathbb{R}$ *be an R-integrable function for some* $n \ge 1$. *Let* $K_n(s)$ *be defined by* (3.3) *in case* $a \le a+\lambda \le b$ *and by* (3.4) *in case* $a < b \le a+\lambda$. *Then we have*

$$\left|\int\limits_a^b w(t)f(t)dt - \int\limits_a^{a+\lambda} f(t)dt - T_{w,n}^{[a,b]} + T_{1,n}^{[a,a+\lambda]}\right| \le$$

$$\le \left\|f^{(2n)}\right\|_p \left(\int\limits_a^{\max\{b,a+\lambda\}} |K_n(s)|^q\,ds\right)^{\frac{1}{q}}. \tag{4.3}$$

The constant $\left(\int_a^{\max\{b,a+\lambda\}} |K_n(s)|^q\,ds\right)^{1/q}$ *in the inequality* (4.3) *is sharp for* $1 < p \le \infty$ *and the best possible for* $p = 1$.

Proof. We take $c = a$, $d = a+\lambda$ and $u(t) = 1$ in Theorem 4.1.

Theorem 4.3. *Suppose that all assumptions of Theorem* 3.2 *hold. Assume* (p,q) *is a pair of conjugate exponents, that is* $1 \le p, q \le \infty$, $1/p + 1/q = 1$. *Let* $\left|f^{(2n)}\right|^p : [a,b]\cup[b-\lambda,b] \to \mathbb{R}$ *be an R-integrable function for some* $n \ge 1$. *Let* $K_n(s)$ *be defined by* (3.7) *in case* $a \le b-\lambda \le b$ *and by* (3.8) *in case* $b-\lambda \le a \le b$. *Then we have*

$$\left|\int\limits_{b-\lambda}^{b} f(t)dt - \int\limits_{a}^{b} w(t)f(t)dt - T_{1,n}^{[b-\lambda,b]} + T_{w,n}^{[a,b]}\right| \leq$$

$$\leq \left\|f^{(2n)}\right\|_p \left(\int\limits_{\min\{a,b-\lambda\}}^{b} |K_n(s)|^q\, ds\right)^{\frac{1}{q}}. \tag{4.4}$$

The constant $\left(\int_{\min\{a,b-\lambda\}}^{b} |K_n(s)|^q\, ds\right)^{1/q}$ *in the inequality* (4.4) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Proof. First we change $a \leftrightarrow c$, $b \leftrightarrow d$ and $w \leftrightarrow u$ in Theorem 2.1 and then we take $c = b - \lambda$, $d = b$ and $u(t) = 1$. The rest of the proof is similar to the proof of Theorem 4.1.

5. Mean value theorems and exponential convexity. Motivated by inequalities (2.7), (3.2) and (3.6) under assumptions of Theorems 2.2, 3.1 and 3.2, respectively, we define following linear functionals:

$$L_1(f) = \int\limits_{a}^{b} w(t)f(t)dt - \int\limits_{c}^{d} u(t)f(t)dt-$$

$$-\sum_{k=0}^{n-1}(b-a)^{2k}\int\limits_{a}^{b} w(x)\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]dx+$$

$$+\sum_{k=0}^{n-1}(d-c)^{2k}\int\limits_{c}^{d} u(x)\left[f^{(2k)}(c)\Lambda_k\left(\frac{d-x}{d-c}\right) + f^{(2k)}(d)\Lambda_k\left(\frac{x-c}{d-c}\right)\right]dx, \tag{5.1}$$

$$L_2(f) = \int\limits_{a}^{b} w(t)f(t)dt - \int\limits_{a}^{a+\lambda} f(t)dt-$$

$$-\sum_{k=0}^{n-1}(b-a)^{2k}\int\limits_{a}^{b} w(x)\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right) + f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]dx+$$

$$+\sum_{k=0}^{n-1}\lambda^{2k}\int\limits_{a}^{a+\lambda}\left[f^{(2k)}(a)\Lambda_k\left(\frac{a+\lambda-x}{\lambda}\right) + f^{(2k)}(a+\lambda)\Lambda_k\left(\frac{x-a}{\lambda}\right)\right]dx, \tag{5.2}$$

$$L_3(f) = \int\limits_{b-\lambda}^{b} f(t)dt - \int\limits_{a}^{b} w(t)f(t)dt-$$

$$-\sum_{k=0}^{n-1}\lambda^{2k}\int\limits_{b-\lambda}^{b}\left[f^{(2k)}(b-\lambda)\Lambda_k\left(\frac{b-x}{\lambda}\right) + f^{(2k)}(b)\Lambda_k\left(\frac{x-b+\lambda}{\lambda}\right)\right]dx+$$

$$+\sum_{k=0}^{n-1}(b-a)^{2k}\int_a^b w(x)\left[f^{(2k)}(a)\Lambda_k\left(\frac{b-x}{b-a}\right)+f^{(2k)}(b)\Lambda_k\left(\frac{x-a}{b-a}\right)\right]dx. \tag{5.3}$$

Also, we define $I_1=[a,b]\cup[c,d]$, $I_2=[a,b]\cup[a,a+\lambda]$, and $I_3=[a,b]\cup[b-\lambda,b]$.

***Remark* 5.1.** Under assumptions of Theorems 2.2, 3.1 and 3.2 respectively, it holds $L_i(f)\geq 0$, $i=1,2,3$, for all $(2n)$-convex functions f.

First we will state and prove mean value theorems for defined functionals.

Theorem 5.1. *Let $f\colon I_i\to\mathbb{R}$, $i=1,2,3$, be such that $f\in C^{2n}(I_i)$. If inequalities in* (2.6), $i=1$, (3.1), $i=2$, *and* (3.5), $i=3$, *hold, then there exist $\xi_i\in I_i$ such that*

$$L_i(f)=f^{(2n)}(\xi_i)L_i(\varphi),\quad i=1,2,3, \tag{5.4}$$

where $\varphi(x)=\dfrac{x^{2n}}{(2n)!}$.

Proof. Let us denote $m=\min f^{(2n)}$ and $M=\max f^{(2n)}$. For a given function $f\in C^{2n}(I_i)$ we define functions $F_1,F_2:I_i\to\mathbb{R}$ with

$$F_1(x)=\frac{Mx^{2n}}{(2n)!}-f(x)\qquad\text{and}\qquad F_2(x)=f(x)-\frac{mx^{2n}}{(2n)!}.$$

Now $F_1^{(2n)}(x)=M-f^{(2n)}(x)\geq 0$, $x\in I_i$, so we conclude $L_i(F_1)\geq 0$ and then $L_i(f)\leq M\cdot L_i(\varphi)$. Similarly, from $F_2^{(2n)}(x)=f^{(2n)}(x)-m\geq 0$ we conclude $m\cdot L_i(\varphi)\leq L_i(f)$.

If $L_i(\varphi)=0$, (5.4) holds for all $\xi_i\in I_i$. Otherwise, $m\leq\dfrac{L_i(f)}{L_i(\varphi)}\leq M$. Since $f^{(2n)}(x)$ is continuous on I_i there exist $\xi_i\in I_i$ such that (5.4) holds.

Theorem 5.2. *Let $f,g:I_i\to\mathbb{R}$, $i=1,2,3$, be such that $f,g\in C^{2n}(I_i)$ and $g^{(2n)}(x)\neq 0$ for every $x\in I_i$. If inequalities in* (2.6), $i=1$, (3.1), $i=2$, *and* (3.5), $i=3$, *hold, then there exist $\xi_i\in I_i$ such that*

$$\frac{L_i(f)}{L_i(g)}=\frac{f^{(2n)}(\xi_i)}{g^{(2n)}(\xi_i)},\quad i=1,2,3. \tag{5.5}$$

Proof. We define functions $\phi_i(x)=f(x)L_i(g)-g(x)L_i(f)$, $i=1,2,3$. Applying Theorem 5.1 on ϕ_i, there exist $\xi_i\in I_i$ such that $L_i(\phi_i)=\phi_i^{(2n)}(\xi_i)L_i(\varphi)$. Since $L_i(\phi_i)=0$ it follows $f^{(2n)}(\xi_i)L_i(g)-g^{(2n)}(\xi_i)L_i(f)=0$ and (5.5) is proved.

Now we will use previously defined functionals to construct exponentially convex functions. We will start this part of the section with some definitions and properties which are used in our results (see [6]).

Definition 5.1. *A function $\psi:I\to\mathbb{R}$ is n-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n}\xi_i\xi_j\,\psi\left(\frac{x_i+x_j}{2}\right)\geq 0,$$

holds for all choices $\xi_1,\ldots,\xi_n\in\mathbb{R}$ and all choices $x_1,\ldots,x_n\in I$. A function $\psi\colon I\to\mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

***Remark* 5.2.** It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N},\ k \leq n$.

Definition 5.2. *A function $\psi\colon I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.*

A function $\psi\colon I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

***Remark* 5.3.** It is known that $\psi\colon I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geq 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

Proposition 5.1. *If f is a convex function on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function f is concave, the inequality is reversed.

Lemma 5.1. *A function Φ is* log*-convex on an interval I if and only if, for all $a, b, c \in I$, $a < b < c$, it holds*

$$[\Phi(b)]^{c-a} \leq [\Phi(a)]^{c-b}[\Phi(c)]^{b-a}.$$

Definition 5.3. *Let f be a real-valued function defined on the segment $[a, b]$. The divided difference of order n of the function f at distinct points $x_0, \ldots, x_n \in [a, b]$, is defined recursively* (*see* [2, 7]) *by*

$$f[x_i] = f(x_i), \quad i = 0, \ldots, n,$$

and

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0, \ldots, x_n]$ is independent of the order of the points $x_0, \ldots, x_n$.

The definition may be extended to include the case in which some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x, \ldots, x}_{j\ times}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \tag{5.6}$$

An elegant method of producing n-exponentially convex and exponentially convex functions is given in [4]. We use this method to prove the n-exponential convexity for above defined functionals. In the sequel the notion log denotes the natural logarithm function.

Theorem 5.3. *Let $\Omega = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1, 2, 3$, in $\mathbb{R}$ such that the function $p \mapsto f_p[x_0, \ldots, x_{2m}]$ is n-exponentially convex in the Jensen sense on J for every $(2m + 1)$ mutually different points $x_0, \ldots, x_{2m} \in I_i$, $i = 1, 2, 3$. Let L_i, $i = 1, 2, 3$, be linear functionals defined by* (5.1)–(5.3)*. Then $p \mapsto L_i(f_p)$ is n-exponentially convex function in the Jensen sense on J.*

If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is n-exponentially convex on J.

Proof. For $\xi_j \in \mathbb{R}$, $j = 1, \ldots, n$, and $p_j \in J$, $j = 1, \ldots, n$, we define the function

$$g(x) = \sum_{j,k=1}^{n} \xi_j \xi_k f_{\frac{p_j + p_k}{2}}(x).$$

Using the assumption that the function $p \mapsto f_p[x_0, \ldots, x_{2m}]$ is n-exponentially convex in the Jensen sense, we have

$$g[x_0, \ldots, x_{2m}] = \sum_{j,k=1}^{n} \xi_j \xi_k f_{\frac{p_j + p_k}{2}}[x_0, \ldots, x_{2m}] \geq 0,$$

which in turn implies that g is a $(2m)$-convex function on J, so $L_i(g) \geq 0$, $i = 1, 2, 3$. Hence

$$\sum_{j,k=1}^{n} \xi_j \xi_k L_i\left(f_{\frac{p_j + p_k}{2}}\right) \geq 0.$$

We conclude that the function $p \mapsto L_i(f_p)$ is n-exponentially convex on J in the Jensen sense.

If the function $p \mapsto L_i(f_p)$ is also continuous on J, then $p \mapsto L_i(f_p)$ is n-exponentially convex by definition.

The following corollaries are an immediate consequences of the above theorem:

***Corollary* 5.1.** *Let $\Omega = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1, 2, 3$, in $\mathbb{R}$, such that the function $p \mapsto f_p[x_0, \ldots, x_{2m}]$ is exponentially convex in the Jensen sense on J for every $(2m + 1)$ mutually different points $x_0, \ldots, x_{2m} \in I_i$, $i = 1, 2, 3$. Let L_i, $i = 1, 2, 3$, be linear functionals defined by* (5.1)–(5.3)*. Then $p \mapsto L_i(f_p)$ is exponentially convex function in the Jensen sense on J. If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is exponentially convex on J.*

***Corollary* 5.2.** *Let $\Omega = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I_i, $i = 1, 2, 3$, in $\mathbb{R}$, such that the function $p \mapsto f_p[x_0, \ldots, x_{2m}]$ is* 2*-exponentially convex in the Jensen sense on J for every $(2m + 1)$ mutually different points $x_0, \ldots, x_{2m} \in I_i$, $i = 1, 2, 3$. Let L_i, $i = 1, 2, 3$, be linear functionals defined by* (5.1)–(5.3)*. Then the following statements hold*:

(i) *If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is* 2*-exponentially convex function on J. If $p \mapsto L_i(f_p)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true*:

$$[L_i(f_s)]^{t-r} \leq [L_i(f_r)]^{t-s} [L_i(f_t)]^{s-r} \tag{5.7}$$

for every choice $r, s, t \in J$, such that $r < s < t$.

(ii) *If the function $p \mapsto L_i(f_p)$ is strictly positive and differentiable on J, then for every $p, q, u, v \in \in J$, such that $p \leq u$ and $q \leq v$, we have*

$$\mu_{p,q}(L_i, \Omega) \leq \mu_{u,v}(L_i, \Omega), \tag{5.8}$$

where

$$\mu_{p,q}(L_i, \Omega) = \begin{cases} \left(\dfrac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\dfrac{\dfrac{d}{dp}L_i(f_p)}{L_i(f_p)}\right), & p = q, \end{cases} \tag{5.9}$$

for $f_p, f_q \in \Omega$.

Proof. (i) This is an immediate consequence of Theorem 5.3, Remark 5.3 and Lemma 5.1.

(ii) Since $p \mapsto L_i(f_p)$ is continuous and strictly positive, by (i) we have that $p \mapsto L_i(f_p)$ is log-convex on J, that is, the function $p \mapsto \log L_i(f_p)$ is convex on J. Applying Proposition 5.1 we get

$$\frac{\log L_i(f_p) - \log L_i(f_q)}{p - q} \leq \frac{\log L_i(f_u) - \log L_i(f_v)}{u - v} \tag{5.10}$$

for $p \leq u, q \leq v, p \neq q, u \neq v$. Hence, we conclude that

$$\mu_{p,q}(L_i, \Omega) \leq \mu_{u,v}(L_i, \Omega).$$

Cases $p = q$ and $u = v$ follow from (5.10) as limit cases.

***Remark* 5.4.** Note that the results from above theorem and corollaries still hold when two of the points $x_0, \ldots, x_{2m} \in I_i$, $i = 1, 2, 3$, coincide, say $x_1 = x_0$, for a family of differentiable functions f_p such that the function $p \mapsto f_p[x_0, \ldots, x_{2m}]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $2m + 1$ points coincide for a family of $2m$ differentiable functions with the same property. The proofs use (5.6) and suitable characterization of convexity.

6. Applications to Stolarsky type means. In this section we will apply general results from previous section to several families of functions which fulfil conditions of obtained general results to get other exponentially convex functions and Stolarsky means.

***Example* 6.1.** Consider a family of functions

$$\Omega_1 = \{f_p : \mathbb{R} \to [0, \infty) : p \in \mathbb{R}\}$$

defined by

$$f_p(x) = \begin{cases} \dfrac{e^{px}}{p^{2n}}, & p \neq 0, \\ \dfrac{x^{2n}}{(2n)!}, & p = 0. \end{cases}$$

Here, $\dfrac{d^{2n}f_p}{dx^{2n}}(x) = e^{px} > 0$ which shows that f_p is $(2n)$-convex on $\mathbb{R}$ for every $p \in \mathbb{R}$ and $p \mapsto \dfrac{d^{2n}f_p}{dx^{2n}}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 5.3 we also have that $p \mapsto f_p[x_0, \ldots, x_{2m}]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 5.1 we conclude that $p \mapsto L_i(f_p)$, $i = 1, 2, 3$, are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping $p \mapsto f_p$ is not continuous for $p = 0$), so it is exponentially convex. For this family of functions, $\mu_{p,q}(L_i, \Omega_1)$, $i = 1, 2, 3$, from (5.9), becomes

$$\mu_{p,q}(L_i, \Omega_1) = \begin{cases} \left(\dfrac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\dfrac{L_i(id \cdot f_p)}{L_i(f_p)} - \dfrac{2n}{p}\right), & p = q \neq 0, \\ \exp\left(\dfrac{1}{2n+1} \dfrac{L_i(id \cdot f_0)}{L_i(f_0)}\right), & p = q = 0, \end{cases}$$

where id is the identity function. Also, by Corollary 5.2 it is monotonic function in parameters p and q.

Theorem 5.2 applied on functions $f_p, f_q \in \Omega_1$ and functionals L_i, $i = 1, 2, 3$, implies that there exist $\xi_i \in I_i$ such that

$$e^{(p-q)\xi_i} = \frac{L_i(f_p)}{L_i(f_q)}$$

so it follows that:

$$M_{p,q}(L_i, \Omega_1) = \log \mu_{p,q}(L_i, \Omega_1), \quad i = 1, 2, 3,$$

satisfies

$$\min\{a, b - \lambda, c\} \leq M_{p,q}(L_i, \Omega_1) \leq \max\{a + \lambda, b, d\}, \quad i = 1, 2, 3.$$

So, $M_{p,q}(L_i, \Omega_1)$ is monotonic mean.

***Example* 6.2.** Consider a family of functions

$$\Omega_2 = \{g_p \colon (0, \infty) \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$g_p(x) = \begin{cases} \dfrac{x^p}{p(p-1)\ldots(p-2n+1)}, & p \notin \{0, 1, \ldots, 2n-1\}, \\ \dfrac{x^j \log x}{(-1)^{2n-1-j} j!(2n-1-j)!}, & p = j \in \{0, 1, \ldots, 2n-1\}. \end{cases}$$

Here, $\dfrac{d^{2n}g_p}{dx^{2n}}(x) = x^{p-2n} > 0$ which shows that g_p is $(2n)$-convex for $x > 0$ and $p \mapsto \dfrac{d^{2n}g_p}{dx^{2n}}(x)$ is exponentially convex by definition. Arguing as in Example 6.1 we get that the mappings $p \mapsto L_i(g_p)$, $i = 1, 2, 3$, are exponentially convex. For this family of functions $\mu_{p,j}(L_i, \Omega_2)$, $i = 1, 2, 3$, from (5.9), is now equal to

$$\mu_{p,q}(L_i,\Omega_2)=\begin{cases}\left(\dfrac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}}, & p\neq q,\\ \exp\left((-1)^{2n-1}(2n-1)!\dfrac{L_i(g_0g_p)}{L_i(g_p)}+\displaystyle\sum_{i=0}^{2n-1}\dfrac{1}{i-p}\right), & p=q\notin\{0,1,\ldots,2n-1\},\\ \exp\left((-1)^{2n-1}(2n-1)!\dfrac{L_i(g_0g_p)}{2L_i(g_p)}+\displaystyle\sum_{\substack{i=0\\ i\neq p}}^{2n-1}\dfrac{1}{i-p}\right), & p=q\in\{0,1,\ldots,2n-1\}.\end{cases}$$

Again, using Theorem 5.2 we conclude that

$$\min\{a,b-\lambda,c\}\leq\left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}}\leq\max\{a+\lambda,b,d\},\quad i=1,2,3.$$

So, $\mu_{p,q}(L_i,\Omega_2), i=1,2,3$ is mean.

***Example* 6.3.** Consider a family of functions

$$\Omega_3=\{\phi_p\colon(0,\infty)\to(0,\infty)\colon p\in(0,\infty)\}$$

defined by

$$\phi_p(x)=\begin{cases}\dfrac{p^{-x}}{(\log p)^{2n}}, & p\neq 1,\\ \dfrac{x^{2n}}{(2n)!}, & p=1.\end{cases}$$

Since $\dfrac{d^{2n}\phi_p}{dx^{2n}}(x)=p^{-x}$ is the Laplace transform of a nonnegative function (see [11]) it is exponentially convex. Obviously ϕ_p are $(2n)$-convex functions for every $p>0$. For this family of functions, $\mu_{p,q}(L_i,\Omega_3)$, $i=1,2,3$, from (5.9) is equal to

$$\mu_{p,q}(L_i,\Omega_3)=\begin{cases}\left(\dfrac{L_i(\phi_p)}{L_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p\neq q,\\ \exp\left(-\dfrac{L_i(id\cdot\phi_p)}{p\,L_i(\phi_p)}-\dfrac{2n}{p\log p}\right), & p=q\neq 1,\\ \exp\left(-\dfrac{1}{2n+1}\dfrac{L_i(id\cdot\phi_1)}{L_i(\phi_1)}\right), & p=q=1,\end{cases}$$

where id is the identity function. This is monotone function in parameters p and q by (5.8). Using Theorem 5.2 it follows that

$$M_{p,q}(L_i,\Omega_3)=-L(p,q)\log\mu_{p,q}(L_i,\Omega_3),\quad i=1,2,3,$$

satisfies

$$\min\{a,b-\lambda,c\}\leq M_{p,q}(L_i,\Omega_3)\leq\max\{a+\lambda,b,d\}.$$

So $M_{p,q}(L_i,\Omega_3)$ is monotonic mean. $L(p,q)$ is logarithmic mean defined by

$$L(p,q) = \begin{cases} \dfrac{p-q}{\log p - \log q}, & p \neq q, \\ p, & p = q. \end{cases}$$

***Example* 6.4.** Consider a family of functions

$$\Omega_4 = \{\psi_p \colon (0,\infty) \to (0,\infty) \colon p \in (0,\infty)\}$$

defined by

$$\psi_p(x) = \frac{e^{-x\sqrt{p}}}{p^n}.$$

Since $\dfrac{d^{2n}\psi_p}{dx^{2n}}(x) = e^{-x\sqrt{p}}$ is the Laplace transform of a nonnegative function (see [11]) it is exponentially convex. Obviously ψ_p are $(2n)$-convex functions for every $p > 0$. For this family of functions, $\mu_{p,q}(L_i, \Omega_4)$, $i = 1, 2, 3$, from (5.9) is equal to

$$\mu_{p,q}(L_i, \Omega_4) = \begin{cases} \left(\dfrac{L_i(\psi_p)}{L_i(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(-\dfrac{L_i(id \cdot \psi_p)}{2\sqrt{p}L_i(\psi_p)} - \dfrac{n}{p}\right), & p = q, \end{cases}$$

where id is the identity function. This is monotone function in parameters p and q by (5.8). Using Theorem 5.2 it follows that

$$M_{p,q}(L_i, \Omega_4) = -\left(\sqrt{p} + \sqrt{q}\right) \log \mu_{p,q}(L_i, \Omega_4), \quad i = 1, 2, 3,$$

satisfies $\min\{a, b-\lambda, c\} \leq M_{p,q}(L_i, \Omega_4) \leq \max\{a+\lambda, b, d\}$, so $M_{p,q}(L_i, \Omega_4)$ is monotonic mean.

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